Formal Methods for Software Development

Reasoning about Programs with Loops and Method Calls

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Program Logic Calculus – Repetition

Calculus realises symbolic interpreter:

$$\Gamma \Longrightarrow \langle \texttt{i=j++;}\mathtt{\textbf{if}}\texttt{(j>10)\{ok=}\mathtt{\textbf{true}}\texttt{;\}...}\rangle \phi$$

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$$\overline{\qquad\ldots\qquad}$$

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Method Call: Example

```
\javaSource "src/";

\programVariables{
 Person p;
 int j;
}

\problem {
  (\forall int i;
    (!p=null ->
      ({j := i}\<{p.setAge(j);}\>(p.age = i))))
}
```

Method Calls

Method Call with actual parameters $arg_0, \ldots, arg_n$

$$\langle \mathtt{o.m}(arg_0, \ldots, arg_n)\mathtt{;}\ \ \omega\rangle\phi$$

assume `m` declared as $\mathbf{void}\ \mathtt{m}(\tau_0\ \mathrm{p}_0, \ldots, \tau_\mathrm{n}\ \mathrm{p}_\mathrm{n})$

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Actions of rule methodCall

1. Declare new local variables $\texttt{p\#i}$, initialize them with actual parameter: $\tau_i\ \texttt{p\#i} = arg_i;$

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   split proof if implementation cannot be uniquely determined.

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Actions of rule methodCall

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2. Look-up implementing class C of m;
   split proof if implementation cannot be uniquely determined.
3. Replace method call with implementation invocation
   o.m(p#0, ..., p#n)@C

Method Calls Cont'd

After executing the initialisers: $\tau_{\mathtt{i}}$ `p#i` $= arg_i;$ apply:

Method Body Expand

Rule methodBodyExpand (simplified)

$$\frac{\Gamma \Longrightarrow \langle \mathtt{method\text{-}frame(source=m(}\tau_0,\ldots,\tau_n\mathtt{)@C,\ this=o):\{}\mathit{body}\mathtt{\}}\ \omega\rangle\phi, \Delta}{\Gamma \Longrightarrow \langle \mathtt{o.m(p\#0,\ldots,p\#n)@C;}\ \omega\rangle\phi, \Delta}$$

1. Replaces method invocation by method frame with method body
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Required in proof to represent call stack

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Demo

```
methods/instanceMethodInlineSimple.key
methods/inlineDynamicDispatch.key
```

Localisation of Fields and Method Implementations

Java has complex rules for localisation of fields and method implementations

- Overloading
- Late binding (dynamic dispatch)
- Scoping (class vs. instance)
- Visibility (private, protected, public)

Proof split into cases if implementation not statically determined

Object initialization

JAVA has complex rules for object initialization

- Chain of constructor calls until Object
- Implicit calls to super()
- Visibility issues
- Initialization sequence

Coding of initialization rules in methods `<createObject>()`, `<init>()`,...
which are then symbolically executed

Limitations of Method Inlining: methodBodyExpand

- Source code might be unavailable
  - API method implementation vendor-specific
  - Source code often unavailable for commercial APIs
- Method is invoked multiple times in a program
  - Avoid multiple symbolic execution of identical code
- Cannot handle unbounded recursion
- Not modular:
  Changing a method requires re-verification of all callers

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Use method contract instead of method implementation:

1. Show that requires clause is satisfied before method call
2. Remove method call, and:
   - assume ensures clause
   - forget prestate values of modifiable locations

