1 Graph definitions

The elements of a graph

Definition 1 (Graph)**.** A graph $G = (V, E)$ is a tuple where V is a set of nodes and E is a set of edges. Each edge e is a pair of nodes (i, j), with $i, j \in V$. If $(i, j) \in E$ then nodes i, j are *connected*.

Remark 1. If the pairs are ordered, then the graph is directed and edges are depicted with arrows. Source nodes are called *parents* and sink nodes *children* Otherwise it is undirected, and edges are depicted with lines.

Glossary

graph	graphe
node	nœd
edge	arête, lien, arc
directed graph	graphe orienté
undirected graph	graphe non-orienté

Walks, paths and trails

Definition 2 (Walk (promenade))**.** A walk from node k_1 to node k_n is a sequence of nodes $k_1, \ldots, k_n, j$, where $(k_t, k_{t+1}) \in E$.

Definition 3 (Trail (chemin non-elementaire))**.** A trail from node k_1 to node k_n is a walk with no edge (k_t, k_{t+1}) appearing twice.

Definition 4 (Path (chemin elementaire))**.** A path from node k_1 to node k_n is a walk with no node in $k_2, \ldots, k_{n-1}$ appearing twice.

Definition 5 (Connected graph)**.** A graph is connected when there is a path between any two pairs of nodes.

Definition 6 (Cycle)**.** A cycle is a path $k_1, \ldots, k_n, k_1$.

Definition 7 (Tree)**.** A tree is a connected graph with no cycles.

Remark 2 (Isomorphism). The same graph can be depicted in many different ways geometrically. This is called isomorphism.

Assigning meaning to graphs

- By itself, the graph structure may not be very useful.
- We can assign different properties to nodes and edges

Graph applications

There are many graph applications, as graphs can be used to represent arbitrary relations between things. Examples are

Example graphs

- Chemical structures
- Biological/Computer/Social networks
- Probabilistic graphical models
- Decision diagrams.
- Preference graphs.
- Decision trees.

Preferences

Definition 8 (Preference graph)**.** A preference graph is a graph $G = (V, E)$ such that $x \to y$ if and only if we prefer x to y.

If, for any two nodes $x, y \in V$, there is a *directed path* from x to y or from y to x, then we have a total preference order, i.e. we can always say whether we prefer x to y.

Example 1. Elections where voters vote for a preference list of candidates are of this type. However, combining voters' preferences in a "fair" way is not possible.

Graph problems

- Shortest path
- Matching
- Largest clique
- Colouring problem
- ...

2 Graph characteristics

Definition 9 (Degree)**.** For undirected graphs, the degree of a node i is defined as

$$\deg(i) = \sum_{(i,j) \in E} 1,$$

i.e. the number of edges containing i.

Definition 10 (In-degree and out-degree)**.** For directed graphs, the indegree of a node i is defined as

$$\deg^+(i) = \sum_{(j,i) \in E} 1, \qquad \deg^-(i) = \sum_{(i,j) \in E} 1,$$

i.e. the number of incoming and outgoing edges to and from i.

3 Random graphs

Random graphs

Definition 11 (Erdõs-Rényi Model)**.** A graph G generated from an $\mathcal{ER}(n, M)$ model is chosen uniformly from all graphs with n nodes and M edges.

Definition 12 (Gilbert model)**.** A graph G generated from an $\mathcal{G}(n, p)$ has n nodes and the probability of any edge (i, j) being in the graph is p.

Remark 3. The Gilbert model is equivalent to an ER model where M is also chosen randomly with probability

$$p^M (1-p)^{\binom{n}{2}-M}.$$

4 Graph representations

The Laplacian of a graph

Definition 13 (Graph Laplacian)**.** The Laplacian of graph $G = (V, E)$ with n vertices is the $n \times n$ matrix L:

$$L_{i,j} \triangleq \begin{cases} \deg(i), & i = j \\ -1, & i \neq j, e(i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$