Discrete Optimization: Home Exam

Chalmers, Period 3, 2017 (TDA206/DIT370)
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Instructions:

- There are 35 points in total for this exam. For the overall grading of this class, please refer to the course website.
- You have until March 16, 2017, 10:00 am to finish this exam and upload it to the FIRE system, just as you did for the homework. Typed submissions and handwritten scans are both fine. Submissions must be legible after printing on A4 paper. All submissions must be in PDF format.
- Please start each problem on a new page (if you submit a LaTeX solution, the command for that is \newpage). Subproblems may be on the same page. For instance, (1a) and (1c) may be on the same page, but (1a) and (2c) must be on different pages.
- Include a cover sheet containing your name as the first page (you may use the page you are reading right now). Do NOT write any solutions on the cover sheet, it will not be considered for grading. Do NOT write your name or other identifying information on any other page.
- All work must be your own. You MAY use whatever tools and sources are available to you. However, you may NOT invoke the help of others, be it your classmates or people on the internet. For example, you MAY use existing answers on StackExchange.com to help you solve the problems, but you may NOT post exam questions there and ask for help. It is your responsibility to ensure that sources are reliable and information found there is correct, so use external sources at your own risk (Exception: Potential errors in the lecture notes or the suggested literature will not be counted against you, of course). Please cite your sources!

Question:	1	2	3	4	5	Total
Points:	6	4	6	13	6	35
Score:						

Name: ______________________________

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Do not write on this page, it will
not be considered for grading!

Question 1 [*6 points total*]

Consider the following LP:

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^3} & -5x_1 + 8x_2 + 4x_3 \\ \text{s.t.} \quad x_1 + x_2 &= 2 \\ x_2 - x_3 &\leq 3 \\ 2x_1 - x_3 &\geq -1 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \\ x_3 &\leq 0\end{aligned}$$

(a) [*1 pts*] Formulate the dual for this LP.

(b) [*2 pts*] Rewrite the primal LP so that constraints are in standard form $\mathbf{Ax} \geq \mathbf{b}$. Vectors may be extended to accommodate slack variables if necessary.

(c) [*2 pts*] If $\mathbf{x}$ is a feasible solution to the primal and the j-th dual constraint is binding, what, if anything, do we know about the primal variable x_j? Justify your answer.

(d) [*1 pts*] Write down the primal coefficient matrix for the original LP. Is this matrix totally unimodular, and why (not)?

Question 2 [*4 points total*]

You are given an NP-hard minimization problem for which the best known exact algorithm has exponential running time. You therefore attempt to solve it as an ILP using branch-and-bound. Your ILP has polynomially many constraints. To improve the lower bounds obtained by solving the LP relaxations, you use a cutting plane method. You want to avoid Chvátal-Gomory cuts, because you have found a formulation for problem-specific cutting planes you want to use.

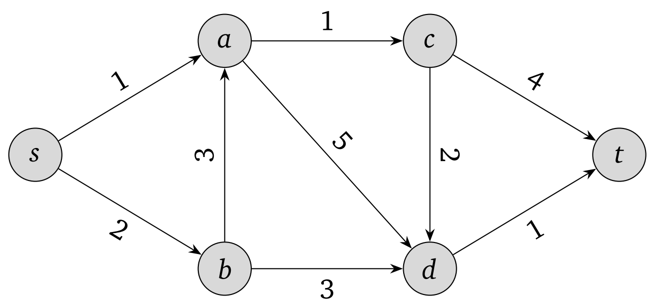
(a) [*1 pts*] What is the purpose and effect of improving lower bounds using cutting planes in this setting?

(b) [*2 pts*] Assume you were able to prove that, for any given instance of your problem, there are polynomially many of your cutting planes, and that adding them to its LP relaxation will create an integer polytope as its feasible region, allowing you to solve the ILP optimally using an LP solver. Assuming $P \neq NP$, what do we know about the separation problem in this case?

(c) [*1 pts*] Given your answer in (b), is the approach described above a good idea, and why (not)?

Question 3 [*6 points total*]

Consider the shortest path problem in a weighted, directed graph, i.e. given a (connected) directed graph $G = (V, E)$ with nonnegative cost c(e) for every edge $e \in E$, as well as vertices $s, t \in V$, we want to find a minimum-cost path from s to t in G.



(a) [*2 pts*] Give an integer linear programming formulation of the problem for the graph above. Clearly explain the variables and constraints you are using.

(b) [*2 pts*] Solve the LP relaxation of (a) using CVX. Given that solution, what is the shortest path?

(c) [*1 pts*] Write down the dual of (a).

(d) [*1 pts*] Typically, we provide a problem description and ask you to formulate it as an ILP. Now we ask the opposite: given the dual in (c), provide a description of the problem that the dual expresses.

Question 4 [*13 points total*]

Suppose you run a store that is open all day and night, all days of the week. You need to have a certain number of personnel in the store at any given time. You know the need of personnel for each 4-hour time slot, see table below:

Time slots	personnel
00:00 – 04:00	5
04:00 – 08:00	7
08:00 – 12:00	12
12:00 – 16:00	8
16:00 – 20:00	12
20:00 – 24:00	9

The union rules do not allow working shifts of 4 hours; if you call in personnel, they always work 8-hour shifts. You would like to minimize the total number of 8-hour shifts.

(a) [*2 pts*] Formulate the problem as an ILP. Clearly explain the variables and constraints you are using.

(b) [*1 pts*] Write down the dual of the LP relaxation. Explain the notation for the dual variables.

(c) [*2 pts*] Solve the LP-relaxation of the primal and the dual with CVX (include your code and the answer from CVX for objective function value and the variable vectors). What do you observe?

(d) [*2 pts*] Let $\mathbf{x}^*$, $\mathbf{y}^*$ denote primal and dual optimal solutions for the LPs of (c). Write down the complementary slackness conditions.

(e) [*3 pts*] Describe a primal-dual algorithm for the problem, in clear pseudocode.

(f) [*3 pts*] Solve the problem depicted above using your PDM, and describe what you do in each step.

Question 5 [*6 points total*]

Let $G = (V, E)$ be a complete, undirected graph with node set V and edge set E, such that each edge $e = (u, w)$ has a non-negative weight $c(u, w)$ between any two nodes u, w. We have seen in the lecture that if $c()$ is metric, i.e. obeys the triangle inequality

$$\forall u, v, w \in V : c(u, w) \leq c(u, v) + c(v, w),$$

then the MST heuristic yields a solution H which is a 2-approximation for the shortest round trip H^* in the Traveling Salesperson Problem. In some applications, there actually exists a stronger version of the triangle inequality:

$$\forall u, v, w \in V : c(u, w) \leq \max \left\{ c(u, v), c(v, w) \right\}.$$

In this case, the weight function is called *ultrametric*. Show that if the edge weights are ultrametric, the MST heuristic yields an approximate solution H with approximation factor

$$\frac{|V|\,\tilde{c}}{c(\text{MST})}$$

where $\tilde{c}$ is the cost of the heaviest edge in the MST, and $c(\text{MST})$ is the cost of the MST.