Method Contract Rule: Normal Behavior Case

Simplified version

```
// implementation contract of m():
/*@ public normal_behavior
  @ requires normalPre;
  @ ensures normalPost;
  @ assignable mod;
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JML Method Contracts Revisited

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Implicit Preconditions and Postconditions

- The object referenced by **this** is not **null**: **this!=null** (precondition only; **this** cannot be changed by method)

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- Invariant for **this**: `\invariant_for(this)`

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- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising Heap Locations

Define anonymising function `anon`: Heap $\times$ LocSet $\times$ Heap $\rightarrow$ Heap

The resulting heap `anon(...)` coincides with the first heap on all locations except for those specified in the location set. Those locations attain the value specified by the second heap.

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Definition:

$$\mathtt{select}(\mathtt{anon}(h1, locs, h2), o, f) = \begin{cases} \mathtt{select}(h2, o, f) & \text{if } (o, f) \in locs \\ \mathtt{select}(h1, o, f) & \text{otherwise} \end{cases}$$

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Effect: After $\mathcal{V}_{mod}$, modfied locations have unknown values

Anonymising Heap Locations: Example

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@ assignable o.a, this.*;
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To erase all knowledge about the values of the locations of the assignable expression:

- Anonymise the current heap on the designated locations:

$$\mathtt{anon}(\mathtt{heap}, \{(\mathtt{o}, \mathtt{a})\} \cup \mathtt{allFields(this)}, \mathtt{h}_{an})$$

- Make that anonymised current heap the new current heap.

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(background only, no need to remember)

KeY uses actually one rule for both kinds of cases.

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```
class Person {
 private /*@ spec_public @*/ int age;
 /*@ public normal_behavior
   @ requires age < 29;
   @ ensures age == \old(age) + 1;
   @ assignable age;
   @ also
   @ public exceptional_behavior
   @ requires age >= 29;
   @ signals_only ForeverYoungException;
   @ assignable \nothing;
   @//allows object creation (not \strictly_nothing)
   @*/
 public void birthday() {
   if (age >= 29) throw new ForeverYoungException();
   age++;
 } }
```

Method Contract Rule: Example Cont'd

Demo

`methods/useContractForBirthday.key`

- Prove without contracts
  - Method treatment: Expand
- Prove with contracts (until method contract application)
  - Method treatment: Contract
- Prove used contracts
  - Method treatment: Expand
  - Select contracts for `birthday()` in `src/Person.java`
  - Prove both specification cases

Verification of Loops

Symbolic execution of loops: unwind

$$\text{unwindLoop} \quad \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\, \texttt{if(b)\{p; while(b) p\}}\, \omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\, \texttt{while(b) p}\, \omega]\phi, \Delta}$$

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How to handle a loop with...

- 0 iterations?

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How to handle a loop with...

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How to handle a loop with...

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- 10 iterations?

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How to handle a loop with...

- 0 iterations? Unwind $1\times$
- 10 iterations? Unwind $11\times$

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- 10000 iterations? Unwind $10001\times$
- an unknown number of iterations?

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- an unknown number of iterations?

We need an invariant rule (or some form of induction)

Loop Invariants

Idea behind loop invariants

- A formula *Inv* whose validity is preserved by loop body whenever the loop guard is true

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How to Derive Loop Invariants Systematically?

Example (Active statement of symbolic execution is loop)

```
    n >= 0 & wellFormed(heap)
->  {i := 0}
       \[{ while (i < n) {
              i = i + 1;
          }
        }\] i = n
```

Look at desired postcondition `i = n`

What, in addition to negated guard `i >= n`, is needed?

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Yes! We have found a suitable loop invariant!

Demo `loops/simple.key` (auto after inv)

