Formal Methods for Software Development

Propositional and (Linear) Temporal Logic

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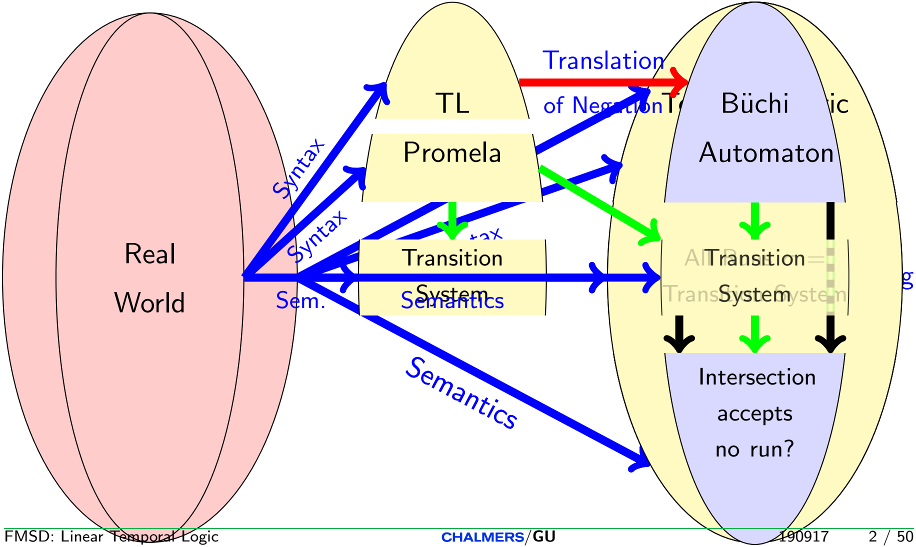
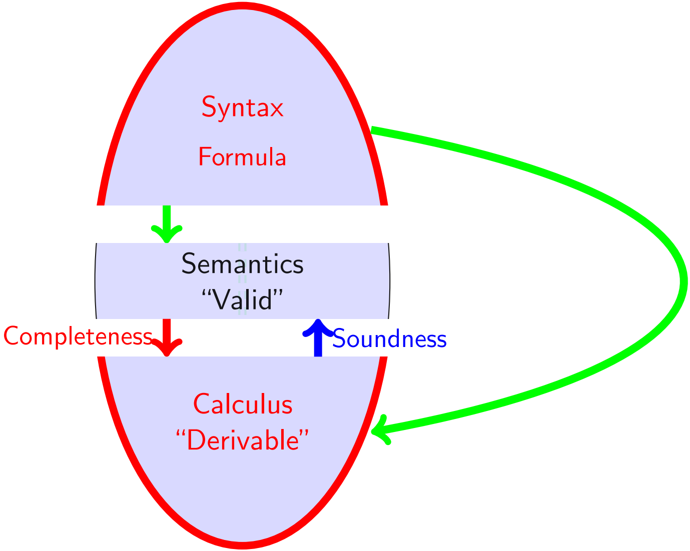
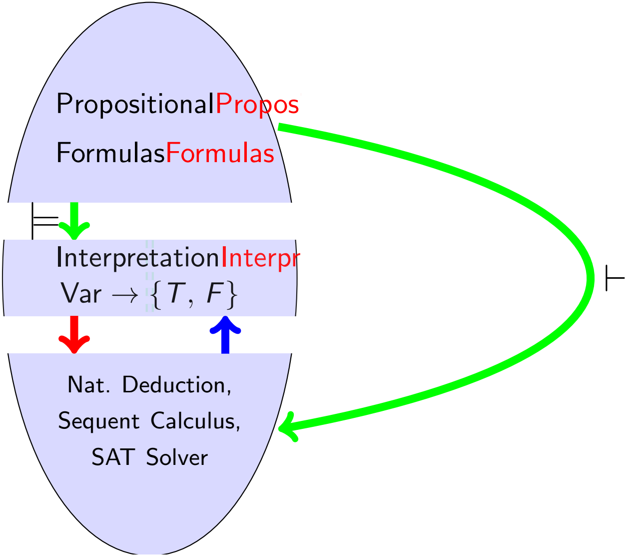
Revisit: FormalisationFormalisation: Syntax, SemanticsFormalisation: Syntax, Semantics, ProvingFormal Verification: Model Checking



The Big Picture: Syntax, Semantics, Calculus



Simplest Case: Propositional Logic—Syntax



Syntax of Propositional Logic

Signature

A set of *atomic propositions AP*
(with typical elements $p, q, r, \ldots$)

Propositional Connectives

$\mathrm{true},\ \mathrm{false},\ \wedge,\ \vee,\ \neg,\ \rightarrow,\ \leftrightarrow$

