A Core Calculus for Covering Copatterns

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1 Terms and Types

Types $A, B, C ::= D$ (Inductive data type) $|$ R (Coinductive record type) $|$ $A \to B \mid A \times B \mid 1$

Datatype $D = c_1\ A_1 \mid \cdots \mid c_n\ A_n$ where $c_i : A_i \to D$. More precisely, $D = \mu X. \langle c_1 A_1 | \ldots | c_n A_n \rangle$ where $c_i = A_i[D/X] \to D$.

Recordtype $R = \{d_1 : A_1, \ldots d_n : A_n\}$ where $d_i : R \to A_i$. More precisely, $R = \nu Y.\{d_1 : A_1, \ldots, d_n : A_n\}$ where $d_i : R \to A_i[R/Y]$.

Terms $t, s ::= f$ (Functions) $|$ x (Variables) $\mid t_1\ t_2 \mid c\ t \mid t.d \mid (t_1, t_2) \mid ()$

1.1 Typing Rules

$$\frac{}{\Delta, x : A \vdash x : A}\ \mathrm{T_{Var}} \qquad \frac{}{\Delta \vdash f : \Sigma(f)}\ \mathrm{T_{Fun}} \qquad \frac{}{\Delta \vdash () : 1}\ \mathrm{T_{Unit}}$$

$$\frac{\Delta \vdash t_1 : A_1 \to A_2 \quad \Delta \vdash t_2 : A_1}{\Delta \vdash t_1\ t_2 : A_2}\ \mathrm{T_{App}} \qquad \frac{\Delta \vdash t : \nu X.R}{\Delta \vdash t.d : R_d[\nu X.R/X]}\ \mathrm{T_{Dest}}$$

$$\frac{\Delta \vdash t_1 : A_1 \quad \Delta \vdash t_2 : A_2}{\Delta \vdash (t_1, t_2) : A_1 \times A_2}\ \mathrm{T_{Pair}} \qquad \frac{\Delta \vdash t : D_c[\mu X.D/X]}{\Delta \vdash c\ t : \mu X.D}\ \mathrm{T_{Const}}$$

We note that $\Sigma(f)$ is the type of f as defined in the signature Σ, a global predefined context.

Lemma 1 (Inversion lemmas for typing)**.** *The following holds:*

1. *If $\Delta \vdash x : A$, then $\Delta = \Delta', x : A$ for some Δ';*
2. *If $\Delta \vdash f : A$, then $A = \Sigma(f)$;*
3. *If $\Delta \vdash () : A$, then $A = 1$;*
4. *If $\Delta \vdash t_1\ t_2 : A$, then there is a type B such that $\Delta \vdash t_1 : B \to A$ and $\Delta \vdash t_2 : B$;*

5. *If* $\Delta \vdash (t_1, t_2) : A$*, then* $A = A_1 \times A_2$ *for some* A_1, A_2 *and* $\Delta \vdash t_1 : A_1$ *and* $\Delta \vdash t_2 : A_2$*;*

6. *If* $\Delta \vdash t.d : A$*, then* $A = R_d[\nu X.R/X]$ *and* $\Delta \vdash t : \nu X.R$*;*

7. *If* $\Delta \vdash c\ t : A$*, then* $A = \mu X.D$ *and* $\Delta \vdash t : D_c[\mu X.D/X]$*.*

Proof. All the statements are proved by case analysis on the derivation rule.

1. There is only one case: $\mathrm{T}_{\mathrm{Var}}$. Then, $\Delta = \Delta', x : A$.

2. There is only one case: $\mathrm{T}_{\mathrm{Fun}}$. Then, $A = \Sigma(f)$.

3. There is only one case: $\mathrm{T}_{\mathrm{Unit}}$. Then, $A = 1$.

4. There is only one case: $\mathrm{T}_{\mathrm{App}}$. Then, there must be a type A_1 such that $\Delta \vdash t_1 : A_1 \to A_2$ and $\Delta \vdash t_2 A_1$.

5. There is only one case: $\mathrm{T}_{\mathrm{Pair}}$. Then, $A = A_1 \times A_2$ and we have that $\Delta \vdash t_1 : A_1$ and $\Delta \vdash t_2 : A_2$.

6. There is only one case: $\mathrm{T}_{\mathrm{Dest}}$. Then $\Delta \vdash t : \nu X.R$ and $A = R_d[\nu X.R/X]$.

7. There is only one case: $\mathrm{T}_{\mathrm{Const}}$. Then, $A = \mu X.D$ and $t : D_c[\mu X.D/X]$.

□

1.2 Typechecking Rules

Inference mode is described with $\Rightarrow$ and checking mode is described with $\Leftarrow$.

$$\frac{}{\Delta, x : A \vdash x \Rightarrow A}\ \mathrm{TC}_{\mathrm{Var}} \qquad \frac{}{\Delta \vdash f \Rightarrow \Sigma(F)}\ \mathrm{TC}_{\mathrm{Fun}} \qquad \frac{}{\Delta \vdash () \Leftarrow 1}\ \mathrm{TC}_{\mathrm{Unit}}$$

$$\frac{\Delta \vdash t_1 \Rightarrow A_1 \to A_2 \quad t_2 \Leftarrow A_1}{\Delta \vdash t_1\ t_2 \Rightarrow A_2}\ \mathrm{TC}_{\mathrm{App}} \qquad \frac{\Delta \vdash t \Rightarrow \nu X.R}{\Delta \vdash t.d \Rightarrow R_d[R/X]}\ \mathrm{TC}_{\mathrm{Dest}}$$

$$\frac{\Delta \vdash t_1 \Leftarrow A_1 \quad \Delta \vdash t_2 \Leftarrow A_2}{\Delta \vdash (t_1, t_2) \Leftarrow A_1 \times A_2}\ \mathrm{TC}_{\mathrm{Pair}} \qquad \frac{\Delta \vdash t \Leftarrow D_c[D/X]}{\Delta \vdash c\ t \Leftarrow \mu X.D}\ \mathrm{TC}_{\mathrm{Const}}$$

$$\frac{\Delta \vdash t \Rightarrow A \quad A = C}{\Delta \vdash t \Leftarrow C}\ \mathrm{TC}_{\mathrm{Switch}}$$

The missing elimination and introduction rules for our types are described through pattern matching. We thus need to define patterns.

2 Patterns

Patterns $p ::= x \mid (p_1, p_2) \mid c\ p \mid ()$

Destructor Patterns $q ::= \cdot \mid q\ p \mid q.d$

2.1 Typechecking Rules

Pattern typing always returns a context representing all the variables in the pattern. The patterns must be linear, that is, a variable appears only once. There are again two modes for pattern typing. The checking mode, denoted by $\Delta \vdash p \Leftarrow A$, follows the checking mode for regular typing. The inference mode, denoted by $\Delta \mid A \vdash q \Rightarrow C$ is a bit more complicated. In this case, both Δ and C are returned. We need to provide the type of the head of the pattern. We note that $[\cdot]$ acts as a placeholder for the head. $[f]$ means the instantiation of the head by f.

$$\frac{}{x : A \vdash x \Leftarrow A}\ \text{PC}_{\text{Var}} \qquad \frac{\Delta \vdash p \Leftarrow D_c[\mu X.D/X]}{\Delta \vdash c\ p \Leftarrow \mu X.S}\ \text{PC}_{\text{Const}}$$

$$\frac{}{\vdash () \Leftarrow 1}\ \text{PC}_{\text{Unit}} \qquad \frac{\Delta_1 \vdash p_1 \Leftarrow A_1 \quad \Delta_2 \vdash p_2 \Leftarrow A_2}{\Delta_1, \Delta_2 \vdash (p_1, p_2) \Leftarrow A_1 \times A_2}\ \text{PC}_{\text{Pair}}$$

$$\frac{}{\cdot \mid A \vdash [\cdot] \Rightarrow A}\ \text{PC}_{\text{Head}} \qquad \frac{\Delta \mid A \vdash q \Rightarrow \nu X.R}{\Delta \mid A \vdash q.d \Rightarrow R_d[\nu X.R/X]}\ \text{PC}_{\text{Dest}}$$

$$\frac{\Delta_1 \mid A \vdash q \Rightarrow B \to C \quad \Delta_2 \vdash p \Leftarrow B}{\Delta_1, \Delta_2 \mid A \vdash q\ p \Rightarrow C}\ \text{PC}_{\text{App}}$$

$$\frac{\Delta \mid \Sigma(f) \vdash q \Rightarrow C \quad \Delta \vdash u \Leftarrow C}{\vdash q[f] \to u}\ \text{D}_{\text{Pattern}}$$

Lemma 2 (Inversion lemmas for patterns)**.** *The following holds:*

1. *If* $\Delta \vdash x \Leftarrow A$ *then* $\Delta = x : A$*;*
2. *If* $\Delta \vdash () \Leftarrow A$ *then* $\Delta = \cdot$ *and* $A = 1$*;*
3. *If* $\Delta \vdash c\ p \Leftarrow A$ *then* $A = \mu X.D$ *and* $\Delta \vdash p \Leftarrow D_c[\mu X.D/X]$*;*
4. *If* $\Delta \vdash (p_1, p_2) \Leftarrow A$ *then* $A = A_1 \times A_2$ *for some* A_1, A_2 *and there are* Δ_1, Δ_2 *such that* $\Delta = \Delta_1, \Delta_2$*,* $\Delta_1 \vdash p_1 \Leftarrow A_1$ *and* $\Delta_2 \vdash p_2 \Leftarrow A_2$*;*
5. *If* $\Delta \mid A \vdash [\cdot] \Rightarrow B$ *then* $\Delta = \cdot$ *and* $A = B$*;*
6. *If* $\Delta \mid A \vdash q.d \Rightarrow B$ *then* $B = R_d[\nu X.R/X]$ *and* $\Delta \mid A \vdash q \Rightarrow \nu X.R$*;*
7. *If* $\Delta \mid A \vdash q\ p \Rightarrow C$ *then there a type* B *and contexts* Δ_1 *and* Δ_2 *such that* $\Delta = \Delta_1, \Delta_2$*,* $\Delta_1 \mid A \vdash q \Rightarrow B \to C$ *and* $\Delta_2 \vdash p \Leftarrow B$*;*
8. *If* $\vdash q[f] \to u$ *then there is a type* C *and a context* Δ *such that* $\Delta \mid \Sigma(f) \vdash q \Rightarrow C$ *and* $\Delta \vdash u \Leftarrow C$*.*

Proof. All the statements are proved by case analysis on the possible rules allowing us to obtain such derivation.