Obtaining Invariants by Strengthening

Example (Slightly changed problem)

```
    n >= 0 & n = m & wellFormed(heap)
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Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

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First attempt: use postcondition `x = x0 + y0`

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Finding the invariant

First attempt: use postcondition `x = x0 + y0`

- Not true at start whenever `y0 > 0`
- Not preserved by loop, because `x` is increased

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What stays invariant?

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```

Finding the invariant

What stays invariant?

- The sum of `x` and `y`: `x + y = x0 + y0` "Generalization"
- Think of delta between `x` and `x0 + y0` within loop

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

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Is `x + y = x0 + y0` a good invariant?

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- Holds in the beginning and is preserved by loop

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Checking the invariant

Is `x + y = x0 + y0` a good invariant?

- Holds in the beginning and is preserved by loop
- But postcondition not implied by `x + y = x0 + y0` and exit condition `y <= 0`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

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```

Strenghtening the invariant

Postcondition holds if `y = 0`

- Add `y >= 0` to invariant: `x + y = x0 + y0 & y >= 0`

Demo `loops/simple3.key`

Basic Loop Invariant: Context Loss

Problems with the Basic Invariant Rule

$$\text{loopInvariant}\ \frac{\begin{array}{ll}\Gamma \Longrightarrow \mathcal{U}\, Inv, \Delta & \text{(initially valid)}\\ Inv,\ b = \mathtt{TRUE} \Longrightarrow [\mathrm{p}]Inv & \text{(preserved)}\\ Inv,\ b = \mathtt{FALSE} \Longrightarrow [\pi\ \omega]\phi & \text{(use case)}\end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \mathtt{\textbf{while}(b)\ p}\ \omega]\phi, \Delta}$$

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- Context Γ, Δ, $\mathcal{U}$ must be omitted in 2nd and 3rd premise:

  $\Gamma, \neg\Delta$ cannot be assumed for arbitrary iterations or at loop exit

  2nd premise State after some loop iterations is not $\mathcal{U}$

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  3rd premise State at loop exit is not $\mathcal{U}$
- Context contains preconditions and class invariants
- Only way to propagate context: add to loop invariant *Inv*

Example

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Example

Precondition: $a \neq \texttt{null}$

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Loop invariant: $0 \leq \texttt{i} \ \& \ \texttt{i} \leq \texttt{a.length}$

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Precondition: $\texttt{a} \neq \texttt{\textbf{null}}$ & *ClassInv*

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- How to erase all values of **assignable** locations?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising JAVA Locations

```
@ assignable i, a[*];
```

To erase all knowledge about these assignable locations:

- introduce a new (not yet used) constant of type `int`, e.g., `c`
- introduce a new (not yet used) constant of type Heap, e.g., $\mathtt{h}_{an}$
  - anonymise the current heap: $\mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{a}), \mathtt{h}_{an})$
- compute anonymizing update for assignable locations

$$\mathcal{V} = \{\mathtt{i} := \mathtt{c} \mid\mid \mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{a}), \mathtt{h}_{an})\}$$

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For local program variables (e.g., `i`) KeY computes assignable clause automatically

Loop Invariants Cont'd

Improved Invariant Rule

$$\frac{}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{\textbf{while}}\texttt{(b) p}\ \omega]\phi, \Delta}$$

Loop Invariants Cont'd

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$$\Gamma \Longrightarrow \mathcal{U}\mathit{Inv}, \Delta \qquad \text{(initially valid)}$$

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- Context is kept as far as possible: $\mathcal{V}$ erases only information in locations assignable in the loop
- Invariant *Inv* does not need to include unmodified locations
- For **`assignable \everything`** (the default):
  - $\mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allLocs}, \mathtt{h}_{an})$ wipes out **all** heap information
  - Equivalent to basic invariant rule
  - Avoid this! Always give a specific **`assignable`** clause

Example with Improved Invariant Rule

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Example with Improved Invariant Rule

Precondition: $a \neq \mathbf{null}$

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Example with Improved Invariant Rule

Precondition: $\mathtt{a} \neq \mathbf{null}$

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int i = 0;
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```

Postcondition: $\forall\, \mathbf{int}\ x;\ (0 \leq x\ \&\ x < \mathtt{a.length} \rightarrow \mathtt{a}[x] = 1)$

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Loop invariant: $0 \leq \texttt{i}\ \ \&\ \ \texttt{i} \leq \texttt{a.length}$

Example with Improved Invariant Rule

Precondition: $\mathtt{a} \neq \mathbf{null}$

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int i = 0;
while(i < a.length) {
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```

Postcondition: $\forall\, \mathbf{int}\ x;\ (0 \leq x \ \&\ x < \mathtt{a.length} \rightarrow \mathtt{a}[x] = 1)$

Loop invariant: $0 \leq \mathtt{i} \ \ \&\ \ \mathtt{i} \leq \mathtt{a.length}$
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Example with Improved Invariant Rule

Precondition: $\texttt{a} \neq \texttt{null}$

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while(i < a.length) {
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$\&\ \ \forall\, \texttt{int}\ x;\ (0 \leq x\ \&\ x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$

Example with Improved Invariant Rule

Precondition: $\texttt{a} \neq \textbf{null}$ & *ClassInv*

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall\, \textbf{int}\ x;\ (0 \leq x \ \&\ x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

Loop invariant: $0 \leq \texttt{i} \ \ \& \ \ \texttt{i} \leq \texttt{a.length}$
$\&\ \ \forall\, \textbf{int}\ x;\ (0 \leq x \ \&\ x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$

Example in JML/JAVA – Loop.java **Demo**