Set of Propositional Formulas For_0

- All elements of $AP \cup \{\mathrm{true}, \mathrm{false}\}$ are formulas
- If ϕ and ψ are formulas then

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi, \quad \phi \leftrightarrow \psi$$

  are also formulas
- There are no other formulas (inductive definition)

Remark on Concrete Syntax

<table>
<tr><th></th><th>Text book</th><th>SPIN</th></tr>
<tr><td>Negation</td><td>$\neg$</td><td>!</td></tr>
<tr><td>Conjunction</td><td>$\wedge$</td><td>&&</td></tr>
<tr><td>Disjunction</td><td>$\vee$</td><td>||</td></tr>
<tr><td>Implication</td><td>$\rightarrow$, $\supset$</td><td>-></td></tr>
<tr><td>Equivalence</td><td>$\leftrightarrow$</td><td><-></td></tr>
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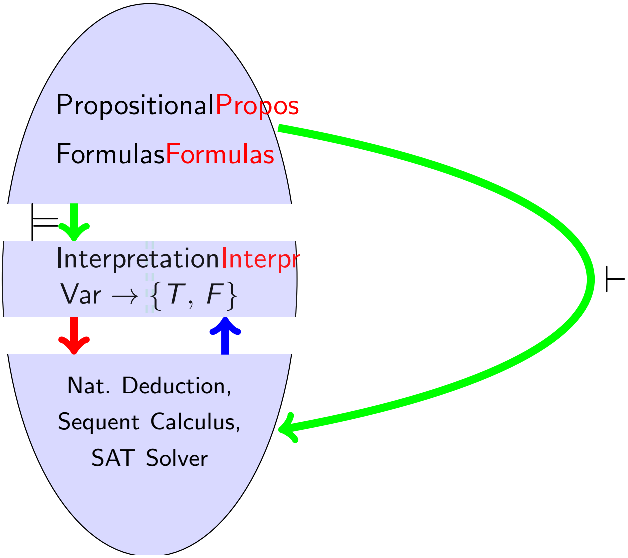
We use mostly the textbook notation,
except for tool-specific slides, input files.

Simplest Case: Propositional Logic—Syntax



Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each atomic proposition

$$\mathcal{I} : AP \rightarrow \{T, F\}$$

Example

Let $AP = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
$\vdots$	$\vdots$	$\vdots$

Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each atomic proposition

$$\mathcal{I} : AP \rightarrow \{T, F\}$$

Valuation Function

$val_{\mathcal{I}}$: Continuation of $\mathcal{I}$ on For_0

$$val_{\mathcal{I}} : For_0 \ \rightarrow \ \{T, F\}$$

$val_{\mathcal{I}}(\mathrm{true}) = T$
$val_{\mathcal{I}}(\mathrm{false}) = F$
$val_{\mathcal{I}}(p_i) = \mathcal{I}(p_i)$

(cont'd on next page)

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \rightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \leftrightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = val_{\mathcal{I}}(\psi) \\ F & otherwise \end{cases}$$

Valuation Examples

Example

Let $AP = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$\ldots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) \quad = \quad T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$
$$val_{\mathcal{I}_2}(p) \quad = \quad \mathcal{I}_2(p) \quad = \quad T$$
$$val_{\mathcal{I}_2}(\ q \rightarrow p\ ) \quad = \quad T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T$$
$$val_{\mathcal{I}_2}(q) \quad = \quad \mathcal{I}_2(q) \quad = \quad F$$

Semantic Notions of Propositional Logic

Let $\phi \in For_0$, $\Gamma \subseteq For_0$

Definition (Satisfying Interpretation, Consequence Relation)

$\mathcal{I}$ satisfies ϕ (write: $\mathcal{I} \models \phi$) iff $val_{\mathcal{I}}(\phi) = T$

ϕ follows from Γ (write: $\Gamma \models \phi$) iff for all interpretations $\mathcal{I}$:

If $\mathcal{I} \models \psi$ for all $\psi \in \Gamma$, then also $\mathcal{I} \models \phi$

Definition (Satisfiability, Validity)

A formula is satisfiable if it is satisfied by some interpretation.
If every interpretation satisfies ϕ (write: $\models \phi$) then ϕ is called valid.

