Spaces as Distributive Lattices

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Spaces as Distributive Lattices

**Axiom of Choice**

\[
\{0, 1\}^\mathbb{N} \to [0, 1]
\]

\[
(b_n) \mapsto \sum b_n / 2^n
\]

This is a surjective map, but it has *no* continuous section

Unfortunately, the fact that it is surjective is not constructively valid
Axiom of Choice

\[ \{-1, 0, 1\}^\mathbb{N} \to [-1, 1] \]

\[ (b_n) \mapsto \sum b_n/2^n \]

This is a surjective map, but it has no continuous section

This time, the fact that it is onto is constructively valid
Spaces as Distributive Lattices

Completeness Theorem

“If something can be stated simply, it has a simple proof”

$$\forall z.inv(1 - uz) \land \forall z.inv(1 - vz) \rightarrow \forall z.inv(1 - (u + v)z)$$
There are a lot of examples of use of topology in algebra

Zariski spectrum of a ring, the space of valuations, the notion of scheme, . . .

Problem: the existence of the elements of these spaces is usually proved using Zorn’s Lemma. How can we represent them computationally?
In constructive mathematics, it is possible to keep these rich topological intuitions by defining a (formal) space to be a *distributive lattice*.

The elements of this lattice have to be thought of as basic open of the space. We are going to present two examples: the Zariski spectrum of a ring and the space of valuations of a field.
A distributive lattice can be thought of as a logical approximation of rings: we replace $+$ by $\lor$ and $\times$ by $\land$.

We have a duality between $\lor$ and $\land$ which is invisible in the theory of rings.

We think of the elements $U$ of the lattice as basic open of a topological space.
Points

What should be a point? We represent it as a predicate $\alpha(U)$ meaning that the point is in $U$. We should have

$$\alpha(1), \quad \neg \alpha(0)$$

$$\alpha(U_1 \lor U_2) \rightarrow \alpha(U_1) \lor \alpha(U_2)$$

$$\alpha(U_1 \land U_2) \leftrightarrow \alpha(U_1) \lor \alpha(U_2)$$

Classically we can think of $\alpha$ as a lattice map $L \rightarrow 2$ where $2$ is the two element lattice

A point is similar to the complement of a prime ideal

We write $Sp(L)$ the space of points of $L$
The topology is in general non separated and we have an order on points

\[ \alpha_1 \leq \alpha_2 \text{ iff } \alpha_1(U) \leq \alpha_2(U) \text{ for all } U \]

The \textit{Krull dimension} \(n\) of a lattice is the length of maximal proper chain

\[ \alpha_0 < \cdots < \alpha_n \]
Spaces as Distributive Lattices

Morphisms

Any lattice map $\psi : L_1 \rightarrow L_2$ defines (by composition) a continuous map $\psi^* : Sp(L_2) \rightarrow Sp(L_1)$

**Proposition:** *The map $\psi^*$ is surjective iff the map $\psi$ is injective*

This can easily proved using Zorn’s Lemma. We understand this result as the fact that we can express the surjectivity of a map in an algebraic way.

An example of this situation will be provided by the *center* map in algebraic geometry.
Zariski spectrum

Fundamental object in abstract algebra, usually defined as a set of prime ideals of a ring $R$ with the basic open

$$D(a) = \{ p \mid a \notin p \}$$

This is a spectral space

The compact open form a distributive lattice. They are exactly the finite union $D(a_1) \lor \cdots \lor D(a_n)$
Zariski spectrum

However, even if the ring $R$ is given concretely (discrete) it may be difficult to show effectively the existence of one prime ideal.

For instance if $N$ is a very large integer, to give a prime ideal of $\mathbb{Z}/N\mathbb{Z}$ is to give a prime factor of $N$.

