Set-based Operators and Fixed-Points by Support

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In Abel and Altenkirch's article *A Predicative Analysis of Structural Recursion* [AA02] one finds on page 34 a definition of the semantics $[\![\sigma]\!]$ of a strictly positive inductive type σ. This definition (5.1) is itself a strictly-positive inductive definition on the meta-level. In the following we exhibit its correspondence to the algorithmic construction of a least fixed-point as found in Pierce's book *Types and Programming Languages* [Pie02] on pages 290–295.

First we give a simplified version of the definition restricted to a single type variable.

Definition 5.1. For $\sigma \in \mathrm{Ty}(X)$ and a closed type τ let

$$[\![\sigma]\!] : \mathcal{P}\left(\mathrm{Val}^\tau\right) \to \mathcal{P}\left(\mathrm{Val}^{\sigma(\tau)}\right)$$

be a monotone operator with a urelement relation

$$\mathcal{U}^\sigma \subseteq \mathrm{Val}^\tau \times \mathrm{Val}^{\sigma(\tau)}$$

such that for all $v \in \mathrm{Val}^{\sigma(\tau)}$ and $u \in \mathrm{Val}^\tau$:

$$\frac{v \in [\![\sigma]\!](V) \qquad u\,\mathcal{U}^\sigma\; v}{u \in V}\,(\mathrm{sb1}) \qquad \frac{v \in [\![\sigma]\!](\mathrm{Val}^\tau)}{v \in [\![\sigma]\!](\mathcal{U}^\sigma(v))}\,(\mathrm{sb2})$$

where $\mathcal{U}^\sigma(v) = \{w \mid w\,\mathcal{U}^\sigma v\}$. The interpretation of μ-types is then obtained by

$$\frac{v \in [\![\sigma]\!](\mathrm{Val}^{\mu X.\sigma}) \qquad \forall u.\; u\,\mathcal{U}^\sigma\; v \to u \in [\![\mu X.\sigma]\!]}{\mathrm{fold}(v) \in [\![\mu X.\sigma]\!]}$$

Pierce (page 290) assumes a universe U and an operator $F : \mathcal{P}(U) \to \mathcal{P}(U)$ and calls a set X *generating for* x if $x \in F(X)$. The support $\mathit{support}_F(x)$ of x is the least generating set for x if there exists one, otherwise undefined. Consider the following correspondences:

Abel, Altenkirch	Pierce
Val	U
$[\![\sigma]\!]$	F
$\mathcal{U}^\sigma$	$\mathit{support}_F$

Then (sb2) expresses that $\mathcal{U}^\sigma(v)$ is a generating set for v if v is $[\![\sigma]\!]$-generateable. Property (sb1) can be rewritten as *if* $v \in [\![\sigma]\!](V)$ *then* $\mathcal{U}^\sigma\ (v) \subseteq V$, expressing the second condition on the support of v that it is the least generating set for v. Hence, for $[\![\sigma]\!]$-generateable elements v, it holds that $\mathcal{U}^\sigma(v) = \mathit{support}_{[\![\sigma]\!]}(v)$.

Extending the urelement relation to sets,

$$\mathcal{U}^\sigma(V) = \bigcup_{v \in V} \mathcal{U}^\sigma(v),$$

property (sb1) becomes $W \subseteq [\![\sigma]\!](V)$ *implies* $\mathcal{U}^\sigma(W) \subseteq V$, exhibiting a Galois connection. This corresponds to Pierce's Lemma 21.5.7 (page 293). Finally, in Exercise 21.5.13 Pierce gives a partial algorithm to test whether a set X is contained in the least-fixed point μF. The logical reading of the algorithm is

$$X \subseteq \mu F \quad \text{iff} \quad X \text{ empty or } (\mathit{support}_F(X) \text{ defined and } \mathit{support}_F(X) \subseteq \mu F)$$

Abel and Altenkirch's rule for introducing elements of an inductive type, generalized to sets reads

$$\frac{V \subseteq [\![\sigma]\!](\mathrm{Val}^{\mu X.\sigma}) \qquad \mathcal{U}^\sigma(V) \subseteq [\![\mu X.\sigma]\!]}{\mathrm{fold}\, V \subseteq [\![\mu X.\sigma]\!]}.$$

The first condition states that all elements of V are $[\![\sigma]\!]$-generateable which means that the support of V is defined. Modulo the folding operation, which comes from iso-recursive types, the rule now exactly expresses Pierce's algorithm.

References

[AA02] Andreas Abel and Thorsten Altenkirch. A predicative analysis of structural recursion. *Journal of Functional Programming*, 12(1):1–41, January 2002.

[Pie02] Benjamin C. Pierce. *Types and Programming Languages*. MIT Press, 2002.