Semantics of Propositional Logic: Examples

Formula (same as before)

$$p \rightarrow (q \rightarrow p)$$

Is this formula valid?

$$\models p \rightarrow (q \rightarrow p)\ ?$$

Semantics of Propositional Logic: Examples

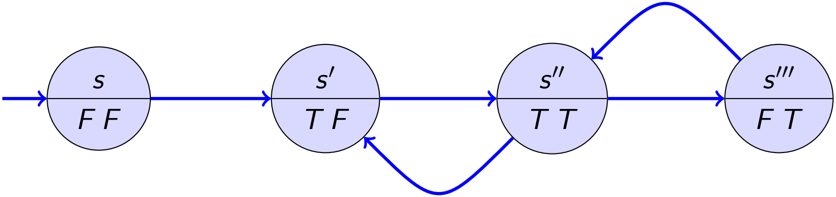
$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔
Satisfying Interpretation? $\mathcal{I}(p) = T$, $\mathcal{I}(q) = T$
Other Satisfying Interpretations? ✘
Therefore, not valid!

$$p \land ((\neg p) \lor q) \models q \lor r$$

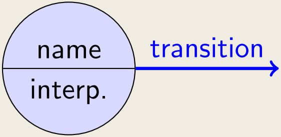
Does it hold? Yes. Why?

Transition Systems (aka Kripke Structures)

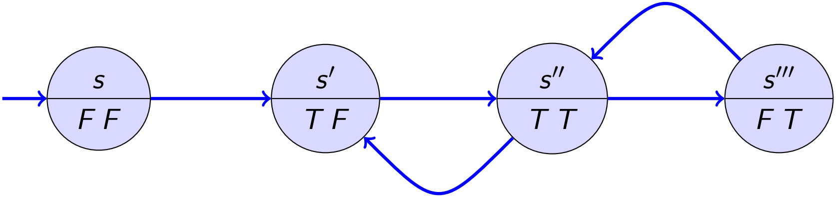


We assume $AP = \{p, q\}$

Notation

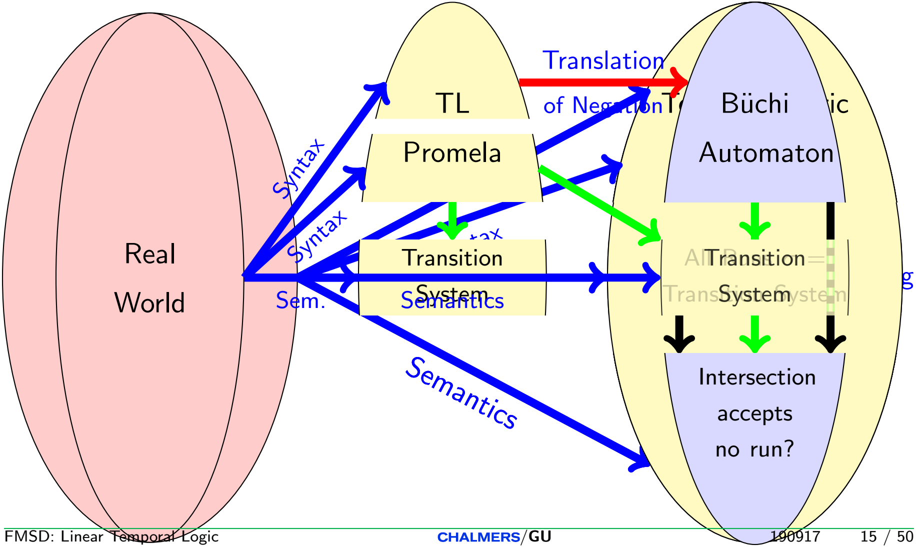


Transition Systems (aka Kripke Structures)



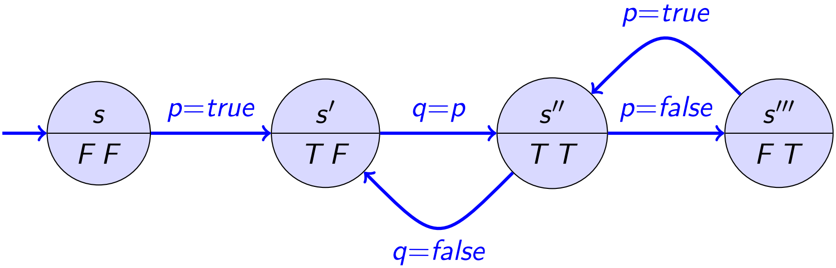
- Each state has *its own* interpretation $\mathcal{I} : \{p, q\} \rightarrow \{T, F\}$
  - Convention: list interpretation of variables in lexicographic order
- Computations, or runs, are *infinite* paths through states
  - 'finite' runs simulated by looping on terminal state
- Prefix of some example runs:
  - $s\,s'\,s''\,s'\,s''\,s'\,s''\,s'''\,\ldots$
  - $s\,s'\,s''\,s'''\,s''\,s'\,s''\,s'\,\ldots$

Revisit: FormalisationFormalisation: Syntax, SemanticsFormalisation: Syntax, Semantics, ProvingFormal Verification: Model Checking



Transition System of some PROMELA Model

```
bool p, q;
p = true; q = p;
do :: q = false; q = p
   :: p = false; p = true
od
```



(assignments only for illustration, not part of transition system)

Transition Systems: Formal Definition

Definition (Transition System)

A transition system $\mathcal{T} = (S, \rightarrow, S_o, L)$ is composed of a set of states S, a transition relation $\rightarrow \subseteq S \times S$, a set $\emptyset \neq S_0 \subseteq S$ of initial states, and a labeling L of each state $s \in S$ with a propositional interpretation $L(s)$.