Often, what matters is not one particular prime ideals, but the collection of all prime ideals.
Zariski spectrum

Zariski spectrum is best seen as a point-free space (cf. Menger, 1940, de Bruijn 1967)

A. Joyal (1972) definition of the Zariski spectrum

We consider the distributive lattice defined by the generators $D(a), \ a \in R$ (seen as formal symbols) and the relations

\[ D(0) = 0 \quad D(1) = 1 \quad D(ab) = D(a) \land D(b) \quad D(a + b) \leq D(a) \lor D(b) \]
In general we define a support of $R$ to be a distributive lattice $L$ with a map $D : R \to L$ satisfying the relations

\[ D(0) = 0 \quad D(1) = 1 \quad D(ab) = D(a) \wedge D(b) \quad D(a + b) \leq D(a) \vee D(b) \]

Intuitively $D(f)$ is the “open set” over which the function $f$ is $\neq 0$

We can distinguish between the properties of an arbitrary support and the properties of the universal support, the Zariski lattice $\text{Zar}(R)$
For an arbitrary support we have $D(a^2) = D(a)$ and $D(a^n) = D(a)$ if $n \geq 1$

We have $D(a, b) = D(a + b, ab)$

If $D(ab) = 0$ then $D(a + b) = D(a, b)$

For the Zariski lattice, all elements can be written on the form

$$D(a_1, \ldots, a_n) = D(a_1) \lor \cdots \lor D(a_n)$$

$D(a) \leq D(b_1, \ldots, b_m)$ if $a$ is in the radical of the ideal generated by $b_1, \ldots, b_m$
Nullstellensatz

**Theorem:** \( D(a_1) \land \cdots \land D(a_n) \leq D(b_1, \ldots, b_m) \) holds iff the product \( a_1 \ldots a_n \) is in the radical of the ideal generated by \( b_1, \ldots, b_m \)

This is also known as the *formal* version of the Nullstellensatz. This can be seen as a *cut-elimination* result: any proof can be reduced to a direct proof.

If \( R \) polynomial ring over \( \mathbb{Q} \), \( D(p) \) can be thought of as the complement of the set of zeros of \( p \) (in some algebraic closure).
The proof of the Nullstellensatz is an explicit construction of the Zariski spectrum (by opposition to a purely abstract universal characterisation)

We consider the (distributive) lattice of radicals of finitely generated ideal and we define $D(a)$ to be $\sqrt{<a>}$

Notice that in the general the lattice of ideals of a ring is not distributive
Spaces as Distributive Lattices

**Zariski spectrum**

This definition is purely algebraic: we manipulate only rings and lattices, $R \mapsto \text{Zar}(R)$ is a functorial construction.

Even if $R$ is discrete (we have an algorithm to decide the equality in $R$), the lattice $\text{Zar}(R)$ does not need to be discrete.

Counter-example with Kripke model: $\mathbb{Z} \rightarrow \mathbb{Z}[1/2]$ is injective but $\text{Zar}(\mathbb{Z}) \rightarrow \text{Zar}(\mathbb{Z}[1/2])$ is not.
Any element of the Zariski lattice is of the form $D(a_1, \ldots, a_n)$. We have seen that $D(a, b) = D(a + b)$ if $D(ab) = 0$

In general we cannot write $D(a_1, \ldots, a_n)$ as $D(a)$ for one element $a$

We can ask: what is the least number $m$ such that any element of $\text{Zar}(R)$ can be written on the form $D(a_1, \ldots, a_m)$. An answer is given by the following version of Kronecker’s Theorem: this holds if $\text{Kdim } R < m$
Sheaf over lattices

If $L$ is a distributive lattice, a *presheaf of rings over $L$* is a family $\mathcal{F}(U)$ of rings for each element $U$ of $L$ with a map $\mathcal{F}(U) \to \mathcal{F}(V)$, $x \mapsto x|V$ whenever $V \leq U$

We require furthermore $x|U = x$ for $x \in \mathcal{F}(U)$ and $(x|V)|W = x|W$ whenever $W \leq V \leq U$
We say that $\mathcal{F}$ is a sheaf iff

(1) whenever $U = U_1 \vee U_2$ and $x_i \in \mathcal{F}(U_i)$ and $x_1|U_1 \wedge U_2 = x_2|U_1 \wedge U_2$ then there exists one and only one $x$ in $\mathcal{F}(U)$ such that $x|U_i = x_i$.