```
public int[] a;
/*@ public normal_behavior
  @  ensures (\forall int x; 0<=x && x<a.length; a[x]==1);
  @  diverges true;
  @*/
public void m() {
  int i = 0;
  /*@ loop_invariant
    @  0 <= i && i <= a.length &&
    @  (\forall int x; 0<=x && x<i; a[x]==1);
    @ assignable a[*];
    @*/
  while(i < a.length) {
    a[i] = 1;
    i++;
  }
}
```

Example from an earlier Lecture

$\forall$ **int** x;
  $(x = \mathtt{n} \land x >= 0 \rightarrow$
    [i = 0; r = 0;
      while (i<n) { i = i + 1; r = r + i;}
      r=r+r-n;
    ] $(\mathtt{r} = x * x)$

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Example from an earlier Lecture

$\forall$ **int** x;

$(x = \mathtt{n} \wedge x >= 0 \rightarrow$

```
[ i = 0; r = 0;
  while (i<n) { i = i + 1; r = r + i;}
  r=r+r-n;
] 
```

$(\mathtt{r} = x * x)$

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Needed Invariant:

Example from an earlier Lecture

$\forall$ **int** x;
$(x = \texttt{n} \land x >= 0 \rightarrow$
$[$ `i = 0; r = 0;`
```
while (i<n) { i = i + 1; r = r + i;}
r=r+r-n;
```
$]\ (\texttt{r} = x * x)$

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Needed Invariant:

```
@ loop_invariant
@     i>=0  && i <= n && 2*r == i*(i + 1);
@ assignable \nothing; // no heap locations changed
```

Example from an earlier Lecture

$\forall$ **int** x;
$(x = \mathtt{n} \land x >= 0 \rightarrow$
$[$
```
i = 0; r = 0;
while (i<n) { i = i + 1; r = r + i;}
r=r+r-n;
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Needed Invariant:

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```

Demo Loop2.java

Hints

Proving `assignable`

- Invariant rule above assumes that **`assignable`** is correct
  E.g., possible to prove nonsense with incorrect
  `assignable \nothing;`
- Invariant rule of KeY generates proof obligation that ensures correctness of **`assignable`**
  This proof obligation is part of 'Body Preserves Invariant' branch

Hints

Proving `assignable`

- Invariant rule above assumes that **`assignable`** is correct
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  This proof obligation is part of 'Body Preserves Invariant' branch

Setting in the KeY Prover when proving loops w. given invariant

- Loop treatment: Invariant
- Quantifier treatment: No Splits with Progs
- If program contains `*`, `/`: Arithmetic treatment: DefOps
- Is search limit high enough (time out, rule apps.)?
- To prove only partial correctness, add **`diverges true;`**

Total Correctness

Is the sequent

$$\Longrightarrow [\texttt{i = -1; while (true)\{\}}]\texttt{i} = 4711$$

provable?

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Yes, e.g.,

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@ loop_invariant true;
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```

Total Correctness

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```
@ loop_invariant true;
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With this, correctness of non-terminating loop is provable:

- Invariant trivially initially valid and preserved:
  Initial Case and Preserved Case close immediately
- Negated loop condition is `false`: Use case close immediately

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With this, correctness of non-terminating loop is provable:

- Invariant trivially initially valid and preserved:
  Initial Case and Preserved Case close immediately
- Negated loop condition is `false`: Use case close immediately