Definition (Run of Transition System)

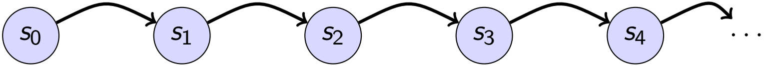
A run of $\mathcal{T} = (S, \rightarrow, S_o, L)$ is a sequence of states
$\sigma = s_0\, s_1\, s_2 \ldots$
such that $s_0 \in S_0$ and $s_i \rightarrow s_{i+1}$ for all $i \geq 0$.

Definition (Trace)

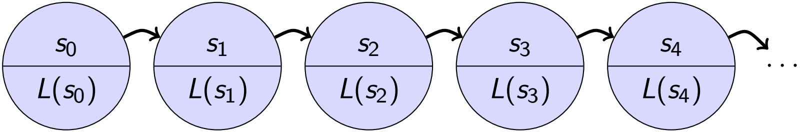
The trace $tr(\sigma)$ of a run $\sigma = s_0\, s_1\, s_2 \ldots$ is the sequence
$\tau = \mathcal{I}_0\, \mathcal{I}_1\, \mathcal{I}_2 \ldots$
such that $\mathcal{I}_i = L(s_i)$.
A trace of transition system $\mathcal{T}$ is $tr(\sigma)$ for any run σ of $\mathcal{T}$.

Runs and Traces Visually

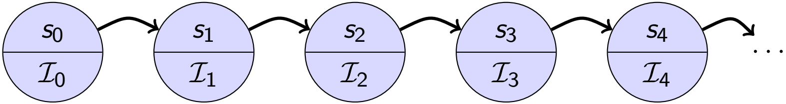
- Given a run $\sigma = s_0\, s_1\, s_2\, s_3\, s_4 \ldots$



- Each state s of a transition system is labelled, via $L(s)$, with an interpretation



- If we name each interpretations $L(s_i)$ as $\mathcal{I}_i$, we have



- The trace $tr(\sigma)$ is: $\tau = \mathcal{I}_0\, \mathcal{I}_1\, \mathcal{I}_2\, \mathcal{I}_3\, \mathcal{I}_4 \ldots$

Notations: Power Set and Sequences

Assume sets X and Y.

Power Set

2^X is the set of all subsets of X (called 'power set of X').

Finite Sequences

Y^* is the set of all finite sequences (words) of elements of Y.

Infinite Sequences

Y^ω is the set of all infinite sequences (words) of elements of Y.

Examples of Power Sets and Sequences

Given the set of atomic propositions $AP = \{p, q\}$.

Power Set

$2^{AP} = \{\,\{\}, \{p\}, \{q\}, \{p, q\}\,\}$

Finite Sequences

$(2^{AP})^*$: set of all finite sequences of elements of 2^{AP}.
E.g.: $\{p\}\{\}\{p, q\}\{p\} \in (2^{AP})^*$

(and infitely many others)

Infinite Sequences

$(2^{AP})^\omega$: set of all infinite sequences of elements of 2^{AP}.
E.g.: $\{p\}\{p, q\}\{p\}\{\}\{p\}\{p, q\}\{p\}\{\} \ldots \in (2^{AP})^\omega$

(and uncountably many others)

Interpretations as Sets

Interpretations over atomic propositions AP can be represented as elements of 2^{AP}.

E.g., assume $AP = \{p, q\}$
I.e., $2^{AP} = \{\, \{\}, \{p\}, \{q\}, \{p, q\} \,\}$

	p	q
$\mathcal{I}_1$	F	F

represented as $\{\}$

	p	q
$\mathcal{I}_2$	T	F

represented as $\{p\}$

	p	q
$\mathcal{I}_3$	F	T

represented as $\{q\}$

	p	q
$\mathcal{I}_4$	T	T

represented as $\{p, q\}$

Runs and Traces revisited

Given states S and atomic propositions AP.