(2) $\mathcal{F}(0)$ is the trivial ring 0.

If $\mathcal{F}$ is a sheaf over a lattice $L$ and $U$ is an element of $L$ then $\mathcal{F}$ defines a sheaf by restriction on the lattice $\downarrow U$. 
Spaces as Distributive Lattices

**Structure sheaf**

To simplify we assume that $R$ is an integral domain

**Lemma:** If $D(b) \leq D(a_1, \ldots, a_n)$ in $\text{Zar}(R)$ then we have

$$R[1/a_1] \cap \cdots \cap R[1/a_n] \subseteq R[1/b]$$
Any element of $\text{Zar}(R)$ can be written $D(b_1, \ldots, b_m) = D(b_1) \lor \cdots \lor D(b_m)$

We define the *structure sheaf* $\mathcal{O}$ on $\text{Zar}(R)$ by

$$\mathcal{O}(D(b_1, \ldots, b_m)) = R[1/b_1] \cap \cdots \cap R[1/b_m]$$

This is well-defined by the previous Lemma
An example of a local-global principle

Classically the point of the space \( \text{Zar}(R) \) are the prime ideals of \( R \) and the fiber of the sheaf \( \mathcal{O} \) at a point \( p \) is the localisation \( R_p \)

One intuition is that we have a continuous family of local rings \( R_p \), and any element of \( R \) defines a global section of this family

We can see \( R[1/a] \) for \( p \) in \( D(a) \) as an “approximation” of \( R_p \) and indeed \( R_p \) can be defined as the inductive limit of all \( R[1/a] \) for \( p \) in \( D(a) \)

We have \( \Gamma(D(a), \mathcal{O}) = R[1/a] \)
Local-global principle

Let us consider a linear system $MX = A$ with $M$ in $\mathbb{R}^{n \times m}$ and $X$ in $\mathbb{R}^{m \times 1}$ and $A$ in $\mathbb{R}^{n \times 1}$

A local-global principle is that if in each $\mathbb{R}_p$ the linear system $MX = A$ has a solution then it has a global solution

If $MX = A$ has a solution in $\mathbb{R}_p$ then we find $a$ such that $p$ in $D(a)$ and $MX = A$ has a solution in $\mathbb{R}[1/a]$.

By compactness we find a finite sequence $a_1, \ldots, a_n$ such that $1 = D(a_1, \ldots, a_n)$ and $MX = A$ has a solution in each $\mathbb{R}[1/a_i]$. 
Local-global principle

The constructive expression of this local-global principle is thus

**Proposition:** If we have $a_1, \ldots, a_n$ such that $1 = D(a_1, \ldots, a_n)$ and $MX = A$ has a solution in each $R[1/a_i]$ then the system $MX = A$ has a global solution in $R$

The proof is simple: we have $X_i, k_i$ such that $MX_i = a_i^{k_i} A$

We have $\Sigma u_is_i^{k_i} = 1$ and so $X = \Sigma u_iX_i$ satisfies $MX = A$

Exactly like “partition of unity” in analysis
Local-global principle

Example: if $M$ is in $R^{k \times l}$ matrix and $1 = \Delta_k(M)$ then $MX = A$

Indeed for each $k \times k$ minor $\delta$ of $M$, we have a solution of $MX = A$ in $R[1/\delta]$
Another local-global principal

If $M$ is an idempotent matrix over a local ring we know that $M$ is similar to a canonical projection matrix.

Hence if $M$ is an idempotent matrix over any ring $R$ the matrix $M$ is locally over any prime $p$ of $R$ similar to a canonical projection matrix.

