Exercises in Agda

Catarina Coquand

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In the library `/usr/local/share/summerschool/agda/SummerSchool05` (on the Summer School's computers) you will find a small library with some standard files. It is good to look around in the different files.

1. In the file with `Exercises/Logic.agda`, there are many basic exercises on the Curry-Howard correspondence between propositions and sets and in the file `Exercises/ListProps.agda` there is some exercises on lists

2. Formulate and prove in type theory

$$((\forall x : A)(\exists y : B)R\ x\ y) \to (\exists f : A \to B)(\forall x : A)R\ x\ (f\ x)$$

   This is a possible formulation of the axiom of choice. It was stressed by Bishop that this form of axiom of choice is constructively valid.

3. In the file `Nat.agda` you can find the definition of natural numbers, Nat. Define the equality, $eqNat$, on Nat by double recursion and show that this equality is substitutive (if $eqNat\ x\ y$ and $P\ x$., then $P\ y$

4. Define the set of well-founded trees (well-orderings), W: given a family of sets $B(x)$ over a set A, the set $W\ A\ B$ has one constructor sup, where $sup\ x\ f : W\ A\ B$ if $x : A$ and $f : B(x) \to W\ A\ B$. Write the corresponding elimination rule in Agda. Prove that

$$W\ A\ B \to \neg(\forall x : A.B\ x)$$

   that is, the two propositions $W\ A\ B$ and $\Pi\ A\ B$ are incompatible. Explain intuitively why

$$W\ A\ B \to \exists x : A.\neg(B\ x)$$

   should not be provable in type theory.

5. Define the type

$$F\ A\ n\ =\ \underbrace{A \to \ldots \to A}_{n} \to A$$

   i.e. a function F that takes a set A, a natural number, n, and returns a set.

Use this type to define a tautology function, i.e. a function that takes as arguments a number n, a boolean function with n arguments, and returns $True$ if and only if this boolean function is a tautology.

6. Let A be a set with a well-founded relation, $<$, on it. Show that if $f : Nat \rightarrow A$ then $\neg((\forall n : Nat)\ f\ (n+1) < f\ n)$, i.e. there is no infinite decreasing sequence. Show that if $<$ is decidable then we have $(\exists n : Nat)\neg(f\ (n+1) < f\ n)$

7. Let A be a set with a well-founded relation, $<$, on it (see `WellOrder.agda`). Prove that if A is inhabited we have

$$((\forall x : A)(P\ x \vee (f\ x < x))) \rightarrow (\exists x : A) P\ x$$

Explain how this proof corresponds to a "for-loop" program.