1. There is only one case: PC_{Var}. Thus, $\Delta = x : A$.

2. There is only one case: PC_{Unit}. Thus, $A = 1$ and $\Delta = \cdot$.

3. There is only one case: PC_{Const}. Thus, $A = \mu X.D$ and $\Delta \vdash p \Leftarrow D_c[\mu X.D/X]$.

4. There is only one case: PC_{Pair}. Thus, $A = A_1 \times A_2$ for some A_1, A_2 there is Δ_1, Δ_2 such that $\Delta = \Delta_1, \Delta_2$ and $\Delta_1 \vdash p_1 \Leftarrow A_1$ and$\Delta_2 \vdash p_2 \Leftarrow A_2$.

5. There is only one case: PC_{Head}. Thus, $A = B$ and $\Delta = \cdot$.

6. There is only one case: PC_{Dest}. Thus, $B = R_d[\nu X.R/X]$ and $\Delta \mid A \vdash q \Rightarrow \nu X.R$.

7. There is only one case: PC_{App}. Thus, there are Δ_1, Δ_2 and a type B such that $\Delta = \Delta_1, \Delta_2$ and $\Delta_1 \mid A \vdash q \Rightarrow B \to C$ and $\Delta_2 \vdash p \Leftarrow B$.

8. There is only one case: $\text{D}_{\text{Pattern}}$. Thus, there is Δ and C such that $\Delta \mid \Sigma(f) \vdash q \Rightarrow C$ and $\Delta \vdash u \Leftarrow C$.

□

3 Pattern Matching

We use the judgment $t =^? p \searrow \sigma$ to mean that the term t matches with the pattern p with resulting substitution σ. More generally, $t =^? q[f] \searrow \sigma$ is used when it is applied to a function.

$$\frac{}{t =^? x \searrow t/x}\,\text{PM}_{\text{Var}} \qquad \frac{}{f =^? f \searrow \cdot}\,\text{PM}_{\text{Fun}} \qquad \frac{}{() =^? () \searrow \cdot}\,\text{PM}_{\text{Unit}}$$

$$\frac{e =^? q \searrow \sigma \quad e' =^? p \searrow \sigma'}{e\ e' =^? q\ p \searrow \sigma, \sigma'}\,\text{PM}_{\text{App}} \qquad \frac{t =^? p \searrow \sigma}{c\ t =^? c\ p \searrow \sigma}\,\text{PM}_{\text{Const}}$$

$$\frac{t_1 =^? p_1 \searrow \sigma_1 \quad t_2 =^? p_2 \searrow \sigma_2}{(t_1, t_2) =^? (p_1, p_2) \searrow \sigma_1, \sigma_2}\,\text{PM}_{\text{Pair}} \qquad \frac{e =^? q \searrow \sigma}{e.d =^? q.d \searrow \sigma}\,\text{PM}_{\text{Dest}}$$

4 Reductions

$$\frac{e =^? q[f] \searrow \sigma \quad q[f] \to u}{e \mapsto u[\sigma]} \qquad \frac{e \mapsto e'}{e \to e'}$$

$$\frac{e_1 \to e_1'}{(e_1, e_2) \to (e_1', e_2)}\,\text{R}_{\text{Pairl}} \qquad \frac{e_2 \to e_2'}{(e_1, e_2) \to (e_1, e_2')}\,\text{R}_{\text{Pairr}} \qquad \frac{e \to e'}{c\ e \to c\ e'}\,\text{R}_{\text{Const}}$$

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2}\,\text{R}_{\text{Appl}} \qquad \frac{e_2 \to e_2'}{e_1\ e_2 \to e_1\ e_2'}\,\text{R}_{\text{Appr}} \qquad \frac{e \to e'}{e.d \to e'.d}\,\text{R}_{\text{Dest}}$$

5 Subject Reduction

Before proving subject reduction, we need to prove a few results first.

Lemma 3 (Substitution Lemma)**.** *If* $\mathcal{D} :: \Delta \vdash u : C$ *and* $\mathcal{E} :: \Gamma \vdash \sigma : \Delta$ *then* $\mathcal{F} :: \Gamma \vdash u[\sigma] : C$ *for some* $\mathcal{F}$.

Proof. The proof is done by induction on the derivation $\mathcal{D} :: \Delta \vdash u : C$.

Base case : $\mathcal{D} :: \overline{\Delta', x : C \vdash x : C}$.

$\mathcal{E}$ contains $\Gamma \vdash \sigma(x) : C$ — by assumption.

$\mathcal{F} :: \Gamma \vdash x[\sigma] : C = \Gamma \vdash \sigma(x) : C$ — by definition of substitution.

Base case: $\mathcal{D} :: \overline{\Delta \vdash f : \Sigma(f)}$

$\Gamma \vdash f[\sigma] : \Sigma = \Gamma \vdash f : \Sigma(f)$ — by $\mathrm{T_{Fun}}$ and definition of substitution.

Base case: $\mathcal{D} :: \overline{\Delta \vdash () : 1}$

$\Gamma \vdash ()[\sigma] : 1 = \Gamma \vdash () : 1$ — by $\mathrm{T_{Fun}}$ and definition of substitution.

Induction step

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}_1}{\Delta \vdash t_1 : A_1 \to A_2} \quad \overset{\mathcal{D}_2}{\Delta \vdash t_2 : A_1}}{\Delta \vdash t_1\ t_2 : A_2}$

$\mathcal{D}_1' :: \Gamma \vdash t_1[\sigma] : A_1 \to A_2$ — by induction hypothesis on $\mathcal{D}_1$.

$\mathcal{D}_2' :: \Gamma \vdash t_2[\sigma] : A_1$ — by induction hypothesis on $\mathcal{D}_2$.

$\mathcal{F} :: \Gamma \vdash t_1[\sigma]\ t_2[\sigma] : A_2$ — by $\mathrm{T_{App}}$.

$\Gamma \vdash (t_1\ t_2)[\sigma] : A_2$ — by definition of substitution for the application.

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{\Delta \vdash t \Rightarrow \nu X.R}}{\Delta \vdash t.d \Rightarrow A_d[R/X]}$

$\mathcal{E} :: \Gamma \vdash t[\sigma] : A_d[R/X]$ — by induction hypothesis on $\mathcal{D}'$.

$\mathcal{E}' :: \Gamma \vdash t[\sigma].d : \nu X.R$ — by $\mathrm{T_{Dest}}$.

$\Gamma \vdash t.d[\sigma] : \nu X.R$ — by definition of substitution.

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}_1}{\Delta \vdash t_1 \Leftarrow A_1} \quad \overset{\mathcal{D}_2}{\Delta \vdash t_2 \Leftarrow A_2}}{\Delta \vdash (t_1, t_2) \Leftarrow A_1 \times A_2}$

$\mathcal{E}_i :: \Gamma \vdash t_i[\sigma] : A_i$ for $i = 1, 2$ — by induction hypothesis on $\mathcal{D}_i$.

$\mathcal{E} :: \Gamma \vdash (t_1[\sigma], t_2[\sigma]) : A_1 \times A_2$ — by $\mathrm{T_{Pair}}$.

$\Gamma \vdash (t_1, t_2)[\sigma] : A_1 \times A_2$ — by definition of substitution.

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{\Delta \vdash t \Leftarrow A_c[D/X]}}{\Delta \vdash c\ t \Leftarrow \mu X.D}$

$\mathcal{E} :: \Gamma \vdash t[\sigma] : A_c[D/X]$ — by induction hypothesis on $\mathcal{D}$.

$\mathcal{E}' :: \Gamma \vdash c\ t[\sigma] : \mu X.D$ — by $\mathrm{T_{Const}}$.

$\Gamma \vdash (c\ t)[\sigma] : \mu X.D$ — by definition of substitution.

□

Lemma 4. *If* $\mathcal{D} :: \Delta \vdash p \Leftarrow A$*,* $\mathcal{E} :: \Gamma \vdash e : A$ *and* $\mathcal{F} :: e =^? p \searrow \sigma$ *then* $\Gamma \vdash \sigma : \Delta$

Proof. The proof is done by induction on the derivation $\mathcal{F} :: e =^? p \searrow \sigma$.