But need a method to prove termination of loops

Mapping Loop Execution to Well-Founded Order

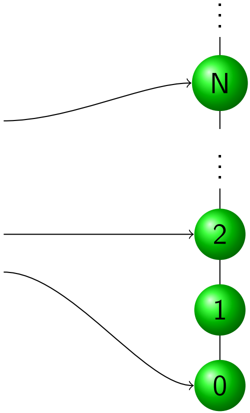
```
while (b) {
  body
}
```

if (b) { body }$_1$

⋮

if (b) { body }$_{17}$

if (b) { body }$_{18}$



Need to find expression getting smaller wrt $\mathbb{N}$ in each iteration

Such an expression is called a decreasing term or variant

Total Correctness: Decreasing Term (Variant)

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
- $v \geq 0$ is preserved by the loop body
- v is strictly decreased by the loop body

Total Correctness: Decreasing Term (Variant)

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
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Proving termination in JML/Java

- Remove `diverges true;` from contract
- Add `decreasing v;` to loop invariant
- KeY creates suitable invariant rule and PO (with $\langle \ldots \rangle \phi$)

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Example (The `array` loop)

```
@ decreasing
```

Total Correctness: Decreasing Term (Variant)

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Add the following premises to the invariant rule:

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Example (The `array` loop)

```
@ decreasing  a.length - i;
```

Total Correctness: Decreasing Term (Variant)

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Example (The `array` loop)

```
@ decreasing  a.length - i;
```

Files:

- `LoopT.java`
- `Loop2T.java`

Final Example: Computing the GCD (see 16.3.8 [KeYbook])

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @  (_big % \result == 0 && _small % \result == 0 &&
   @    (\forall int x; x>0 && _big % x == 0
   @       && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {
   int big = _big; int small = _small;
   while (small != 0) {
     final int t = big % small;
     big = small;
     small = t;
   }
   return big;
 }
}
```

Computing the GCD: Method Specification

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public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
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   @ (_big % \result == 0 && _small % \result == 0 &&
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   @ assignable \nothing;
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 private static int gcdHelp(int _big, int _small) {...}
```

requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

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ensures if `_big` positive, then

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ensures if `_big` positive, then

- the return value `\result` is a divisor of both arguments

Computing the GCD: Method Specification

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public class Gcd {
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   @   (\forall int x; x>0 && _big % x == 0
   @      && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {...}
```

requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

ensures if `_big` positive, then

- the return value `\result` is a divisor of both arguments
- all other divisors `x` of the arguments are also divisors of `\result` and thus smaller or equal to `\result`

Computing the GCD: Specify the Loop Body

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Which locations are changed (at most)?

Computing the GCD: Specify the Loop Body

```
int big = _big; int small = _small;
while (small != 0) {
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```

Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

Computing the GCD: Specify the Loop Body

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int big = _big; int small = _small;
while (small != 0) {
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}
return big;
```

Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

```
@ decreases small;
```

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration?

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration? No.

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Adding `big>0` to loop invariant?

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Adding `big>0` to loop invariant? No. Not initially valid.

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
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Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve?

Computing the GCD: Specify the Loop Body Cont'd

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Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of divisors!
  All common divisors of `_big, _small` are also divisors of `big, small`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
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return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of divisors!

  All common divisors of `_big`, `_small` are also divisors of `big`, `small`

```
(\forall int x; x > 0;
                (_big%x == 0 && _small%x == 0)
                <==>
                (big%x == 0 && small%x == 0));
```

Computing the GCD: Final Specification

```
int big = _big; int small = _small;
/*@ loop_invariant small >= 0 && big >= small &&
  @   (big == 0 ==> _big == 0) &&
  @   (\forall int x; x > 0; (_big % x == 0 && _small % x == 0)
  @                          <==>
  @                          (big % x == 0 && small % x == 0));
  @ decreases small;
  @ assignable \nothing;
  @*/
  while (small != 0) {
    final int t = big % small;
    big = small;
    small = t;
  }
  return big; // assigned to \result
```

Computing the GCD: Final Specification

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  @                            <==>
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```

Why does big divides _small and _big follow from the loop invariant?

Computing the GCD: Final Specification