- A run $\sigma = s_0\, s_1\, s_2\, s_3\, s_4 \ldots$ is an element of S^ω
- A trace $\tau = \mathcal{I}_0\, \mathcal{I}_1\, \mathcal{I}_2\, \mathcal{I}_3 \ldots$ is an element of of $(2^{AP})^\omega$

An example of a trace $\tau = \mathcal{I}_0\, \mathcal{I}_1\, \mathcal{I}_2\, \mathcal{I}_3 \ldots$ may look like:

$\tau = \{p\}\{p, q\}\{p\}\{\} \ldots$

Linear Time Properties

Definition (Linear Time Property)

Given a set of atomic propositions AP.
Each subset $P \subseteq (2^{AP})^{\omega}$ is a linear time (LT) property over AP.

Intuition:

- Assume a trace property $P \subseteq (2^{AP})^{\omega}$.
- A trace τ fulfils the property P iff $\tau \in P$.
- A trace τ violates the property P iff $\tau \in (2^{AP})^{\omega} \setminus P$ (i.e., $\tau \notin P$).

Classes of LT Properties

The LT properties can be devided in three classes:

- Safety properties
- Liveness properties
- Properties that are neither safety nor liveness properties

Safety Properties

Definition (Safety Properties, Bad Prefixes)

An LT property P_{safe} over AP is called a *safety property* if for all traces $\tau \in (2^{AP})^\omega \setminus P_{safe}$, there exists a finite prefix $\hat{\tau}$ of τ such that

$$\left\{\tau' \in (2^{AP})^\omega \mid \hat{\tau} \text{ is a finite prefix of } \tau'\right\} \cap P_{safe} = \emptyset$$

- Each violating trace τ has a finite, 'bad prefix' $\hat{\tau}$ that cannot be extended to a safe trace.
- A safety violation manifests itself in finite time, and cannot be repaired thereafter.

Liveness Properties

Let $pref(P)$ be the set of finite prefixes of elements of P.

Definition (Liveness Properties)

An LT property P_{live} over AP is called a liveness property whenever $pref(P_{live}) = (2^{AP})^*$

A liveness property

- allows every finite prefix
- cannot be refuted in finite time

Linear Temporal Logic—Syntax

An extension of propositional logic that
allows to specify properties of all traces

Syntax

Based on propositional signature and syntax.

Extension with three connectives (in this course):

Always If ϕ is a formula, then so is $\Box\phi$

Eventually If ϕ is a formula, then so is $\Diamond\phi$

Until If ϕ and ψ are formulas, then so is $\phi\mathcal{U}\psi$

Concrete Syntax

	text book	SPIN
Always	$\Box$	[]
Eventually	$\Diamond$	<>
Until	$\mathcal{U}$	U

Linear Temporal Logic Syntax: Examples

Let $AP = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \rightarrow q$
- $\Diamond p$
- $\Box q$
- $\Diamond \Box (p \rightarrow q)$
- $(\Box p) \rightarrow ((\Diamond p) \vee \neg q)$
- $p \mathcal{U} (\Box q)$

Temporal Logic—Semantics

Valuation of temporal formula relative to a trace (infinite sequence of interpretations)

Definition (Validity Relation)

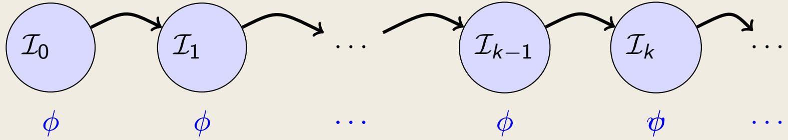
Validity of temporal formula depends on traces $\tau = \mathcal{I}_0\,\mathcal{I}_1\,\mathcal{I}_2\ldots$

$\tau \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in AP$.
$\tau \models \neg\phi$	iff	not $\tau \models \phi$ (write $\tau \not\models \phi$)
$\tau \models \phi \land \psi$	iff	$\tau \models \phi$ and $\tau \models \psi$
$\tau \models \phi \lor \psi$	iff	$\tau \models \phi$ or $\tau \models \psi$
$\tau \models \phi \rightarrow \psi$	iff	$\tau \not\models \phi$ or $\tau \models \psi$

Temporal connectives?

Temporal Logic—Semantics (Cont'd)

Trace τ



If $\tau = \mathcal{I}_0\,\mathcal{I}_1\,\mathcal{I}_2\ldots$, then $\tau|_i$ denotes the suffix $\mathcal{I}_i\,\mathcal{I}_{i+1}\,\mathcal{I}_{i+2}\ldots$ of τ.