Base case $\mathcal{F} :: \overline{e =^? x \searrow e/x}$.

$\mathcal{D}' :: x : A \vdash x \Leftarrow A$ by inversion on $\mathcal{D}$.

$\Gamma \vdash \sigma(x) : A$ by $\mathcal{F}$.

Base case: $\mathcal{F} :: \overline{() =^? () \searrow \cdot}$

By inversion on $\mathcal{D}$, $\Delta = \cdot$, so there is nothing to show.

Induction step.

Case $\mathcal{F} :: \dfrac{\overset{\mathcal{F}_1}{e_1 =^? p_1 \searrow \sigma_1} \quad \overset{\mathcal{F}_2}{e_2 =^? p_2 \searrow \sigma_2}}{(e_1, e_2) =^? (p_1, p_2) \searrow \sigma_1, \sigma_2}$

$\mathcal{D} :: \Delta \vdash (p_1, p_2) \Leftarrow A$ by assumption.

$\mathcal{E} :: \Gamma \vdash (e_1, e_2) : A$ by assumption.

$\mathcal{D}_i :: \Delta_i \vdash p_i : A_i$ for $i = 1, 2$

where $A = A_i \times A_2$ and $\Delta = \Delta_1, \Delta_2$ by inversion on PC_{Pair}.

$\mathcal{E}_i :: \Gamma \vdash e_i : A_i$ for $i = 1, 2$ by inversion on T_{Pair}.

$\Gamma \vdash \sigma_i : \Delta_i$ by induction hypothesis on $\mathcal{F}_i$.

$\Gamma \vdash \sigma_1, \sigma_2 : \Delta_1, \Delta_2$.

Case $\mathcal{F} :: \dfrac{\overset{\mathcal{F}'}{e =^? p \searrow \sigma}}{c\ e =^? c\ p \searrow \sigma}$

$\mathcal{D} :: \Delta \vdash c\ p \Leftarrow A$ by assumption.

$\mathcal{E} :: \Gamma \vdash c\ e : A$ by assumption.

$\mathcal{D}' :: \Delta \vdash p \Leftarrow D_c[\mu X.D/X]$ and $A = \mu X.D$ by inversion on PC_{Const}.

$\mathcal{E}' = \Gamma \vdash e : D_c[\mu X.D/X]$ by inversion on T_{Const}.

$\Gamma \vdash \sigma : \Delta$ by induction hypothesis on $\mathcal{F}'$.

□

Lemma 5. *If* $\Delta \mid \Sigma(f) \vdash q \Rightarrow C$*,* $\Gamma \vdash e : D$ *and* $e =^? q[f] \searrow \sigma$ *then* $C = D$ *and* $\Gamma \vdash \sigma : \Delta$

Proof. The proof is done by induction on the derivations of the pattern matching.

Base case: $\mathcal{D} :: \overline{f =^? f \searrow \cdot}$

$\mathcal{E} :: \cdot \mid \Sigma(f) \vdash [\cdot] \Rightarrow C$ by assumption.

$\mathcal{E}' :: \cdot \mid \Sigma(f) \vdash [\cdot] \Rightarrow \Sigma(f)$ by inversion on PC_{Head}.

$\mathcal{F} :: \Gamma \vdash f : D$ by assumption.

$\mathcal{F}' :: \Gamma \vdash f : \Sigma(f)$ by inversion on T_{Fun}.

Thus, $C = D = \Sigma(f)$ and $\Gamma \vdash \sigma : \Delta$, trivially.

Induction step

Case: $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e =^? q \searrow \sigma}}{e.d =^? q.d \searrow \sigma}$

$\mathcal{E} :: \Delta \mid \Sigma(f) \vdash q.d \Rightarrow C$ by assumption.
$\mathcal{F} :: \Gamma \vdash e.d : D$ by assumption.
$\mathcal{E}' :: \Delta \mid \Sigma(f) \vdash q \Rightarrow \nu X.R$ and $C = R_d[\nu X.R/X]$ by inversion on $\mathrm{PC}_{\mathrm{Dest}}$.
$\mathcal{F}' :: \Gamma \vdash e : \nu X.R'$ and $D = R'_d[\nu X.R'/X]$ by inversion on $\mathrm{T}_{\mathrm{Dest}}$.
$\Gamma \vdash \sigma : \Delta$ and $\nu X.R = \nu X.R'$ by induction hypothesis on $\mathcal{D}'$.
Thus, $R = R'$ and so $C = D$.

Case : $\mathcal{D} :: \dfrac{\overset{\mathcal{D}_1}{e =^? q \searrow \sigma} \quad \overset{\mathcal{D}_2}{e' =^? p \searrow \sigma'}}{e\ e' =^? q\ p \searrow \sigma, \sigma'}$

$\mathcal{E} :: \Delta \mid \Sigma(f) \vdash q\ p \Rightarrow C$ by assumption.
$\mathcal{E}_1 :: \Delta_1 \mid \Sigma(f) \vdash q \Rightarrow B \to C$ and $\mathcal{E}_2 :: \Delta_2 \vdash p \Leftarrow B$
for some type B, and where $\Delta = \Delta_1, \Delta_2$ by inversion on $\mathrm{PC}_{\mathrm{App}}$.
$\mathcal{F} :: \Gamma \vdash e\ e' : D$ by assumption.
$\mathcal{F}_1 :: \Gamma \vdash e : D' \to D$ and $\mathcal{F}_2 :: \Gamma \vdash e' : D'$
for some type D' by inversion on $\mathrm{T}_{\mathrm{App}}$.
$B \to C = D' \to D$ and $\Gamma \vdash \sigma : \Delta_1$ by induction hypothesis on $\mathcal{D}_1$, using $\mathcal{E}_1$ and $\mathcal{F}_1$.
Thus, $B = D'$ and $C = D$
$\Gamma \vdash \sigma' : \Delta_2$ by lemma 4 on $\mathcal{D}_2$, $\mathcal{E}_2$, $\mathcal{F}_2$.
$\Gamma \vdash \sigma, \sigma' : \Delta_1, \Delta_2$

□

Lemma 6 (Correctness of Contraction). *If* $\Gamma \vdash e : C$, $\vdash [f]\ q \to u$ *and* $e =^? q[f] \searrow \sigma$ *then* $\Gamma \vdash u[\sigma] : C$.

Proof. By assumption, we have $\mathcal{D} :: \dfrac{\overset{\mathcal{D}_1}{\Delta \mid \Sigma(f) \vdash q \Rightarrow D} \quad \overset{\mathcal{D}_2}{\Delta \vdash u \Leftarrow D}}{\vdash q[f] \to u}$ since it is the only rule that could have been used.

By lemma 5, using $\mathcal{D}_1$ and both assumptions we have that $C = D$ and $\Gamma \vdash \sigma : \Delta$. Then, by substitution lemma and $\mathcal{D}_2$, we conclude that $\Gamma \vdash u[\sigma] : C$.

□

Theorem 7 (Subject Reduction). *If* $\Gamma \vdash e : A$ *and* $e \mapsto e'$ *then* $\Gamma \vdash e' : A$

Proof. The proof is done by induction on the reduction rules.

Base Case : $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e =^? q \searrow \sigma} \quad q[f] \to u}{e \mapsto u[\sigma]}$.

By assumption, $\mathcal{D}'$ and wellformedness of $q[f] \to u$, we obtain from the correctness of contraction lemma that $\Gamma \vdash u[\sigma] : C$.

Induction Step

Case : $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e \mapsto e'}}{e \to e'}$

$\mathcal{E} :: \Gamma \vdash e : A$ by assumption.
$\mathcal{E}' :: \Gamma \vdash e' : A$ by induction hypothesis on $\mathcal{D}'$.

Case : $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e_1 \to e_1'}}{(e_1, e_2) \to (e_1', e_2)}$

$\mathcal{E} :: \Gamma \vdash (e_1, e_2) : A$ by assumption.
$\mathcal{E}_i :: \Gamma \vdash e_i : A_i$ where $i = 1, 2$ and $A = A_1 \times A_2$ by inversion on $\mathrm{T}_{\mathrm{Pair}}$.
$\mathcal{E}_1' :: \Gamma \vdash e_1' : A_1$ by induction hypothesis on $\mathcal{D}'$
$\Gamma \vdash (e_1', e_2) : A_1 \times A_2$ by $\mathrm{T}_{\mathrm{Pair}}$.

Case : $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e_2 \to e_2'}}{(e_1, e_2) \to (e_1, e_2')}$

$\mathcal{E} :: \Gamma \vdash (e_1, e_2) : A$ by assumption.
$\mathcal{E}_i :: \Gamma \vdash e_i : A_i$ where $i = 1, 2$ and $A = A_1 \times A_2$ by inversion on $\mathrm{T}_{\mathrm{Pair}}$.
$\mathcal{E}_2' :: \Gamma \vdash e_2' : A_2$ by induction hypothesis on $\mathcal{D}'$
$\Gamma \vdash (e_1, e_2') : A_1 \times A_2$ by $\mathrm{T}_{\mathrm{Pair}}$.

