# Constructive algebra

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# Structure sheaf

Classically we have a family of rings $R_\alpha$ varying continuously in $\alpha$ point of the Zariski spectrum of $R$

$$R_\alpha = \varinjlim_{\alpha \in D_Z(a)} R[1/a]$$

We defined sheaf $F_R(D(a)) = R[1/a]$ only on basic open; in general

$$F_R(U) = \varprojlim_{D(a) \subseteq U} R[1/a]$$

# Double negation

We defined $a \Vdash \varphi$

$[\![\varphi]\!]$ as an open set is the union of $D(a)$ such that $a \Vdash \varphi$

$\Vdash \neg\neg\varphi$ iff $[\![\varphi]\!]$ is *dense*

$U$ is dense iff $U \cap D(a) = 0$ implies $D(a) = 0$

# Double negation

**Theorem:** (Ingo Blechschmidt, 2016) *If* $M$ *is a finitely generated module on* $R$ *then the set of* $a$ *such that* $M[1/a]$ *is free over* $R[1/a]$ *is dense in this sense*

The proof is direct by induction on the number of generators of $R$

This generalizes the Grothendieck generic freeness Lemma, EGA IV, lemme 6.9.2 stated there only for *Noetherian* integral domain

# Sheaf property

If we have $D(a) = D(a_1, \ldots, a_l)$ and

$x = \frac{u_1}{a_1^N} = \cdots = \frac{u_l}{a_l^N}$

then we have $M$ such that $a^M = \Sigma v_i a_i^N$ and then

$x = \frac{\Sigma v_i u_i}{\Sigma v_i a_i^N} = \frac{\Sigma v_i u_i}{a^M}$

is in $R[1/a]$

High-school fact: if $m/n = p/q$ then $m/n = p/q = (m+n)/(p+q)$

# Local-global property

A property $\psi(R)$ of a ring $R$ is *local-global* iff

if $D(a) = D(a_1, \dots, a_n)$ then $\psi$ holds for $R$ iff it holds for all $R[1/a_i]$

# Local-global property

(1) "To be a local ring" if *not* a local-global property!

(2) That a given linear system of equations $AX = B$ has a solution is a local-global propery

(3) That a given element is a square is not a local-global property

(4) To be integrally closed is a local-global property

# Prüfer domain are integrally closed

We follow the proof that valuation domains are integrally closed and use that this is a local global property

Given $x = a/b$ such that $x^2 = u_1 x + u_2$ we have (Prüfer property)

$a/b = s/u$ and $b/a = t/v$ with $u + v = 1$

We have $x = u_1 + u_2/x$ so $a/b = u_1 + u_2 t/v = (u_1 v + u_2 t)/v$

Hence $a/b = s/u = (u_1 v + u_2 t)/v = s + u_1 v + u_2 t$

High-school fact: if $m/n = p/q$ then $m/n = p/q = (m + n)/(p + q)$