Advanced Algorithms Course.
Lecture Notes. Part 13

Extra Topic

Coloring the Edges of Bipartite Graphs

One motivation of maximum bipartite matchings was to find best assignments between two types of entities. Now we consider another assignment problem that leads to a certain coloring problem.

Assume that we have to arrange lectures for disjoint groups of students. Every lecture needs one time slot, and for every lecture we know the group and the teacher. How many slots are needed to schedule all lectues?

Clearly, a group or a teacher cannot have different lectues at the same time. We define a bipartite graph $G = (X, Y, E)$ whre the nodes in X and Y represent groups and teachers, respectively, and edges in E represent lectures. Furthermore we represent the slots by colors of edges. The task is now to color the edges such that no two edges of equal color share a node, or equivalently, the edges of each color form a matching. The number of colors shall be minimized.

A trivial lower bound is the maximum node degree Δ: Since the edges incident to any one node must get different colors, we need at least Δ colors. We will show that this trivial bound is tight: An edge coloring with Δ colors always exists, and we can even efficiently compute it.

The basic idea of this surprising result is to compute a matching M that covers all nodes of degree Δ. Then we can assign one color to the edges of M, and remove the edges of M from G. We are left with a bipartite graph of maximum degree $\Delta - 1$, and we iterate this procedure. The only crucial point is to show that M really exists, and to find it. (However, it would be a mistake to successively remove maximum matchings in a greedy fashion.)

First we construct "half of the solution": a matching that covers the set $S \subseteq X$ of all nodes of degree Δ in X. (For the moment we do not care about such nodes in Y.) To this end we compute a maximum matching, but in the bipartite graph containing only the nodes of S and Y. In the following we will prove that this maximum matching covers all of S. We will need some small but famous theorems that are of independent interest.

The first one is **König's theorem**: In a bipartite graph $G = (X, Y, E)$, a maximum matching M and a minimum vertex cover C have the same size.

The inequality $|C| \geq |M|$ holds in every graph, since the complement of any vertex cover is an independent set. In order to prove the nontrivial inequality $|C| \leq |M|$ we consider the network used for computing a mximum matching, with source s and sink t. A maximum matching M corresponds to a maximum flow in this network, hence there exists a minimum cut (A, B) of capacity $|M|$. The set A contain s and some nodes of $X \cup Y$. If an edge xy from A to B belongs to E, we may move y from B to A: Since yt is the only outgoing edge from y, this does not increase the cut capacity, that is, we retain a minimum cut. By iteration it follows the existence of a minimum cut (A, B) where no edge from A to B belongs to E. Now define $C = (B \cap X) \cup (A \cap Y)$. It is easy to check that C is a vertex cover in G, and $|C|$ equals the cut capacity.

The second prerequisite is Hall's theorem. Still, let $G = (X, Y, E)$ be a bipartite graph. For any $T \subseteq X$ we denote by $N(T) \subseteq Y$ the neighborhood of T, that is, the set of nodes in Y being adjacent to some node in T.

Hall's theorem says: If every $T \subseteq X$ satisfies $|N(T)| \geq |T|$, then G has a matching that covers X.

The necessity of the condition is obvious. In order to prove sufficiency, let C be some minimum vertex cover, and $T = X \setminus C$. Then we must have $N(T) \subseteq C$, to cover all edges incident with T. More precisely, we have $N(T) = C \cap Y$, since C is minimal. Hence $C = (X \setminus T) \cup N(T)$, and $|X| - |T| + |N(T)| \geq |X|$ holds by assumption. By König's theorem, G has a matching of size $|X| - |T| + |N(T)| \geq |X|$, which concludes the proof.

Back to our problem where $G = (X, Y, E)$ is bipartite graph and $S \subseteq X$ is the set of nodes of degree Δ: We observe that every $T \subseteq S$ satisfies $|N(T)| \geq |T|$, since the nodes in S have maximum degree. Thus we can apply Hall's theorem to the subgraph (S, Y, E) and conclude that G has a matching that covers S. Remember that we can compute it via a flow. Similarly, let $U \subseteq Y$ be set of nodes in Y with degree Δ. By symmetry, besides a matching M that covers S, we also get another matching N that covers U.

But what we actually want is *one* matching that covers $S \cup U$. This matching can be rather easily constructed in a final step: Note that the edge set $M \cup N$ is a union of disjoint paths and cycles where edges of M and N alternate. Roughly speaking, by taking every other edge in these paths and cycles, thereby removing unnecessary edges, we obtain the desired matching. In particular, any path of even length has an end node with degree smaller than Δ in the graph, and we need not take the edge at that end.

For edge coloring in general graphs we add two informations without proofs: $\Delta + 1$ colors are always enough (this is called Vizing's theorem), but the problem to decide whether the optimal number of colors is Δ or $\Delta + 1$ is NP-complete, even for graphs with $\Delta = 3$.