Case: $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e_1 \to e_1'}}{e_1\ e_2 \to e_1'\ e_2}$

$\mathcal{E} :: \Gamma \vdash e_1\ e_2 : A$ by assumption.
$\mathcal{E}_1 :: \Gamma \vdash e_1 : B \to A$, $\mathcal{E}_2 :: \Gamma \vdash e_2 : B$ for some B by inversion on $\mathrm{T}_{\mathrm{App}}$.
$\mathcal{F} :: \Gamma \vdash e_1' : B \to A$ by induction hypothesis on $\mathcal{D}'$.
$\Gamma \vdash e_1'\ e_2 : A$ by $\mathrm{T}_{\mathrm{App}}$.

Case: $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e_2 \to e_2'}}{e_1\ e_2 \to e_1\ e_2'}$

$\mathcal{E} :: \Gamma \vdash e_1\ e_2 : A$ by assumption.
$\mathcal{E}_1 :: \Gamma \vdash e_1 : B \to A$, $\mathcal{E}_2 :: \Gamma \vdash e_2 : B$ for some B by inversion on $\mathrm{T}_{\mathrm{App}}$.
$\mathcal{F} :: \Gamma \vdash e_2' : B$ by induction hypothesis on $\mathcal{D}'$.
$\Gamma \vdash e_1\ e_2' : A$ by $\mathrm{T}_{\mathrm{App}}$.

Case: $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e \to e'}}{c\ e \to c\ e'}$

$\mathcal{E} :: \Gamma \vdash c\ e : A$ by assumption.
$\mathcal{E}' :: \Gamma \vdash e : D_c[\mu X.D/X]$ and $A = \mu X.D$ by inversion on $\mathrm{T}_{\mathrm{Const}}$.
$\mathcal{F} :: \Gamma \vdash e' : D_c[\mu X.D/X]$ by induction hypothesis on $\mathcal{D}'$.
$\Gamma \vdash c\ e' : \mu X.D$ by $\mathrm{T}_{\mathrm{Const}}$.

Case: $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{e \to e'}}{e.d \to e'.d}$

$\mathcal{E} :: \Gamma \vdash e.d : A$ by assumption.
$\mathcal{E}' :: \Gamma \vdash e : \nu X.R$ and $A = R_d[\nu X.R/X]$ by inversion on $\mathrm{T}_{\mathrm{Dest}}$.
$\mathcal{F} :: \Gamma \vdash e' : \nu X.R$ by induction hypothesis on $\mathcal{D}'$.
$\Gamma \vdash e'.d : R_d[\nu X.R/X]$ by $\mathrm{T}_{\mathrm{Dest}}$.
□

6 Values

We now define values. We represent them with a new judgment $\Gamma \vdash_v e : A$. With this judgment, we will often denote e as v to obtain $\Gamma \vdash_v v : A$.

The rules are

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash_v x : A}\ \mathrm{V}_{\mathrm{Var}} \qquad \frac{\Gamma \vdash_v v : D_c[\mu X.D/X]}{\Gamma \vdash_v c\ v : \mu X.D}\ \mathrm{V}_{\mathrm{Const}} \qquad \frac{\Gamma \vdash e : \nu X.R}{\Gamma \vdash_v e : \nu X.R}\ \mathrm{V}_{\mathrm{Record}}$$

$$\frac{}{\Gamma \vdash_v () : 1}\ \mathrm{V}_{\mathrm{Unit}} \qquad \frac{\Gamma \vdash_v v_1 : A_1 \quad \Gamma \vdash_v v_2 : A_2}{\Gamma \vdash_v (v_1, v_2) : A_1 \times A_2}\ \mathrm{V}_{\mathrm{Pair}} \qquad \frac{\Gamma \vdash e : A \to B}{\Gamma \vdash_v e : A \to B}\ \mathrm{V}_{\mathrm{Arrow}}$$

We also have some inversion lemmas for values.

Lemma 8. *The following hold for $v \neq x$.*

1. *If $\Gamma \vdash_v v : A_1 \times A_2$ then $v = (v_1, v_2)$, $\Gamma \vdash_v v_1 : A_1$ and $\Gamma \vdash_v v_2 : A_2$;*
2. *If $\Gamma \vdash_v v : 1$ then $v = ()$;*
3. *If $\Gamma \vdash_v v : \mu X.D$ then $v = c\ v'$ and $\Gamma \vdash v' : D_c[\mu X.D/X]$.*

Proof. All the statements are proved by case analysis on the rules for values.

1. The only possible case is $\mathrm{V}_{\mathrm{Pair}}$. Thus, $v = (v_1, v_2)$ for some v_1, v_2 and $\Gamma \vdash_v v_1 : A_1$ and $\Gamma \vdash_v v_2 : A_2$.
2. The only possible case is $\mathrm{V}_{\mathrm{Unit}}$. Thus, $v = ()$.
3. The only possible case is $\mathrm{V}_{\mathrm{Const}}$. Thus, $v = c\ v'$ for some v' and $\Gamma \vdash_v v' : D_c[\mu X.D/X]$.

□

7 Coverage

We introduce a judgment to indicate that a series of patterns cover a given type. The goal is to prove that if a series of patterns cover a given type and that we have a term of that type, then this term will match against one of the patterns. The judgment is $A \lhd (\Delta_1 \vdash p_1) \ldots (\Delta_n \vdash p_n)$, or for convenience, $A \lhd \vec{\Delta} \vdash \vec{p}$ or $A \lhd \vec{P}$.

We introduce the following rules

$$\frac{}{A \triangleleft (x : A \vdash x)}\ \mathrm{C_{Var}} \qquad \frac{A \triangleleft \vec{P}\ (\Delta, x : \mu X.D \vdash p(x))}{A \triangleleft \vec{P}\ (\Delta, x : D_c[\mu X.D/X] \vdash p(c\ x))_{c \in D}}\ \mathrm{C_{Const}}$$

$$\frac{A \triangleleft \vec{P}\ (\Delta, x : 1 \vdash p(x))}{A \triangleleft \vec{P}\ (\Delta \vdash p())}\ \mathrm{C_{Unit}} \qquad \frac{A \triangleleft \vec{P}\ (\Delta, x : A_1 \times A_2 \vdash p(x))}{A \triangleleft \vec{P}\ (\Delta, x_1 : A_1, x_2 : A_2 \vdash p(x_1, x_2))}\ \mathrm{C_{Pair}}$$

We note that the rules are used non-deterministically since they imply choosing a pattern among the list we currently have.

We want to prove the following

Theorem 9. *If $\mathcal{D} :: A \triangleleft (\Delta_1 \vdash p_1) \dots (\Delta_n \vdash p_n)$ and $\mathcal{E} :: \vdash_v v : A$, then there is an i such that $v =^? p_i \searrow \sigma$.*

Before proving those, we will need a few lemmas.

Lemma 10. *If $\mathcal{D} :: \Delta, x : A_1 \times A_2 \vdash p(x) \Leftarrow C$, $\mathcal{E} :: \vdash_v v : C$ and $\mathcal{F} :: v =^? p(x) \searrow \sigma$ then $\Delta, x_1 : A_1, x_2 : A_2 \vdash p(x_1, x_2) \Leftarrow C$, $v =^? p(x_1, x_2) \searrow \sigma'$ and $\sigma = \sigma'[x \mapsto (\sigma'(x_1), \sigma'(x_2))]$*

Proof. The proof is done by induction on $\mathcal{F}$.

Base Case $\mathcal{F} :: \overline{v =^? x \searrow v/x}$

$\mathcal{D} :: \Delta, x : A_1 \times A_2 \vdash x \Leftarrow C$ — by assumption.
$C = A_1 \times A_2$ and $\Delta = \cdot$ — by inversion on $\mathrm{PC_{Var}}$.
$\mathcal{D}_i :: x_i : A_i \vdash x \Leftarrow A_i$ for $i = 1, 2$ — by $\mathrm{PC_{Var}}$
$\mathcal{D}' :: x_1 : A_1, x_2 : A_2 \vdash (x_1, x_2) \Leftarrow A_1 \times A_2$ — by $\mathrm{PC_{Pair}}$
$\mathcal{E} :: \vdash_v v : A_1 \times A_2$ — by assumption.
$\mathcal{E}_1 :: \vdash_v v_1 : A_1$, $\mathcal{E}_2 :: \vdash_v v_2 : A_2$, and $v = (v_1, v_2)$ for some v_1, v_2 — by lemma 8
$\mathcal{F}_i :: v_i =^? x_i \searrow v_i/x_i$ for $i = 1, 2$ — by $\mathrm{PM_{Var}}$.
$\mathcal{F}' :: (v_1, v_2) =^? (x_1, x_2) \searrow v_1/x_1, v_2/x_2$ — by $\mathrm{PM_{Pair}}$.
$\mathcal{F}' :: v =^? (x_1, x_2) \searrow \sigma'$ where
$\sigma(x) = v = (v_1, v_2) = (\sigma'(x_1), \sigma'(x_2))$ and $\sigma(y) = \sigma'(y)$ for $y \neq x$.

Induction step

$$\text{Case } \mathcal{F} :: \frac{\overset{\mathcal{F}_1}{v_1 =^? p_1(x) \searrow \sigma_1} \quad \overset{\mathcal{F}_2}{v_2 =^? p_2 \searrow \sigma_2}}{(v_1, v_2) =^? (p_1(x), p_2) \searrow \sigma_1, \sigma_2}$$

Without loss of generality, we chose x to be in p_1. The proof is the same for x in p_2 with 1 and 2 swapped.

