Strong Normalization for Guarded Recursive Types

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Introduction

- Guarded recursive types (Nakano, LICS 2000)
- Negative recursive types while maintaining consistency
  - $\mu X. \blacktriangleright X \to A$
  - $\mathsf{fix} : (\blacktriangleright A \to A) \to A$
- Applications
  - Semantics (abstracting step-indexing)
  - Functional Reactive Programming (causality)
  - Coinduction (productivity, with a "Globally"/"$\Box$" modality)

- This talk: Strong Normalization.

Guarded types

- Types and terms.

$$
\begin{array}{lcl}
A, B & ::= & A \to B \mid \blacktriangleright A \mid X \mid \mu X. A \\
t, u & ::= & x \mid \lambda x.t \mid t\,u \mid \mathsf{next}\,t \mid t * u
\end{array}
$$

- Occurrences of X in $\mu X. A$ must be under a $\blacktriangleright$ "guard".
- Good:
  - $\mu X. \blacktriangleright X$
  - $\mu X. A \times \blacktriangleright X$ and $\mu X. \blacktriangleright (A \times X)$
  - $\mu X. (\blacktriangleright X) \to A$ and $\mu X. \blacktriangleright (X \to A)$.
- Bad:
  - $\mu X. X$ and $\mu X. A \times X$
  - $\mu X. X \to A$ and $\mu X. X \to \blacktriangleright A$
  - $\mu X. \blacktriangleright \mu X. X$.

Typing

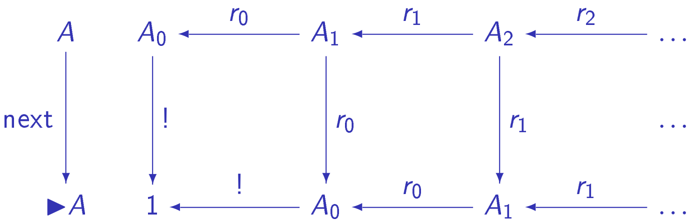
- Type equality: congruence closure of $\vdash \mu X.A = A[\mu X.A/X]$.
- Typing $\Gamma \vdash t : A$.

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{next}\, t : \blacktriangleright A} \qquad \frac{\Gamma \vdash t : \blacktriangleright(A \to B) \qquad \Gamma \vdash u : \blacktriangleright A}{\Gamma \vdash t * u : \blacktriangleright B}$$

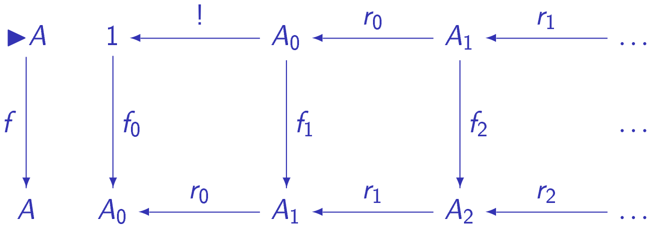
$$\frac{\Gamma \vdash t : A \qquad \vdash A = B}{\Gamma \vdash t : B}$$

Denotational Semantics

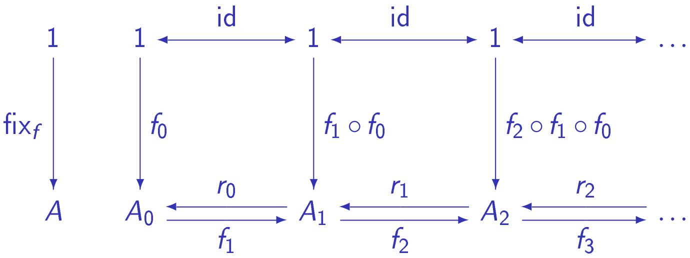
Types as streams of sets:
$A : \mathbb{N} \to Set$ with restriction maps.



Fixed-point construction (intuition)



Any map $f : \blacktriangleright A \to A$ has a fixed-point $\mathrm{fix}_f : 1 \to A$:



Reduction

- Redex contraction $t \mapsto t'$.

$$
\begin{array}{lcl}
(\lambda x.t)\, u & \mapsto & t[u/x] \\
\mathsf{next}\, t * \mathsf{next}\, u & \mapsto & \mathsf{next}\,(t\, u)
\end{array}
$$

- Full one-step reduction $t \longrightarrow t'$: Compatible closure of $\mapsto$.