Hence by compactness we should be able to find $a_1, \ldots, a_n$ such that $1 = D(a_1, \ldots, a_n)$ and $M$ is similar to a canonical projection matrix over each $R[1/a_i]$.

By completeness we expect to be able to find such a sequence $a_1, \ldots, a_n$ from $M$. 
Another local-global principle

Let $M$ be in $R^{l \times l}$ and $C_1, \ldots, C_l$ be the vector column of $M$

Write $E_1, \ldots, E_l$ is the canonical basis of $R^{l \times 1}$

Define $C_i^0 = C_i$ and $C_i^1 = E_i - C_i$

The following argument gives a sequence: write $1 = \det I_l = \det(M + I_l - M)$ as a sum of $2^l$ elements that are the determinants $d_\sigma$ of the matrix $C_{b_1}^{b_1}, \ldots, C_l^{b_l}$ where $\sigma = b_1, \ldots, b_l$ is a sequence of 0, 1

Clearly over $R[1/d_\sigma]$ we have a basis of $\text{Im } M$ formed by the elements $C_i$ for $i$ such that $b_i = 0$ and a basis of $\text{Im } (I_l - M)$ formed by the elements $E_i - C_i$ for $i$ such that $b_i = 1$
Finite local-global principle

A finitely presented module is given by a matrix $M$ over a ring $R$

The $n$-Fitting ideal $I_n$ is the ideal generated by the $n \times n$ minor of $M$

It can be shown that the module defined by $M$ is projective iff $I_n^2 = I_n$ iff $I_n$ is generated by an idempotent

**Proposition:** If we have $a_1, \ldots, a_m$ such that $1 = D(a_1, \ldots, a_m)$ and $M$ defines a projective module over $R[1/a_i]$ for all $i$ then $M$ defines a projective module over $R$
Summary

We can describe topological space used in commutative algebra as *distributive lattices*.

Distributive lattices can be described equationally.

We can also define the notion of *sheaf* over a distributive lattice and express local-global principles.
Let $L$ be a field, and $R$ a subring of $L$

Another spectral space important in commutative algebra is the space $\text{Val}(L, R)$ of *valuation rings* of $L$ containing $R$

Such a ring is a subring $V \subseteq L$ containing $R$ and such that if $s$ in $L$ and $s \neq 0$ then $s$ is in $V$ or $1/s$ is in $V$

We have always the solution $V = L$
We define the lattice $\text{Val}(L, R)$ as the universal solution of the problem $V_R : L \to \text{Val}(L, R)$ with the conditions

\begin{align*}
V_R(r) &= 1 \quad (r \in R) \\
V_R(s_1) \wedge V_R(s_2) &\leq V_R(s_1s_2) \wedge V_R(s_1 + s_2) \\
1 &= V_R(s) \vee V_R(1/s) \quad (s \neq 0)
\end{align*}
In general we cannot simplify $V_R(s_1) \land \cdots \land V_R(s_l)$, but we have

$$V_R(s) \land V_R(1/s) = V_R(s + s^{-1})$$

$$V_R((x + y)^{-1}) \leq V_R(1/x) \lor V_R(1/y)$$

$$1 = V_R(x^{-1}) \lor V_R((1 - x)^{-1})$$
Theorem: $V_R(t_1) \land \cdots \land V_R(t_n) \leq V_R(s_1) \lor \cdots \lor V_R(s_m)$ holds iff we have an equality of the form $1 = \sum 1/s_i P_i(t_j, 1/s_i)$

This is a cut-elimination Theorem, proved by algebraic elimination of variables.
The main Lemma for this result is extracted from the classical proof of existence of valuation rings