$\mathcal{D} :: \Delta, x : A_1 \times A_2 \vdash (p_1(x), p_2) \Leftarrow C$ — by assumption.
$\mathcal{D}_1 :: \Delta_1, x : A_1 \times A_2 \vdash p_1(x) \Leftarrow C_1$, $\mathcal{D}_2 :: \Delta_2 \vdash p_2 \Leftarrow C_2$
where $\Delta = \Delta_1, \Delta_2$ and $C = C_1 \times C_2$ — by inversion on $\mathrm{PC_{Pair}}$.
$\mathcal{E} :: \vdash_v (v_1, v_2) : C_1 \times C_2$ — by assumption.
$\mathcal{E}_i :: \vdash_v v_i : C_i$ — by inversion on $\mathrm{T_{Pair}}$.
$\mathcal{D}_1' :: \Delta_1, x_1 : A_1, x_2 : A_2 \vdash p_1(x_1, x_2) \Leftarrow C_1$,
$\mathcal{F}_1' :: v_1 =^? p_1(x_1, x_2) \searrow \sigma_1'$, and
$\sigma_1 = \sigma_1'[x \mapsto (\sigma_1'(x_1), \sigma_1'(x_2))]$ — by induction hypothesis on $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{F}_1$.

$\mathcal{D}' :: \Delta_1, x_1 : A_1, x_2 : A_2, \Delta_2 \vdash (p_1(x_1, x_2), p_2) \Leftarrow C_1 \times C_2$ by $\mathrm{PC_{Pair}}$.
$\mathcal{F}' :: (v_1, v_2) =^? (p_1(x_1, x_2), p_2) \searrow \sigma_1', \sigma_2$ by $\mathrm{PM_{Pair}}$.

Case $\mathcal{F} :: \dfrac{\begin{matrix}\mathcal{F}' \\ v =^? p(x) \searrow \sigma\end{matrix}}{c\ v =^? c\ p(x) \searrow \sigma}$

$\mathcal{D} :: \Delta, x : A_1 \times A_2 \vdash c\ p(x) \Leftarrow C$ by assumption.
$\mathcal{D}' :: \Delta, x : A_1 \times A_2 \vdash p(x) \Leftarrow D_c[\mu X.D/X]$
and $C = \mu X.D$ for some D by inversion on $\mathrm{PC_{Const}}$.
$\mathcal{E} :: \vdash_v c\ v : \mu X.D$ by assumption.
$\mathcal{E}' :: \vdash_v v : D_c[\mu X.D/X]$ by inversion on $\mathrm{T_{Const}}$.
$\mathcal{D}'' :: \Delta, x_1 : A_1, x_2 : A_2 \vdash p(x_1, x_2) \Leftarrow D_c[\mu X.D/X]$,
$\mathcal{F}'' :: v =^? p(x_1, x_2) \searrow \sigma'$, and
$\sigma(x) = (\sigma'(x_1), \sigma'(x_2))$ and $\sigma(y) = \sigma'(y)$
for $y \neq x$ by induction hypothesis on $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{F}'$.
$\Delta, x_1 : A_1, x_2 : A_2 \vdash c\ p(x_1, x_2) \Leftarrow \mu X.D$ by $\mathrm{PC_{Const}}$.
$c\ v =^? c\ p(x_1, x_2) \searrow \sigma'$ by $\mathrm{PM_{Const}}$.
□

Lemma 11. *If $\mathcal{D} :: \Delta, x : \mu X.D \vdash p(x) \Leftarrow C$, $\mathcal{E} :: \vdash_v v : C$ and $\mathcal{F} :: v =^? p(x) \searrow \sigma$ then there is a $c \in D$ such that $\Delta, x' : D_c[\mu X.D/X] \vdash p(c\ x) \Leftarrow C$, $v =^? p(c\ x') \searrow \sigma'$ and $\sigma = \sigma'[x \mapsto c\ \sigma'(x')]$.*

Proof. The proof is done by induction on $\mathcal{F}$.

Base Case $\mathcal{F} :: \dfrac{}{v =^? x \searrow v/x}$

$\mathcal{D} :: \Delta, x : \mu X.D \vdash x \Leftarrow C$ by assumption.
$C = \mu X.D$ and $\Delta = \cdot$ by inversion on $\mathrm{PC_{Var}}$.
$\mathcal{D}' :: x' : D_c[\mu X.D/X] \vdash x' \Leftarrow D_c[\mu X.D/X]$ by $\mathrm{PC_{Var}}$
$\mathcal{D}'' :: x' : D_c[\mu X.D/X] \vdash c\ x' \Leftarrow \mu X.D$ by $\mathrm{PC_{Const}}$.
$\mathcal{E} :: \vdash_v v : \mu X.D$ by assumption.
$\mathcal{E}' :: \vdash_v v' : D_c[\mu X.D/X]$ and $v = c\ v'$
for some v' and some $c \in D$ by lemma 8.
$\mathcal{F}' :: v' =^? x' \searrow v'/x'$ by $\mathrm{PM_{Var}}$.
$\mathcal{F}'' :: c\ v' =^? c\ x' \searrow v'/x'$ by $\mathrm{PM_{Const}}$.
$\mathcal{F}'' :: v =^? c\ x' \searrow \sigma'$ where
$\sigma(x) = v = c\ v' = c\ \sigma'(x')$ and $\sigma(y) = \sigma'(y)$ for $y \neq x$.

Induction step

Case $\mathcal{F} :: \dfrac{\begin{matrix}\mathcal{F}_1 \\ v_1 =^? p_1(x) \searrow \sigma_1\end{matrix} \quad \begin{matrix}\mathcal{F}_2 \\ v_2 =^? p_2 \searrow \sigma_2\end{matrix}}{(v_1, v_2) =^? (p_1(x), p_2) \searrow \sigma_1, \sigma_2}$

Without loss of generality, we chose x to be in p_1. The proof is the same for x in p_2 with 1 and 2 swapped.
$\mathcal{D} :: \Delta, x : \mu X.D \vdash (p_1(x), p_2) \Leftarrow C$ by assumption.
$\mathcal{D}_1 :: \Delta_1, x : \mu X.D \vdash p_1(x) \Leftarrow C_1$, $\mathcal{D}_2 :: \Delta_2 \vdash p_2 \Leftarrow C_2$
where $\Delta = \Delta_1, \Delta_2$ and $C = C_1 \times C_2$ by inversion on $\mathrm{PC_{Pair}}$.

$\mathcal{E} :: \vdash_v (v_1, v_2) : C_1 \times C_2$ by assumption.
$\mathcal{E}_i :: \vdash_v v_i : C_i$ by inversion on $\mathrm{T}_{\mathrm{Pair}}$.
$\mathcal{D}'_1 :: \Delta_1, x' : D_c[\mu X.D] \vdash p_1(c\ x') \Leftarrow C_1$ for some $c \in D$,
$\mathcal{F}'_1 :: v_1 =^? p_1(c\ x') \searrow \sigma'_1$, and
$\sigma_1 = \sigma'_1[x \mapsto c\ \sigma'_1(x')$ by induction hypothesis on $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{F}_1$.
$\mathcal{D}' :: \Delta_1, x' : D_c[\mu X.D], \Delta_2 \vdash (p_1(c\ x'), p_2) \Leftarrow C_1 \times C_2$ by $\mathrm{PC}_{\mathrm{Pair}}$.
$\mathcal{F}' :: (v_1, v_2) =^? (p_1(c\ x'), p_2) \searrow \sigma'_1, \sigma_2$ by $\mathrm{PM}_{\mathrm{Pair}}$.

Case $\mathcal{F} :: \dfrac{\overset{\mathcal{F}'}{v =^? p(x) \searrow \sigma}}{c'\ v =^? c'\ p(x) \searrow \sigma}$
$\mathcal{D} :: \Delta, x : \mu X.D \vdash c'\ p(x) \Leftarrow C$ by assumption.
$\mathcal{D}' :: \Delta, x : \mu X.D \vdash p(x) \Leftarrow S_{c'}[\mu X.S/X]$
and $C = \mu X.S$ for some S by inversion on $\mathrm{PC}_{\mathrm{Const}}$.
$\mathcal{E} :: \vdash_v c'\ v : \mu X.S$ by assumption.
$\mathcal{E}' :: \vdash_v v : S_{c'}[\mu X.S/X]$ by inversion on $\mathrm{T}_{\mathrm{Const}}$.
$\mathcal{D}'' :: \Delta, x' : D_c[\mu X.D/X] \vdash p(c\ x') \Leftarrow S_{c'}[\mu X.S/X]$,
$\mathcal{F}'' :: v =^? p(c\ x') \searrow \sigma'$, and
$\sigma(x) = \sigma'[x \mapsto c\ \sigma'(x')$ by induction hypothesis on $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{F}'$.
$\Delta, x' : D_c[\mu X.D/X] \vdash c'\ p(c\ x') \Leftarrow \mu X.S$ by $\mathrm{PC}_{\mathrm{Const}}$.
$c'\ v =^? c'\ p(c\ x') \searrow \sigma'$ by $\mathrm{PM}_{\mathrm{Const}}$.
□

Lemma 12. *If* $\mathcal{D} :: \Delta, x : 1 \vdash p(x) \Leftarrow C$, $\mathcal{E} :: \vdash_v v : C$ *and* $\mathcal{F} :: v =^? p(x) \searrow \sigma$ *then* $\Delta \vdash p() \Leftarrow C$, $v =^? p() \searrow \sigma'$ *and* $\sigma = \sigma'[x \mapsto ()]$

Proof. The proof is done by induction on $\mathcal{F}$.