Recursion from recursive types

Guarded recursion combinator can be encoded.
The standard Y combinator would need a type T such that

$$T = T \to A$$

to typecheck the self applications of x and ω:

$$\begin{array}{lclcll} & & f & : & A \to A & \\ & & x\,x & : & A & \text{if } x : T \\ \omega & := & \lambda(x : T).\, f\,(x\,x) & : & T \to A & \\ \mathsf{Y} & := & \omega\,\omega & : & A & \end{array}$$

Recursion from recursive types

We can solve $T = \blacktriangleright T \to A$:

$$T = \mu X. \blacktriangleright X \to A$$

So we get a guarded fixpoint combinator:

$$\begin{array}{lllll}
 & & f & : & \blacktriangleright A \to A \\
 & & x & : & \blacktriangleright(\blacktriangleright T \to A) \quad \text{if } x : \blacktriangleright T \\
 & & x * \mathsf{next}\, x & : & \blacktriangleright A \quad \text{if } x : \blacktriangleright T \\
\omega & := & \lambda(x : \blacktriangleright T).\, f\,(x * \mathsf{next}\, x) & : & \blacktriangleright T \to A \\
\mathsf{Y}_f & := & \omega\,(\mathsf{next}\,\omega) & : & A
\end{array}$$

$$\mathsf{Y}_f \longrightarrow f\,(\mathsf{next}\,\omega * \mathsf{next}\,(\mathsf{next}\,\omega)) \longrightarrow f\,(\mathsf{next}\,(\omega\,(\mathsf{next}\,\omega))) = f\,(\mathsf{next}\,\mathsf{Y}_f)$$

Note: Full reduction $\longrightarrow$ of Y_f diverges.

More Examples

- Streams!?
- RepMin: One pass through binary tree, replacing all labels by their minimum.
- Attribute grammars!?

Restricted reduction

- Restore normalization: do not reduce under next.
- Relaxed: reduce only under next up to a certain depth.
- Family $\longrightarrow_n$ of reduction relations.

$$\frac{t \mapsto t'}{t \longrightarrow_n t'} \qquad \frac{t \longrightarrow_n t'}{\mathsf{next}\, t \longrightarrow_{n+1} \mathsf{next}\, t'}$$

- Plus compatibility rules for all other term constructors.
- $\longrightarrow_n$ is monotone in n (more fuel gets you further).
- Goal: each $\longrightarrow_n$ is strongly normalizing.

Restricted reduction (Example)

$$\mathsf{Y} \longrightarrow_0^* f\,(\mathsf{next}\,\mathsf{Y}) \not\longrightarrow_0$$

$$\mathsf{Y} \longrightarrow_1^* f\,(\mathsf{next}\,(f\,(\mathsf{next}\,\mathsf{Y}))) \not\longrightarrow_1$$

$$\mathsf{Y} \longrightarrow_2^* f\,(\mathsf{next}\,(f\,(\mathsf{next}\,(f\,(\mathsf{next}\,\mathsf{Y}))))) \not\longrightarrow_2$$

$$\vdots$$

Strong normalization as well-foundedness

- $t \in \mathsf{sn}_n$ if $\longrightarrow_n$ reduction starting with t terminates.

$$\frac{\forall t'.\, t \longrightarrow_n t' \implies t' \in \mathsf{sn}_n}{t \in \mathsf{sn}_n}$$

- sn_n is antitone in n, since $\longrightarrow_n$ occurs negatively.
- More reductions $\implies$ less termination.

Inductive SN_n

- Take the inductively defined normal forms:

$$E ::= \_ \mid E\,u \mid E * u \mid \mathsf{next}\,t * E$$

$$\frac{E \in \mathsf{SN}_n}{E[x] \in \mathsf{SN}_n} \qquad \frac{t \in \mathsf{SN}_n}{\lambda x.t \in \mathsf{SN}_n} \qquad \frac{}{\mathsf{next}\,t \in \mathsf{SN}_0} \qquad \frac{t \in \mathsf{SN}_n}{\mathsf{next}\,t \in \mathsf{SN}_{n+1}}$$

- And close them under "Strong head reduction" $t \longrightarrow_n^{\mathsf{SN}} t'$

$$\frac{t \longrightarrow_n^{\mathsf{SN}} t' \qquad t' \in \mathsf{SN}_n}{t \in \mathsf{SN}_n} \qquad \frac{t \mapsto t' \qquad t \in \mathsf{SN}_n}{E[t] \longrightarrow_n^{\mathsf{SN}} E[t']}$$

- $t \longrightarrow_n^{\mathsf{SN}} t'$ is like weak head reduction but erased terms must be s.n.

Notions of s.n. coincide?

- Rules for SN_n are closure properties of sn_n.
- $\mathsf{SN}_n \subseteq \mathsf{sn}_n$ follows by induction on SN_n.
- Converse $\mathsf{sn}_n \subseteq \mathsf{SN}_n$ does not hold!
- Counterexamples are ill-typed s.n. terms, e.g.,

$$(\lambda x.x) * y \qquad \text{or} \qquad (\mathsf{next}\, x)\, y.$$

- Solution: consider only well-typed terms.
- Proof of $t \in \mathsf{sn}_n \implies t \in \mathsf{SN}_n$ by case distinction on t: neutral ($E[x]$), introduction ($\lambda x.t, \mathsf{next}\, t$), or weak head redex.