Definition (Validity Relation for Temporal Connectives)

Given a trace $\tau = \mathcal{I}_0\,\mathcal{I}_1\,\mathcal{I}_2\ldots$

$\tau \models \Box\phi$ iff $\tau|_k \models \phi$ for all $k \geq 0$

$\tau \models \Diamond\phi$ iff $\tau|_k \models \phi$ for some $k \geq 0$

$\tau \models \phi\,\mathcal{U}\,\psi$ iff $\tau|_k \models \psi$ for some $k \geq 0$, and $\tau|_j \models \phi$ for all $0 \leq j < k$ (if $k = 0$ then ϕ needs never hold)

Safety and Liveness Formulas

Safety Formulas

- Formulas describing a safety property
- Example:
  $\Box\,(\neg \texttt{P\_in\_CS} \lor \neg \texttt{Q\_in\_CS})$
  'simultaneous visit to the critical sections never happens'
- Often state that "something bad never happens"

Liveness Formulas

- Formulas describing a liveness property
- Example:
  $\Diamond\,\texttt{P\_in\_CS}$
  'P enters its critical section eventually'
- Often state that "something good happens eventually"

Complex Properties

What does this mean?Infinitely Often

$$\tau \models \Box\Diamond\phi$$

"During trace τ the formula ϕ becomes true infinitely often"

Validity of Temporal Logic

Definition (Validity)

ϕ is valid, write $\models \phi$, iff $\tau \models \phi$ for all traces $\tau = \mathcal{I}_0\,\mathcal{I}_1\,\mathcal{I}_2\ldots$.

Representation of Traces

Can represent a set of traces as a sequence of propositional formulas:

- $\phi_0\,\phi_1\,\phi_2$... represents all traces $\mathcal{I}_0\,\mathcal{I}_1\,\mathcal{I}_2$... such that $\mathcal{I}_i \models \phi_i$ for $i \geq 0$

Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

No, there is a trace where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some trace?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

$$\Box\phi \rightarrow \phi \qquad (\neg\Box\phi) \leftrightarrow (\Diamond\neg\phi) \qquad \Diamond\phi \leftrightarrow (\mathrm{true}\;\mathcal{U}\phi)$$

All are valid! (proof is exercise)

- $\Box$ is reflexive
- $\Box$ and $\Diamond$ are dual connectives
- $\Box$ and $\Diamond$ can be expressed with only using $\mathcal{U}$

Temporal Logic—Semantics (Cont'd)

Extension of validity of temporal formulas to transition systems:

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, \rightarrow, S_0, L)$, a temporal formula ϕ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\tau \models \phi$ for all traces τ of $\mathcal{T}$.

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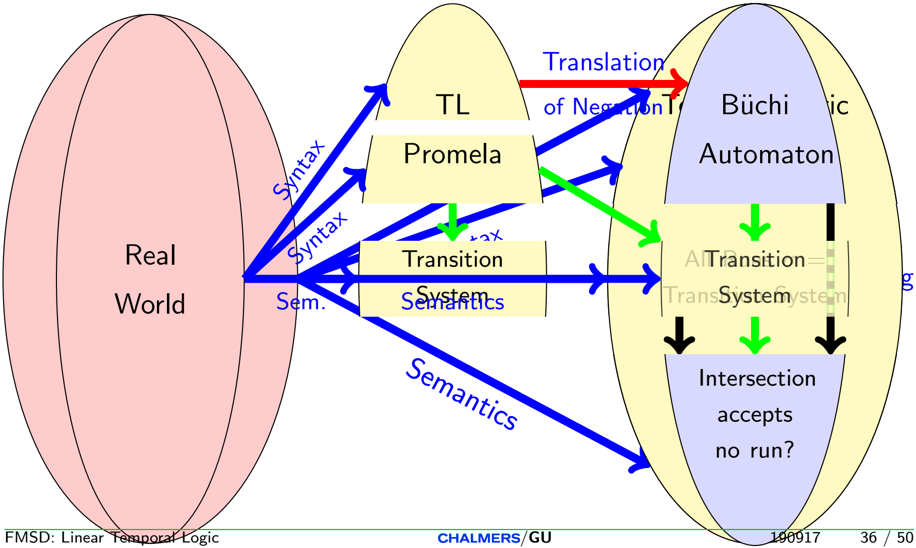


ω-Languages

Given a finite alphabet (vocabulary) Σ

An ω-word $w \in \Sigma^{*\omega}$ is a n infinite sequence

$$w = a_o \dots a_{nk} \dots$$

with $a_i \in \Sigma, i \in \{0, \dots, n\}\mathbb{N}$

$\mathcal{L}^\omega \subseteq \Sigma^{*\omega}$ is called a n ω-language

Büchi Automaton

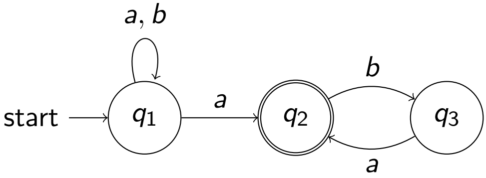
Definition (Büchi Automaton)