**Main Lemma:** If $I$ is an ideal and we have two relations

\[ x^k = b_1 x^{k-1} + \cdots + b_0, \quad 1 = a_l x^l + \cdots + a_0 \]

with $a_0, \ldots, a_l$ in $I$ then $1$ is in $I$

This proved by induction on $k + l$

We can then apply this to the ideal $<1/s_1, \ldots, 1/s_k>$ of $R[1/s_1, \ldots, 1/s_k]$
Special case: \(1 = V_R(s/t_1) \vee \cdots \vee V_R(s/t_n)\) iff \(s\) is integral over the ideal \(I\) generated by \(t_1, \ldots, t_n\) in \(R[t_1, \ldots, t_n, s]\). This means that we have an equality

\[s^l = a_1 s^{l-1} + \cdots + a_l\]

where \(a_k\) is in \(I^k\)

Special case: \(1 = V_R(s)\) iff \(1/s\) is invertible in \(R[1/s]\) iff \(s\) is integral over \(R\)

We get a constructive reading of the fact that the intersection of valuation rings containing \(R\) is the integral closure of \(R\)
**Application: Dedekind Prague’s Theorem**

**Theorem:** If \((\sum a_i X^i)(\sum b_j X^j) = \sum c_k X^k\) then each product \(a_i b_j\) is integral over the coefficients \(c_k\).

This generalises a famous result of Gauss: if all \(a_i, b_j\) are rationals and all \(c_k\) are integers then all products \(a_i b_j\) are integers.

This “may be considered as one of the most basic result in commutative algebra of the XIXth century . . . It ended up as one exercise in Bourbaki, but here it is proved in a non constructive way” (Olaf Neumann)

This appears as an exercise in Bourbaki, Algebra, Chapter 7 (Diviseurs)
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**Application: Dedekind Prague’s Theorem**

We get a proof-theoretic reading of the non constructive argument. We take $L = \mathbb{Q}(a_0, \ldots, a_n, b_0, \ldots, b_m)$, $R = \mathbb{Q}$ and we prove

$$1 = V(a_i b_j/c_0) \lor \cdots \lor V(a_i b_j/c_m)$$

This corresponds to the non constructive argument: prove this for an *arbitrary* valuation
Spaces as Distributive Lattices

**Application: Dedekind Prague’s Theorem**

For \( n = m = 2 \) a proof certificate of \( 1 = V(a_0b_1/c_0) \lor \cdots \lor V(a_0b_1/c_4) \) is

\[
(a_0b_1)^6 = p_1(a_0b_1)^5 + p_2(a_0b_1)^4 + p_3(a_0b_1)^3 + p_4(a_0b_1)^2 + p_5(a_0b_1) + p_6
\]

where

\[
\begin{align*}
p_1 &= 3c_1, \quad p_2 = -3c_1^2 - 2c_0c_2, \quad p_3 = c_1^3 + 4c_0c_1c_2 \\
p_4 &= -c_0^2c_1c_3 - 2c_0c_1^2c_2 - c_0^2c_2^2 + 4c_0^3c_4 \\
p_5 &= c_0^2c_1^2c_3 + c_0^2c_1c_2^2 - 4c_0^3c_1c_4 \\
p_6 &= -c_0^3c_1c_2c_3 + c_0^4c_2^2 + c_0^3c_1^2c_4
\end{align*}
\]
Application: Dedekind Prague’s Theorem

Constructively $L \rightarrow \text{Val}_R(L)$ is seen as a (clever) system of notations which records polynomial identities.

Classically $\text{Val}_R(L)$ is seen as a set of points.
Given any domain $R$ of field of fractions $L$ we have a lattice map

$$\psi : \text{Zar}(R) \to \text{Val}(L, R), \quad D(a) \longmapsto V(1/a) \ (a \neq 0)$$

This is the center map. It is always injective.

The (constructive) proof of this fact requires cut-elimination

Intuitively: the function $f$ is $\neq 0$ iff $1/f$ is finite
The terminology comes from the study of points for algebraic curves.

We look at the local ring at a point of the curve.

If the point is not singular its local ring is a discrete valuation ring.

If the point is singular there is a finite number of discrete valuation rings of center the maximal ideal defined by this point. In this case, it is possible to show directly the existence of these valuations.