Saturated sets (semantic types)

- Types are modeled by sets $\mathscr{A} \subseteq \mathsf{SN}_n$.
- n-closure $\overline{\mathscr{A}}_n$ of $\mathscr{A}$ inductively:

$$\frac{t \in \mathscr{A}}{t \in \overline{\mathscr{A}}_n} \qquad \frac{E \in \mathsf{SN}_n}{E[x] \in \overline{\mathscr{A}}_n} \qquad \frac{t \longrightarrow_n^{\mathsf{SN}} t' \qquad t' \in \overline{\mathscr{A}}_n}{t \in \overline{\mathscr{A}}_n}$$

- $\mathscr{A}$ is n-saturated ($\mathscr{A} \in \mathsf{SAT}_n$) if $\overline{\mathscr{A}}_n \subseteq \mathscr{A}$.
- Saturated sets are non-empty (contain e.g. the variables).

Constructions on semantic types

- Function space and "later":

$$\mathscr{A} \to \mathscr{B} = \{t \mid t\,u \in \mathscr{B} \text{ for all } u \in \mathscr{A}\}$$
$$\blacktriangleright_n \mathscr{A} = \overline{\{\text{next}\,t \mid t \in \mathscr{A} \text{ if } n > 0\}}_n$$

- If $\mathscr{A}, \mathscr{B} \in \text{SAT}_n$ then $\mathscr{A} \to \mathscr{B} \in \text{SAT}_n$.
- $\blacktriangleright_0 \mathscr{A} \in \text{SAT}_0$.
- If $\mathscr{A} \in \text{SAT}_n$ then $\blacktriangleright_{n+1} \mathscr{A} \in \text{SAT}_{n+1}$.

Type interpretation

- Type interpretation $[\![A]\!]_n \in \mathsf{SAT}_n$

$$
\begin{array}{lcl}
[\![A \to B]\!]_n & = & \bigcap_{n' \leq n}([\![A]\!]_{n'} \to [\![B]\!]_{n'}) \\
[\![\blacktriangleright A]\!]_0 & = & \blacktriangleright_0 \mathsf{SN}_0 = \overline{\{\mathsf{next}\ t\}}_0 \\
[\![\blacktriangleright A]\!]_{n+1} & = & \blacktriangleright_{n+1} [\![A]\!]_n \\
[\![\mu X. A]\!]_n & = & [\![A[\mu X. A/X]]\!]_n
\end{array}
$$

- By lex. induction on $(n, \mathsf{size}(A))$ where $\mathsf{size}(\blacktriangleright A) = 0$.
- Requires recursive occurrences of X to be guarded by a $\blacktriangleright$.

Type soundness

- Context interpretation:

$$\rho \in [\![\Gamma]\!]_n \iff \rho(x) \in [\![A]\!]_n \text{ for all } (x{:}A) \in \Gamma$$

- Identity substitution $\mathsf{id} \in [\![\Gamma]\!]_n$ since $x \in [\![A]\!]_n$.
- Type soundness: if $\Gamma \vdash t : A$ then $t\rho \in [\![A]\!]_n$ for all n and $\rho \in [\![\Gamma]\!]_n$.
- Corollary: $t \in \mathsf{SN}_n$ for all n.

Formalization in Agda

Syntax of types as a mixed inductive-coinductive datatype:

$$\mathsf{Ty} = \nu X \mu Y.\,(Y \times Y) + X$$

```
mutual
  data Ty : Set where
    _→̂_ : (a b : Ty)   → Ty
    ▸̂_  : (a∞ : ∞Ty)   → Ty

  record ∞Ty : Set where
    coinductive
    constructor delay_
    field       force_ : Ty
```

- Intensional (propositional) equality too weak for coinductive types.
- $\Longrightarrow$ add an extensionality axiom for our coinductive type.

Well-typed terms

```
data Tm (Γ : Cxt) : (a : Ty) → Set where
  var   : ∀{a}       (x : Var Γ a)                                  → Tm Γ a
  abs   : ∀{a b}     (t : Tm (a :: Γ) b)                            → Tm Γ (a →̂ b)
  app   : ∀{a b}     (t : Tm Γ (a →̂ b))  (u : Tm Γ a)               → Tm Γ b
  next  : ∀{a∞}      (t : Tm Γ (force a∞))                          → Tm Γ (▸̂ a∞)
  _*_   : ∀{a∞ b∞}   (t : Tm Γ (▸̂(a∞ ⇒ b∞))) (u : Tm Γ (▸̂ a∞))      → Tm Γ (▸̂ b∞)
```

- We used intrinsically well-typed terms (data structure indexed by typing context and type expression).
- Second Kripke dimension (context) required "everywhere", e.g., in SN and $[\![A]\!]$.

Conclusions & Further work

- Strong normalization is a new result, albeit expected for the restricted reduction.
- Agda formalization (ca. 3kLoc, 170kB) useful as basis for further research.
- Add modalities to handle (co)inductive types.
- Integrate into Intensional Type Theory.