Base Case $\mathcal{F} :: \overline{v =^? x \searrow v/x}$
$\mathcal{D} :: \Delta, x : 1 \vdash x \Leftarrow C$ by assumption.
$C = 1$ and $\Delta = \cdot$ by inversion on $\mathrm{PC}_{\mathrm{Var}}$.
$\mathcal{D} :: \vdash () \Leftarrow 1$ by $\mathrm{PC}_{\mathrm{Unit}}$.
$\mathcal{E} :: \vdash_v v : 1$ by assumption.
$v = ()$ by lemma 8.
$\mathcal{F}' :: () =^? () \searrow \cdot$ by $\mathrm{PM}_{\mathrm{Unit}}$.
$() =^? () \searrow \sigma'$ where
$\sigma(x) = v = ()$ and $\sigma(y) = \sigma'(y)$ for $y \neq x$.

Induction step
Case $\mathcal{F} :: \dfrac{\overset{\mathcal{F}_1}{v_1 =^? p_1(x) \searrow \sigma_1} \quad \overset{\mathcal{F}_2}{v_2 =^? p_2 \searrow \sigma_2}}{(v_1, v_2) =^? (p_1(x), p_2) \searrow \sigma_1, \sigma_2}$
Without loss of generality, we chose x to be in p_1. The proof is the same for x in p_2 with 1 and 2 swapped.
$\mathcal{D} :: \Delta, x : 1 \vdash (p_1(x), p_2) \Leftarrow C$ by assumption.
$\mathcal{D}_1 :: \Delta_1, x : 1 \vdash p_1(x) \Leftarrow C_1$, $\mathcal{D}_2 :: \Delta_2 \vdash p_2 \Leftarrow C_2$
where $\Delta = \Delta_1, \Delta_2$ and $C = C_1 \times C_2$ by inversion on $\mathrm{PC}_{\mathrm{Pair}}$.

$\mathcal{E} :: \vdash_v (v_1, v_2) : C_1 \times C_2$ by assumption.
$\mathcal{E}_i :: \vdash_v v_i : C_i$ by inversion on $\mathrm{T}_{\mathrm{Pair}}$.
$\mathcal{D}'_1 :: \Delta_1 \vdash p_1() \Leftarrow C_1$,
$\mathcal{F}'_1 :: v_1 =^? p_1() \searrow \sigma'_1$, and
$\sigma_1(x) = ()$ and $\sigma_1(y) = \sigma'_1(y)$
for all $y \neq x$ by induction hypothesis on $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{F}_1$.
$\mathcal{D}' :: \Delta_1, \Delta_2 \vdash (p_1(), p_2) \Leftarrow C_1 \times C_2$ by $\mathrm{PC}_{\mathrm{Pair}}$.
$\mathcal{F}' :: (v_1, v_2) =^? (p_1(), p_2) \searrow \sigma'_1, \sigma_2$ by $\mathrm{PM}_{\mathrm{Pair}}$.

Case $\mathcal{F} :: \dfrac{\overset{\mathcal{F}'}{v =^? p(x) \searrow \sigma}}{c\ v =^? c\ p(x) \searrow \sigma}$

$\mathcal{D} :: \Delta, x : 1 \vdash c\ p(x) \Leftarrow C$ by assumption.
$\mathcal{D}' :: \Delta, x : 1 \vdash p(x) \Leftarrow D_c[\mu X.D/X]$
and $C = \mu X.D$ for some D by inversion on $\mathrm{PC}_{\mathrm{Const}}$.
$\mathcal{E} :: \vdash_v c\ v : \mu X.D$ by assumption.
$\mathcal{E}' :: \vdash_v v : D_c[\mu X.D/X]$ by inversion on $\mathrm{T}_{\mathrm{Const}}$.
$\mathcal{D}'' :: \Delta \vdash p() \Leftarrow D_c[\mu X.D/X]$,
$\mathcal{F}'' :: v =^? p() \searrow \sigma'$, and
$\sigma(x) = ()$ and $\sigma(y) = \sigma'(y)$
for $y \neq x$ by induction hypothesis on $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{F}'$.
$\Delta \vdash c\ p(2) \Leftarrow \mu X.D$ by $\mathrm{PC}_{\mathrm{Const}}$.
$c\ v =^? c\ p() \searrow \sigma'$ by $\mathrm{PM}_{\mathrm{Const}}$.
□

Proof (Theorem 9). The proof is done by induction of the derivation of $\mathcal{D}$.

Base case $\mathcal{D} :: \overline{A \triangleleft (x : A \vdash x)}$
$\mathcal{E} :: \vdash_v v : A$ by assumption.
$v =^? x \searrow v/x$ by $\mathrm{PM}_{\mathrm{Var}}$.

Induction step.

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{A \triangleleft \vec{P}\ (\Delta, x : A_1 \times A_2 \vdash p(x))}}{A \triangleleft \vec{P}\ (\Delta, x_1 : A_1, x_2 : A_2 \vdash p(x_1, x_2))}$

$\mathcal{E} :: \vdash_v v : A$ by assumption.
By induction hypothesis, v matches a pattern out of $\vec{P}\ (\Delta, x : A_1 \times A_2 \vdash p(x))$.
If it matches a pattern in $\vec{P}$, we are done. Thus,
$\mathcal{F} :: v =^? p(x) \searrow \sigma$ without loss of generality.
$v =^? p(x_1, x_2) \searrow \sigma'$ where
$\sigma = \sigma'[x \mapsto (\sigma'(x_1)\sigma'(x_2))]$
and $\Delta, x_1 : A_1, x_2 : A_2 \vdash p(x_1, x_2) \Leftarrow A$ by lemma 10.

Case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}'}{A \triangleleft \vec{P}\ (\Delta, x : \mu X.D \vdash p(x))}}{A \triangleleft \vec{P}\ (\Delta, x : D_c[\mu X.D/X] \vdash p(c\ x) \mid c \in D)}$

$\mathcal{E} :: \vdash_v v : A$ by assumption.
By induction hypothesis, v matches a pattern out of $\vec{P}$ $(\Delta, x : \mu X.D \vdash p(x))$. If it matches a pattern in $\vec{P}$, we are done. Thus,
$\mathcal{F} :: v =^? p(x) \searrow \sigma$ without loss of generality.
$v =^? p(c\ x) \searrow \sigma'$ for some $c \in D$ where
$\sigma = \sigma'[x \mapsto c\ \sigma'(x)]$,
and $\Delta, x : D_c[\mu X.D/X] \vdash p(c\ x) \Leftarrow A$ by lemma 11.

$$\text{Case } \mathcal{D} :: \frac{\overset{\mathcal{D}'}{A \triangleleft \vec{P}\ (\Delta, x : 1 \vdash p(x))}}{A \triangleleft \vec{P}\ (\Delta \vdash p())}$$

$\mathcal{E} :: \vdash_v v : A$ by assumption.
By induction hypothesis, v matches a pattern out of $\vec{P}$ $(\Delta, x : 1 \vdash p(x))$. If it matches a pattern in $\vec{P}$, we are done. Thus,
$\mathcal{F} :: v =^? p(x) \searrow \sigma$ without loss of generality.
$v =^? p() \searrow \sigma'$ where
$\sigma[x \mapsto ()]$,
and $\Delta \vdash p() \Leftarrow A$ by lemma 12.
□

8 Evaluation Context

Before we dive in the definition of an evaluation context, we need to have a closer look to the semantics of functions symbols. We have a judgment Rules$(f) = \{(q_i, e_i)_{i=1,\dots,n}\}$ where $\vdash q_i[f] \to e_i$ for all $i = 1, \dots, n$ and such that if $q \neq q_i$ for all i then $\not\vdash q[f] \to e$ for any e. Rules(f) thus defines all possible patterns for f.

An evaluation context is an expression of the following form.

Evaluation Context $E ::= \cdot \mid E\ e \mid E.d$

We say that $E =^? q \searrow \sigma$ if $E[f] =^? q[f] \searrow \sigma$. Then, if $\mathit{Rules}(f)(q) = e$, $E[f] \mapsto e[\sigma]$. We can also compose evaluation contexts such as $E_1 = E_2[E[]]$ Then $E_1[f] \to E_2[e[\sigma]]$. We have a judgment for typing evaluation context. $\Gamma \mid A \vdash E : C$ where A represents the type of f in $E[f]$. The rules are the following.

$$\frac{}{\Gamma \mid A \vdash \cdot : A}\ \mathrm{ET_{Head}} \qquad \frac{\Gamma \mid A \vdash E : \nu X.R}{\Gamma \mid A \vdash E.d : R_d[\nu X.R/X]}\ \mathrm{ET_{Dest}}$$

$$\frac{\Gamma \mid A \vdash E : B \to C \quad \Gamma \vdash e : B}{\Gamma \mid A \vdash E\ e : C}\ \mathrm{ET_{App}}$$

We also have another judgment $\Gamma \mid A \vdash_v E : C$ to denote evaluations contexts applied to values. The rules are very much the same.

$$\frac{}{\Gamma \mid A \vdash_v \cdot : A}\ \mathrm{EV_{Head}} \qquad \frac{\Gamma \mid A \vdash_v E : \nu X.R}{\Gamma \mid A \vdash_v E.d : R_d[\nu X.R/X]}\ \mathrm{EV_{Dest}}$$

$$\frac{\Gamma \mid A \vdash_v E : B \to C \quad \Gamma \vdash_v v : B}{\Gamma \mid A \vdash_v E\ v : C}\ \mathrm{EV}_{\mathrm{App}}$$

Lemma 13 (Composition of contexts)**.** *If $\mathcal{D} :: \Gamma \mid A \vdash E_1 : B$ and $\mathcal{E} :: \Gamma \mid B \vdash E_2 : C$, then $\Gamma \mid A \vdash E_2[E_1[\cdot]] : C$*

Proof. The proof is done by induction on $\mathcal{E}$.

