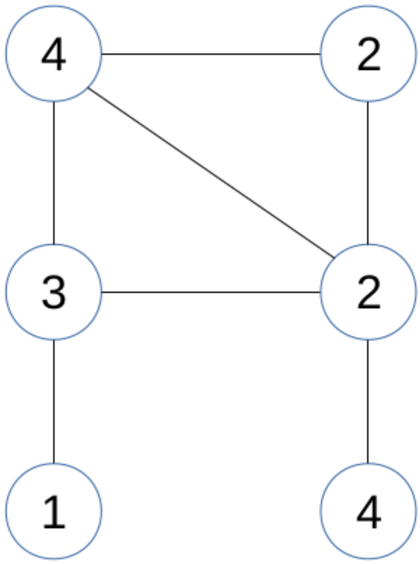
Homework 4

1. The Minimum Weight Vertex Cover Problem is defined as follows: Given an undirected graph $G = (V, E)$, with node set V and edge set E, where each node has a positive weight $w(v)$ associated with it (see figure), the goal is to select a subset $V' \subseteq V$ of nodes such that every edge has at least one node incident to it, and the total selected node weight $\sum_{v \in V'} w(v)$ is minimized.

   (a) (1 point) Formulate the LP relaxation of Minimum Weight Vertex Cover for the example below, and its dual.



> **Solution:** For each node $v \in V$, we introduce a variable $x_v \in \{0, 1\}$. To interpret a solution of the ILP, we take $V' := \{v \mid x_v = 1\}$
> Since $x_v \in \{0, 1\}$, the weight of such a solution is equal to $\sum_{v \in V} x_v.w(v)$, which is the cost function that should be minimized by the ILP problem.
> For each edge (u, v), we introduce a constraint $x_u + x_v \geq 1$. This constraint is respected if and only if V' includes at least one of u or v.
> We get the following ILP (nodes are labeled A tp F from left to right and from

top to bottom):

$$\begin{aligned}
\min \quad & 4x_A + 2x_B + 3x_C + 2x_D + x_E + 4x_F \\
\text{s.t.} \quad & x_A + x_B \geq 1 \\
& x_A + x_C \geq 1 \\
& x_A + x_D \geq 1 \\
& x_B + x_D \geq 1 \\
& x_C + x_D \geq 1 \\
& x_C + x_E \geq 1 \\
& x_D + x_F \geq 1 \\
& \vec{x} \geq \vec{0} \\
& \vec{x} \leq \vec{1} \\
& \vec{x} \in \mathbb{Z}^6
\end{aligned}$$

The relaxation of this ILP can be obtained by omitting the constraint $\vec{x} \in \mathbb{Z}^6$. We can further remove the constraint $\vec{x} \leq \vec{1}$ because no *optimal* solution can contain a variable set to a value greater than 1 (assume you had such an optimal solution s^* with a value $s_i^* > 1$, by setting s_i^* to 1, you would get another feasible solution with a smaller value, contradicting the optimality of s^*).

The dual of this relaxed LP is the following problem:

$$\begin{aligned}
\max \quad & y_{AB} + y_{AC} + y_{AD} + y_{BD} + y_{CD} + y_{CE} + y_{DF} \\
\text{s.t.} \quad & y_{AB} + y_{AC} + y_{AD} \leq 4 \\
& y_{AB} + y_{BD} \leq 2 \\
& y_{AC} + y_{CD} + y_{CE} \leq 3 \\
& y_{AD} + y_{BD} + y_{CD} + y_{DF} \leq 2 \\
& y_{CE} \leq 1 \\
& y_{DF} \leq 4 \\
& \vec{y} \geq \vec{0}
\end{aligned}$$

(b) (0.5 points) Find a feasible dual solution such that none of the index sets describing complementary slackness conditions (i.e. I, I^C, J, and J^C) are empty.

Solution: Let us consider the solution y:

$$(y_{AB} = 0, y_{AC} = 0, y_{AD} = 0, y_{BD} = 0, y_{CD} = 0, y_{CE} = 1, y_{DF} = 0)$$

We can easily check that this solution is feasible, with only the fifth dual constraint binding.

For this solution we have the following index sets:

$$
\begin{aligned}
I &= \{i \mid \ y_i = 0\} = \{1,2,3,4,5,7\} \\
I^C &= \{6\} \\
J &= \{j \mid \sum_i a_{ij} y_i = c_j\} = \{5\} \\
J^C &= \{1,2,3,4,6\}
\end{aligned}
$$

(c) (1 point) Formulate the restricted primal for your dual solution.

Solution: According to the definition and given the sets I^C and J^C, we obtain the following restricted primal:

$$
\begin{aligned}
\min \quad & s_6 + x_A + x_B + x_C + x_D + x_F \\
\text{s.t.} \quad & x_A + x_B \geq 1 \\
& x_A + x_C \geq 1 \\
& x_A + x_D \geq 1 \\
& x_B + x_D \geq 1 \\
& x_C + x_D \geq 1 \\
& x_D + x_F \geq 1 \\
& x_C + x_E - s_5 = 1 \\
& \vec{s} \geq \vec{0} \\
& \vec{x} \geq \vec{0}
\end{aligned}
$$

(d) (0.5 points) Solve your restricted primal using CVX (submit your code and solutions)

Solution: An optimal solution of this restricted primal is:

$$(s_6 = 0, x_A = 1, x_B = 0, x_C = 0, x_D = 1, x_E = 1, x_F = 0)$$

The value of this solution is 2. Since, it is non-zero, we cannot use this solution to obtain an optimal solution for the relaxed LP.