```
int big = _big; int small = _small;
/*@ loop_invariant small >= 0 && big >= small &&
  @   (big == 0 ==> _big == 0) &&
  @   (\forall int x; x > 0; (_big % x == 0 && _small % x == 0)
  @                          <==>
  @                          (big % x == 0 && small % x == 0));
  @ decreases small;
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  @*/
  while (small != 0) {
    final int t = big % small;
    big = small;
    small = t;
  }
  return big; // assigned to \result
```

Why does `big` divides `_small` and `_big` follow from the loop invariant?

If `big` is positive, one can instantiate `x` with it, and use `small == 0`

Computing the GCD: Demo

Demo `loops/Gcd.java`

1. Show `Gcd.java` and `gcd(a,b)`
2. Select "One Step Simplification", "Contract", "DefOps", 10k steps
3. Prove contract of `gcd()`, using contract of `gcdHelp()`
4. Select "Invariant"
5. Prove contract of `gcdHelp()`

Some Hints On Finding Invariants

General Advice

- Invariants must be developed, they don't come out of thin air!
- Be as systematic in deriving invariants as when debugging a program

Some Hints On Finding Invariants, Cont'd

Technical Hints

- Good starting point: desired postcondition (of the loop!)
  - What, in addition to negated loop guard, is needed for it to hold?

Some Hints On Finding Invariants, Cont'd

Technical Hints

- Good starting point: desired postcondition (of the loop!)
  - What, in addition to negated loop guard, is needed for it to hold?
- If the invariant candidate is not preserved by the loop body:
  - Can you add stuff from the precondition?
  - Does it need strengthening?
  - Try to express the relation between partial and final result

Some Hints On Finding Invariants, Cont'd

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- Simulate a few loop body executions to discover invariant patterns

Some Hints On Finding Invariants, Cont'd

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- If the invariant is not initially valid:
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  - Did you forget an assumption in the requires clause?

Some Hints On Finding Invariants, Cont'd

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Some Hints On Finding Invariants, Cont'd

Technical Hints

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- Simulate a few loop body executions to discover invariant patterns
- If the invariant is not initially valid:
  - Can it be weakened such that the postcondition still follows?
  - Did you forget an assumption in the requires clause?
- Several "rounds" of weakening/strengthening might be required
- Use the KeY tool to iteratively try invariants:
  - Loop treatment: None
  - apply **Loop Invariant** $\rightarrow$ **Enter Loop Specification**
  - After each change of invariant make sure all cases are ok
  - If not, prune and retry

Understanding Unclosed Proofs (see also p.528ff [KeYbook])

Reasons why a proof may not close

- Buggy or incomplete specification
- Bug in program
- Maximal number of steps reached: restart or increase # of steps
- Automatic proof search fails: apply some rules manually

Understanding Unclosed Proofs (see also p.528ff [KeYbook])

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- Bug in program
- Maximal number of steps reached: restart or increase # of steps
- Automatic proof search fails: apply some rules manually

Understanding open proof goals

- Follow the control flow from the proof root to the open goal
- Branch labels give useful hints
- Identify unprovable part of post condition or invariant
- Sequent remains always in "pre-state"
  Constraints on program variables refer to value at start of program (exception: formula is behind update or modality)
- NB: $\Gamma \Longrightarrow o = \mathtt{null}, \Delta$ is equivalent to $\Gamma, o \neq \mathtt{null} \Longrightarrow \Delta$

Literature for this Lecture

KeYbook *W. Ahrendt*, B. Beckert, R. Bubel, R. Hähnle, P. Schmitt, M. Ulbrich, editors.
Deductive Software Verification - The KeY Book
Vol 10001 of *LNCS*, Springer, 2016
(E-book at `link.springer.com`)

- *W. Ahrendt*, S. Grebing, Using the KeY Prover
  Chapter 15 in [KeYbook], p.528ff + Section 15.3 (also for Lab2)
- B. Beckert, R. Hähnle, M. Hentschel, P.H. Schmitt,
  Formal Verification with KeY: A Tutorial
  Chapter 16 in [KeYbook], except Section 16.6

further reading:

- B. Beckert, V. Klebanov, B. Weiß, Dynamic Logic for Java
  Chapter 3 in [KeYbook], Section 3.7

Master's Thesis Projects in Formal Methods

see Formal Methods Master Theses on the web (click here).

Thank You