Base case $\mathcal{E} :: \overline{\Gamma \mid B \vdash \cdot : B}$ Then $C = B$ and $\Gamma \mid A \vdash \cdot\ [E_1[\cdot]] : C$.

Induction step

Case $\mathcal{E} :: \dfrac{\overset{\mathcal{E}_1}{\Gamma \mid B \vdash E_2 : D \to C} \quad \overset{\mathcal{E}_2}{\Gamma \vdash e : D}}{\Gamma \mid B \vdash E_2\ e : C}$

By induction hypothesis on $\mathcal{E}_1$, we have $\Gamma \mid A \vdash E_2[E_1[\cdot]] : D \to C$. Thus, $\Gamma \mid A \vdash E_2[E_1[\cdot]]\ e : C$ by $\mathrm{ET}_{\mathrm{App}}$.

Case $\mathcal{E} :: \dfrac{\overset{\mathcal{E}'}{\Gamma \mid B \vdash E_2 : \nu X.R}}{\Gamma \mid B \vdash E_2.d : R_d[\nu X.R/X]}$

By induction hypothesis on $\mathcal{E}'$, we have $\Gamma \mid A \vdash E_2[E_1[\cdot]] : \nu X.R$. Thus $\Gamma \mid A \vdash E_2[E_1[\cdot]].d : R_c[\nu X.R/X]$ by $\mathrm{ET}_{\mathrm{Dest}}$.

□

There is a similar version for values. The proof being the very same, we will not do it.

Lemma 14 (Composition of contexts (values))**.** *If $\Gamma \mid A \vdash_v E_1 : B$ and $\Gamma \mid B \vdash_v E_2 : C$, then $\Gamma_1 \mid A \vdash_v E_2[E_1[\cdot]] : C$*

We now prove the following that will be needed later.

Lemma 15. *If $\mathcal{D} :: \Gamma \mid B \to C \vdash_v E : D$ and $E \neq \cdot$ then $E = E'[\cdot\ v]$ with $\Gamma \vdash_v v : B$ and $\Gamma \mid C \vdash_v E' : D$.*

Proof. The proof is done by induction on the derivation $\mathcal{D}$.

Base case $\mathcal{D} :: \dfrac{\overset{\mathcal{D}_1}{\Gamma \mid B \to C \vdash_v \cdot : B \to C} \quad \overset{\mathcal{D}_2}{\Gamma \vdash_v v : B}}{\Gamma \mid B \to C \vdash_v \cdot\ v : C}$

The statement holds by letting $E' = \cdot$.

Base case $\mathcal{D} :: \Gamma \mid B \to C \vdash_v \cdot.d : D$

The only derivation that allows to obtain $\mathcal{D}$ is $\mathrm{EV}_{\mathrm{Dest}}$ and this would imply that $\nu X.R = B \to C$ for some R which is impossible.

Induction step.

For both cases, we assume that $E \neq \cdot$. Otherwise, we get back to the two base cases.

$$\text{Case } \mathcal{D} :: \frac{\overset{\mathcal{D}_1}{\Gamma \mid B \to C \vdash_v E : D' \to D} \quad \overset{\mathcal{D}_2}{\Gamma \vdash_v v : D'}}{\Gamma \mid B \to C \vdash_v E\ v : D}$$

$E = E'[\cdot\ v']$ with
$\mathcal{E}_1 :: \Gamma \mid C \vdash_v E' : D' \to D$ and
$\mathcal{E}_2 :: \Gamma \vdash_v v' : B$ by induction hypothesis on $\mathcal{D}_1$.
$\mathcal{E} :: \Gamma \mid C \vdash_v E'\ v : D$ by $\mathrm{EV}_{\mathrm{App}}$.

$$\text{Case } \mathcal{D} :: \frac{\overset{\mathcal{D}'}{\Gamma \mid B \to C \vdash_v E : \nu X.R}}{\Gamma \mid B \to C \vdash_v E.d : R_d[\nu X.R/X]}$$

$E = E'[\cdot\ v]$ with
$\mathcal{E}_1 :: \Gamma \mid C \vdash_v E' : \nu X.R$ and
$\mathcal{E}_2 :: \Gamma \vdash_v v : B$ by induction hypothesis on $\mathcal{D}'$.
$\mathcal{E} :: \Gamma \mid C \vdash_v E'.d : R_d[\nu X.R/X]$ by $\mathrm{EV}_{\mathrm{Dest}}$.

□

Lemma 16. *If $\mathcal{D} :: \Gamma \mid \nu X.R \vdash_v E : D$ and $E \neq \cdot$ then $E = E'[\cdot.d]$ with $\Gamma \mid R_d[\nu X.R/X] \vdash_v E' : D$.*

Proof. The proof is done by induction on the derivation $\mathcal{D}$.

Base case $\mathcal{D} :: \Gamma \mid \nu X.R \vdash_v \cdot\ v : D$
This case is also impossible as the only rule that can be applied to obtain $\mathcal{D}$ is $\mathrm{EV}_{\mathrm{App}}$ and this would require us to have $D' \to D = \nu X.R$ for some D' which is impossible.

$$\text{Base case } \mathcal{D} :: \frac{\Gamma \mid \nu X.R \vdash_v \cdot \nu X.R}{\Gamma \mid \nu X.R \vdash_v \cdot.d : R_d[\nu X.R/X]}$$

The statement holds by setting $E' = \cdot$.

Induction step.

For both cases, we assume that $E \neq \cdot$. Otherwise, we get back to the two base cases.

$$\text{Case } \mathcal{D} :: \frac{\overset{\mathcal{D}_1}{\Gamma \mid \nu X.R \vdash_v E : D' \to D} \quad \overset{\mathcal{D}_2}{\Gamma \vdash_v v : D'}}{\Gamma \mid \nu X.R \vdash_v E\ v : D}$$

$E = E'[\cdot.d]$ with
$\mathcal{E}_1 :: \Gamma \mid R_d[\nu X.R/X] \vdash_v E' : D' \to D$ by induction hypothesis on $\mathcal{D}_1$.
$\mathcal{E} :: \Gamma \mid R_d[\nu X.R/X] \vdash_v E'\ v : D$ by $\mathrm{EV}_{\mathrm{App}}$.

$$\text{Case } \mathcal{D} :: \frac{\overset{\mathcal{D}'}{\Gamma \mid \nu X.R \vdash_v E : \nu X.R'}}{\Gamma \mid \nu X.R \vdash_v E.d' : R'_{d'}[\nu X.R'/X]}$$

$E = E'[\cdot.d]$ with
$\mathcal{E}_1 :: \Gamma \mid R_d[\nu X.R/X] \vdash_v E' : \nu X.R'$ by induction hypothesis on $\mathcal{D}'$.
$\mathcal{E} :: \Gamma \mid R_d[\nu X.R/X] \vdash_v E'.d' : R'_{d'}[\nu X.R'/X]$ by $\mathrm{EV}_{\mathrm{Dest}}$.

□

9 Copattern Coverage

We have a different judgment than the one for coverage in the case of copattern. It is the following $A \;\triangleleft|\; (\Delta \vdash q \Rightarrow C)$ or, more generally, $A \;\triangleleft|\; \vec{Q}$ where $\vec{Q} = (\Delta_i \vdash q_i \Leftarrow C_i)_{i=1,\ldots,n}$. The meaning behind this judgment is that C is covered by a list of patterns satisfying the judgments $\Delta_i \mid A \vdash q_i \Rightarrow C_i$. The rules are the following.

$$\frac{}{A \;\triangleleft|\; (\cdot \vdash \cdot \Rightarrow A)}\text{CC}_{\text{Head}} \qquad \frac{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash Q \Rightarrow \nu x.R)}{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash q.d \Rightarrow R_d[\nu X.R/X])_{d \in R}}\text{CC}_{\text{Dest}}$$

$$\frac{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash q \Rightarrow B \to C) \quad B \triangleleft (\Delta_i \vdash p_i)_{i=1,\ldots,n}}{A \;\triangleleft|\; \vec{Q}\; (\Delta, \Delta_i \vdash q\; p_i \Rightarrow C)}\text{CC}_{\text{App}}$$

Theorem 17. *If $\mathcal{D} :: \cdot \mid A \vdash_v E : D$ and $\mathcal{E} :: A \;\triangleleft|\; (\Delta_i \vdash q_i \Rightarrow C_i)_{i=1,\ldots,n}$ but not $\vdash_v E[f] : D$ then there are E_1, E_2 such that $E = E_1[E_2[\cdot]]$, $E_2 =^? q_i \searrow \sigma$ for some i, $\cdot \mid A \vdash_v E_2 : C_i$ and $\cdot \mid C_i \vdash_v E_1 : D$.*

Proof. This is proved by induction on $\mathcal{E}$.