A (non-deterministic) Büchi automaton over an alphabet Σ consists of a

- finite, non-empty set of locations Q
- a transition relation $\delta \subseteq Q \times \Sigma \times Q$
- a non-empty set of initial locations $Q_0 \subseteq Q$
- a set of accepting locations $F = \{f_1, \ldots, f_n\} \subseteq Q$

Example

$\Sigma = \{a, b\}, Q = \{q_1, q_2, q_3\}, I = \{q_1\}, F = \{q_2\}$



Büchi Automaton—Executions and Accepted Words

Definition (Execution)

Let $\mathcal{B} = (Q, \delta, Q_0, F)$ be a Büchi automaton over alphabet Σ.
An execution of $\mathcal{B}$ is a pair (w, v), with

- $w = a_o \dots a_k \dots \in \Sigma^\omega$
- $v = q_o \dots q_k \dots \in Q^\omega$

where $q_0 \in Q_0$, and $(q_i, a_i, q_{i+1}) \in \delta$, for all $i \in \mathbb{N}$

Definition (Accepted Word)

A Büchi automaton $\mathcal{B}$ accepts a word $w \in \Sigma^\omega$, if there exists an execution (w, v) of $\mathcal{B}$ where some accepting location $f \in F$ appears infinitely often in v.

Büchi Automaton—Language

Let $\mathcal{B} = (Q, \delta, Q_0, F)$ be a Büchi automaton, then

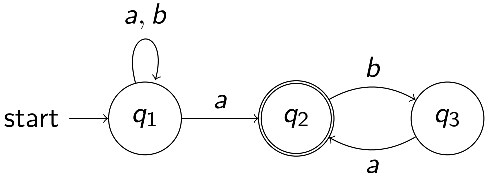
$$\mathcal{L}^\omega(\mathcal{B}) = \{ w \in \Sigma^\omega \,|\, \mathcal{B} \text{ accepts } w \}$$

denotes the ω-language recognised by $\mathcal{B}$.

An ω-language for which an accepting Büchi automaton exists is called ω-regular language.

Example, ω-Regular Expression

Which language is accepted by the following Büchi automaton?



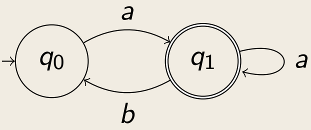
Solution: $(a + b)^*(ab)^\omega$ [NB: $(ab)^\omega = a(ba)^\omega$]

ω-regular expressions similar to standard regular expression

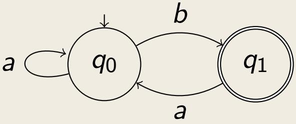
- ab a **followed by** b
- $a + b$ a **or** b
- a^* arbitrarily, but finitely often a
- **new:** a^ω infinitely often a

Büchi Automata—More Examples

Language: $a(a + ba)^\omega$



Language: $(a^* ba)^\omega$



Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

Theorem (Decidability)

It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton $\mathcal{B}$ is empty.

Theorem (Closure properties)

The set of ω-regular languages is closed with respect to intersection, union and complement:

- *if $\mathcal{L}_1, \mathcal{L}_2$ are ω-regular then $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are ω-regular*
- *$\mathcal{L}$ is ω-regular then $\Sigma^\omega \backslash \mathcal{L}$ is ω-regular*

But in contrast to regular finite automata:

Non-deterministic Büchi automata are strictly more expressive than deterministic ones.

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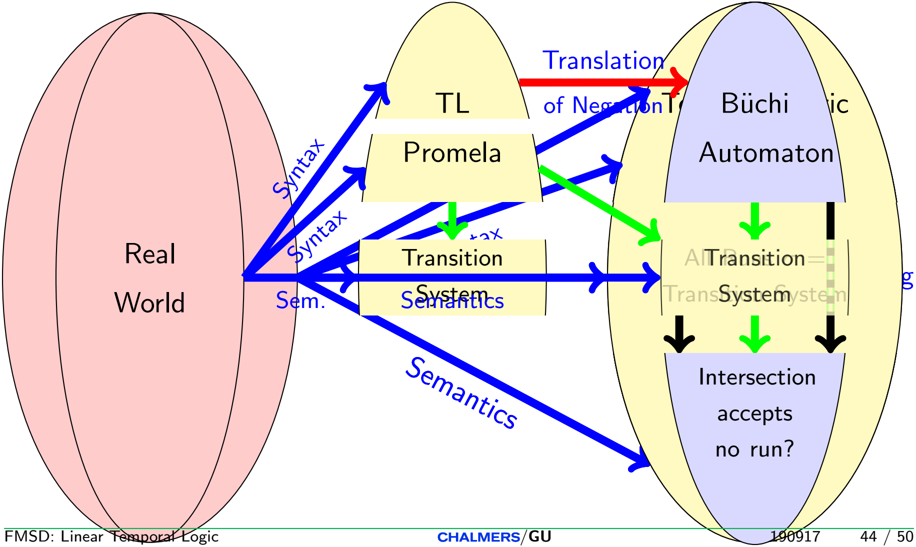


