Optimal Sequential Decisions Under Uncertainty

Part II: Subjective probability and utility

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Subjective probability

- We can use probability to represent a subjective belief.
- One problem is how to *elicit* a quantitative representation of our belifs.
- The concept of relative likelihood can be used to construct a *probability distribution*.

How likely are events

The relative likelihood of two events A and B

- Do you think A is more likely than B? Write $A \succ B$.
- Do you think A is less likely than B? Write $A \prec B$.
- Do you think A is as likely as B? Write $A \simeq B$.

We also use $\succeq$ and $\preceq$ for at least as likely as and for no more likely than.

Relation to probability measures

A probability measure P is said to agree with a relation $A \preceq B$, if it has the property that: $P(A) \leq P(B)$ if and only if $A \preceq B$.

The unique probability distribution P

There is a unique P such that for all events A, B such that $A \preceq B$, $P(A) \leq P(B)$.

But how can we find it?

Subjective probability assumptions I

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Assumption (SP1)

For any events A, B, one of the following must hold: $A \succ B$, $A \prec B$, $A \simeq B$.

It is always possible to say whether one event is more likely than the other.

Subjective probability assumptions I

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Assumption (SP1)

For any events A, B, one of the following must hold: $A \succ B$, $A \prec B$, $A \simeq B$.

Assumption (SP2)

Let $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ *with* $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$.
If $A_i \preceq B_i$ *then* $A \preceq B$.

If we can split A, B in such a way that each part of A is less likely than its counterpart in B, then A is less likely than B.

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Assumption (SP3)

If S is the certain event, then: $\emptyset \preceq A$ *and* $\emptyset \prec S$.

Subjective probability assumptions II

Theorem (Transitivity)

If A, B, D such that $A \preceq B$ and $B \preceq D$, then $A \preceq D$.

Theorem (Complement)

For any A, B: $A \preceq B$ iff $A^{\complement} \succeq B^{\complement}$.

Theorem (Fundamental property of relative likelihoods)

If $A \subset B$ then $A \preceq B$. Furthermore, $0 \preceq A \preceq S$ for any event A.

Assigning probabilities

Assigning probabilities

- How can we assign probabilities to events in an unambiguous manner?
- Assume that we feel that $A \succ A^{\complement}$.
- How can we compare A and $A^{\complement}$ with other events?

Imagine a family of events $\mathcal{F}$.

Requirements for events

- Each event $A \in \mathcal{F}$ must have a known probability.
- For any number $p \in [0, 1]$, there exists an event A with probability p.

To assign a probability to some specific event B, we just need to find $A \in \mathcal{F}$ such that $A \simeq B$, and set $P(B) = P(A)$.

Uniform distributions

Let A be an interval on the real line. Denote the length[1] of A by $\lambda(A)$.

Definition

Let X be a random variable such that $0 \leq X(s) \leq 1$ for all $s \in S$. X has a uniform distribution on the interval $[0, 1]$ if: For any A, B subintervals of $[0, 1]$, $\mathbb{I}\{X \in A\} \preceq \mathbb{I}\{X \in B\}$ iff $\lambda(A) \leq \lambda(B)$.

This means that any larger interval is more likely than any smaller interval.

Assumption (SP5)

There exists a random variable which has uniform distribution on $[0, 1]$.

[1] Recall that the length λ is equivalent to the Lebesgue measure

Constructing the probability distribution

The likelihood of a uniform variable falling within an interval

Let X be a uniform random variable. Let $G(A) \triangleq \mathbb{I}\{X \in A\}$, for any interval $A \subset [0, 1]$.

$$G(A) \preceq G(B) \text{ iff } \lambda(A) \leq \lambda(B)$$

Theorem (Equivalent event)

For any event A, there exists some $\alpha \in [0, 1]$ such that $A \simeq G[0, \alpha]$.

Definition (The probability of A)

If A is any event, then $P(A)$ is defined to be α. Hence,

$$A \simeq G([0, P(A)]).$$

Theorem (Relative likelihood and probability)

For any two events A, B, $A \preceq B$ iff $P(A) \leq P(B)$.

Conditional likelihoods

Define $(A \mid D) \preceq (B \mid D)$ to mean that B is at least as likely as A when it is known that D has occured.

Assumption (CP)

For any events A, B, D,

$$(A \mid D) \preceq (B \mid D) \quad \textit{iff} \quad A \cap D \preceq B \cap D.$$

Theorem

If a relation $\preceq$ satisfies assumptions SP1 to SP6 and CP, then P is the unique probability distribution such that:
For any A, B, D such that $P(D) > 0$,

$$(A \mid D) \preceq (B \mid D) \quad \textit{iff} \quad P(A \mid D) \preceq P(B \mid D)$$

A simple exercise in updating beliefs

The area of Germany

Form a subjective probability for the area a of Germany in km^2.

- A_1 $a < 10^5\ \mathrm{km}^2$
- A_2 $a \in [10^5, 2.5 \cdot 10^5)\ \mathrm{km}^2$
- A_3 $a \in [2.5 \cdot 10^5, 5 \cdot 10^5)\ \mathrm{km}^2$
- A_4 $a \in [5 \cdot 10^5, 10^6)\ \mathrm{km}^2$
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Choose $P(A_i)$ for all i.

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Additional information

- The EU's largest country is France ($6.7 \cdot 10^5 \mathrm{km}^2$) and the smallest is Malta with $316 \mathrm{km}^2$.

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The correct answer is A_3, since $a = 3.57 \cdot 10^5$

A simple exercise in probability elicitation

Temperature prediction

Let τ be the temperature tomorrow at noon in Frankfurt.

Eliciting the prior / forming the subjective probability measure P

- Select a temperature x_0 such that $\mathbb{I}\{\tau \leq x_0\} \simeq \mathbb{I}\{\tau > x_0\}$.

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- Select two temperatures x_1, x_2 such that $P(\tau \in [x_1, x_2]) = 0.9$.