Base case $\mathcal{E} :: \overline{A \;\triangleleft|\; (\cdot \vdash \cdot \Rightarrow A)}$
Choose $E_1 = E$, $E_2 = \cdot$. Then, $E_2 =^? \cdot \searrow \cdot$.

Induction Step.

Case $\mathcal{E} :: \dfrac{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash q \Rightarrow B \to C) \quad B \triangleleft (\Delta_i \vdash p_i)_{i=1,\ldots,n}}{A \;\triangleleft|\; \vec{Q}\; (\Delta, \Delta_i \vdash q\; p_i \Rightarrow C)_{i=1,\ldots,n}}$

By induction hypothesis, the statement holds for one of the patterns in $\vec{Q}\; (\Delta \vdash q \Rightarrow B \to C)$. If the pattern has been chosen in $\vec{Q}$ we are done. Thus, without loss of generality, $E = E_1[E_2[\cdot]]$, $\cdot \mid A \vdash_v E_2 : B \to C$, $\cdot \mid B \to C \vdash_v E_1 : D$, and $E_2 =^? q \searrow \sigma$.

If $E_1 = \cdot$ then $D = B \to C$ and $\vdash_v E[f] : D$ holds, which is a contradiction to our assumptions. If $E_1 \neq \cdot$, then $E_1 = E_1[\cdot\; v]$ with $\cdot \vdash_v v : B$ and $\cdot \mid C \vdash_v E_1' : D$ by lemma 15.

Since $B \triangleleft (\Delta_i \vdash p_i)$, there is a p_i with $v =^? p_i \searrow \rho$ by theorem 9. Thus, $E_2' = E_2\; v$, $\cdot A \vdash_v E_2' : C$ by EV_{App}, and $E_2' =^? q\; p_i \searrow \sigma, \rho$ by PM_{App}.

Case $\mathcal{E} :: \dfrac{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash q \Rightarrow \nu X.R)}{A \;\triangleleft|\; \vec{Q}\; (\Delta \vdash q.d \Rightarrow R_d[\nu X.R/X])_{d \in R}}$

By induction hypothesis, the statement holds for one of the patterns in $\vec{Q}\; (\Delta \vdash q \Rightarrow \nu X.R)$. If the pattern has been chosen in $\vec{Q}$ we are done. Thus, without loss of generality, $E = E_1[E_2[\cdot]]$, $\cdot \mid A \vdash_v E_2 : \nu X.R$, $\cdot \mid \nu X.R \vdash_v E_1 : D$, and $E_2 =^? q \searrow \sigma$.

If $E_1 = \cdot$ then $D = \nu X.R$ and $\vdash_v E[f] : D$ holds, contradicting our assumption. Thus $E_1 \neq \cdot$ and, by lemma 16, $E_1 = E_1'[\cdot.d]$ with $\cdot \mid R_d[\nu X.R/X] \vdash_v E_1' : D$.

□

10 Progress

Before stating and proving the progress theorem, we need to prove the decomposition theorem.

Lemma 18 (Decomposition Theorem)**.** *If $\cdot \vdash e : A$ then either*

1. $e = (), A = 1$,
2. $e = (e_1, e_2), A = A_1 \times A_2$,
3. $e = c\ e', A = \mu X.D$,
4. $e = E_2[E_1[f]\ e']$ *where* $\cdot \mid\vdash_v E_1 : B \to C$, $\cdot \mid C \vdash E_2 : A$ *and* $\not\vdash_v e' : B$ *for some evaluation contexts* E_1, E_2*, some term* e' *and some types* B, C.
5. $e = E[f]$ *and* $\cdot \mid \Sigma(f) \vdash_v E : A$

Proof. The proof is done by induction on e.

Case $\vdash x : A$ is impossible as the term is closed.

Case $\vdash f : A$ matches with case 5 with $E = \cdot$ since we trivially have $\cdot \mid A \vdash_v E : A$.

Case $\vdash () : A$. Then $A = 1$ by inversion.

Case $\vdash (e_1, e_2) : A$. Then, $A = A_1 \times A_2$ by inversion.

Case $\vdash c\ e : A$. Then, $A = \mu X.D$ by inversion.

Case $\vdash e_1\ e_2 : A$. Then by inversion $\vdash e_1 : B \to A$ and $\vdash e_2 : B$. By induction hypothesis $e_1 = E[f]$ with $\cdot \mid \Sigma(f) \vdash_v E : A$ for some E or $e_1 = E_2[E_1[f]\ e']$ for some E_1, E_2, and e' where $\cdot \mid\vdash_v E_1 : B \to C$, $\cdot \mid C \vdash E_2 : A$ and $\not\vdash_v e' : B$, as the 3 other cases are impossible. In the former case, if $\not\vdash_v e_2 : B$, we can obtain case 4 by letting $E_2 = \cdot$ and $E = E_1$. This gives us $e_1\ e_2 = \cdot[E[f]\ e_2]$. If $\vdash_v e_2 : B$, then, by EV_{App}, $\cdot \mid \Sigma(f) \vdash_v E[f]\ e_2 : A$ and $E' = E\ e_2$. In the latter case, we have $E_2[E_1[f]\ e']\ e_2 = E_2'[E_1[f]\ e']$ by setting $E_2[\cdot]\ e_2 = E_2'[\cdot]$.

Case $\vdash e.d : A$. Then by inversion, $\vdash e : \nu X.R$ for some R. By induction hypothesis, $e = E[f]$ and $\cdot \mid \Sigma(f) \vdash_v E : \nu X.R$, or $e = E_2[E_1[f]\ e']$ where e' is not a value. In the former care, $e.d = E[f].d = E'[f]$ and $\cdot \mid \Sigma(f) \vdash_v E.d : R_d[\nu X.R/X]$ by EV_{Dest}. In the latter case, $e.d = E_2[E_1[f]\ e'].d = E_2'[E_1[f]\ e']$.

□

From now on, we assume that the rules of every function f we use cover $\Sigma(f)$, more specifically, $\Sigma(f) \triangleleft| (\Delta_i \vdash q_i \Rightarrow C_i)_{i=1,\dots,n}$ where $q_i \in \text{Rules}(f)$ and for all $q \neq q_i$ for all i $q \notin \text{Rules}(f)$. We will denote this $\Sigma(f) \triangleleft| \text{Rules}(f)$.

Theorem 19. *If $\mathcal{D} :: \vdash e : A$ then either $\vdash_v e : A$ or $e \to e'$ for some e'*

Proof. The proof is done by induction on e. By the decomposition theorem, we have four possible cases.

Base case $e = (), A = 1$. Then $\vdash_v () : 1$ by V_{Var}.

Induction step

Case $e = (e_1, e_2)$, $A = A_1 \times A_2$.
By inversion on T_{Pair}, we have $\vdash e_1 : A_1$ and $\vdash e_2 : A_2$. By induction hypothesis, either $\vdash_v e_1 : A_1$ or $e_1 \to e_1'$. In the latter case, we obtain $(e_1, e_2) \to (e_1', e_2)$ by R_{Pair}.

In the former case, we apply induction hypothesis on e_2 to obtain either $\vdash_v e_2 : A_2$ or $e_2 \to e_2'$. In the former case, we obtain $\vdash_v (e_1, e_2) : A_1 \times A_2$ by V_{Pair}. In the latter case, we have $(e_1, e_2) \to (e_1, e_2')$ by R_{Pair}.

Case $e = c\ e'$, $A = \mu X.D$.
By inversion on T_{Const}, we have $\vdash e' : D_c[\mu X.D/X]$. By induction hypothesis, either $\vdash_v e' : D_c[\mu X.D/X]$ or $e' \to e''$. In the former case, $\vdash_v c\ e' : \mu X.D$ by V_{Const}. In the latter case, $c\ e' \to c\ e''$ by R_{Const}.

Case $e = E_2[E_1[f]\ e']$ where e' is not a value.
Then, by induction hypothesis $e' \to e''$ for some e'' and so $E_2[E_1[f]\ e'] \to E_2[E_1[f]\ e'']$.

Case $e = E[f]$ and $\cdot \mid \Sigma(f) \vdash_v E : A$.
If $\vdash_v E[f] : A$ then we are done. Without loss of generality, we assume it is not the case. By our assumption on f, $\Sigma(f) \triangleleft| \text{Rules}(f)$. Thus we can apply theorem 17 and obtain E_1, E_2 such that $E = E_1[E_2[\cdot]]$, $E_2 =^? q_i \searrow \sigma$ for some $q_i \in \text{Rules}(f)$, $\cdot \mid \Sigma(f) \vdash E_2 : C_i$ and $\cdot \mid C_i \vdash_v E_1 : A$. Thus, by our reduction rules $E_2[f] \mapsto u_i[\sigma]$ where $(q_i, u_i) \in \text{Rules}(f)$ and so $E_2[f] \to u_i[\sigma]$. We conclude that $E_1[E_2[f]] \to E_1[u_i[\sigma]]$.

□