Linear Temporal Logic and Büchi Automata

LTL and Büchi Automata are connected

Recall

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, \rightarrow, S_0, L)$, a temporal formula ϕ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\tau \models \phi$ for all traces τ of $\mathcal{T}$.

A trace of the transition system is an infinite sequence of interpretations.

Intended Connection

Given an LTL formula ϕ:

Construct a Büchi automaton accepting exactly those traces (infinite sequences of interpretations) that satisfy ϕ.

Encoding an LTL Formula as a Büchi Automaton

AP set of propositional variables, e.g., $AP = \{r, s\}$

Suitable alphabet Σ for Büchi automaton?

A state transition of Büchi automaton must represent an interpretation.

Choose Σ to be the set of all interpretations over AP, encoded as 2^{AP}.

(Recall slide 'Interpretations as Sets')

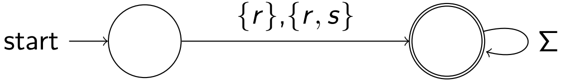
Example

$\Sigma = \big\{\emptyset, \{r\}, \{s\}, \{r, s\}\big\}$

Büchi Automaton for LTL Formula By Example

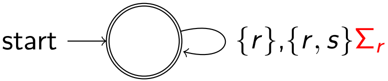
Example (Büchi automaton for formula r over $AP = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those traces τ satisfying r



In the first interpretation $\mathcal{I}_0$ (of τ), r must hold, the rest is arbitrary

Example (Büchi automaton for formula $\Box r$ over $AP = \{r, s\}$)

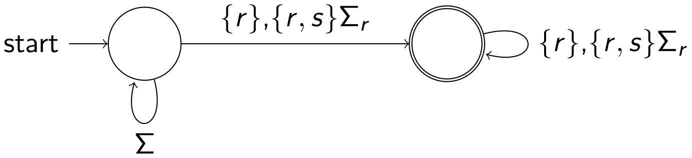


$$\Sigma_r := \{I | I \in \Sigma, r \in I\}$$

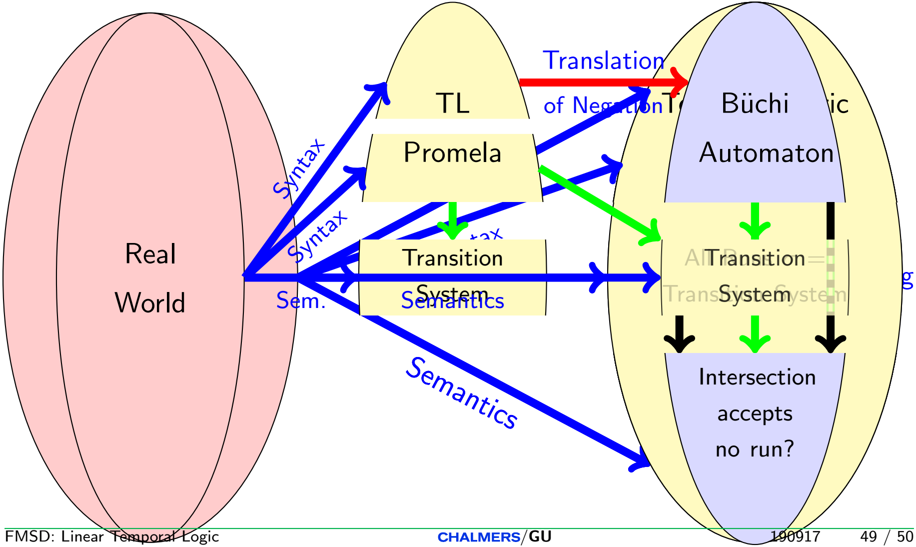
In *all* states $\mathcal{I}_i$ (of τ), r must hold

Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula $\Diamond\Box r$ over $AP = \{r, s\}$)



Revisit: FormalisationFormalisation: Syntax, SemanticsFormalisation: Syntax, Semantics, ProvingFormal Verification: Model Checking



Literature for this Lecture

Ben-Ari Section 5.2.1
(only syntax of LTL)

Baier and Katoen Principles of Model Checking,
May 2008, The MIT Press,
ISBN: 0-262-02649-X
(for in depth theory of model checking)