Homework 2

1. Consider the following LP problem:

$$
\begin{array}{rrcl}
\max & 4x_1 - 2x_2 + 5x_3 + 6x_4 + 7x_5 & & \\
\text{s.t.} & 2x_1 + 2x_2 - 4x_3 + 4x_4 + 8x_5 & \leq & 6 \\
& 2x_1 + \ \ x_2 - 2x_3 - \ \ x_4 - 3x_4 & \geq & -1 \\
& 5x_1 - 2x_2 + 4x_3 + 4x_4 + 2x_5 & = & 5 \\
& 2x_1 - 2x_2 + 5x_3 + 3x_4 + \ \ x_5 & \leq & 4 \\
& \vec{x} & \geq & \vec{0}
\end{array}
$$

(a) (0.5 points) Write this LP in canonical form.

Solution: The second constraint is not in the right form, but we can replace it by the equivalent constraint:

$$-2x_1 - x_2 + 2x_3 + x_4 + 3x_4 \leq 1$$

The third constraint is not in the right form, but it is equivalent to the following two constraints:

$$5x_1 - 2x_2 + 4x_3 + 4x_4 + 2x_5 \leq 5$$

and

$$-5x_1 + 2x_2 - 4x_3 - 4x_4 - 2x_5 \leq -5$$

Therefore, here is the canonical form of the problem:

$$
\begin{array}{rrcl}
\max & 4x_1 - 2x_2 + 5x_3 + 6x_4 + 7x_5 & & \\
\text{s.t.} & 2x_1 + 2x_2 - 4x_3 + 4x_4 + 8x_5 & \leq & 6 \\
& -2x_1 - \ \ x_2 + 2x_3 + \ \ x_4 + 3x_4 & \leq & 1 \\
& 5x_1 - 2x_2 + 4x_3 + 4x_4 + 2x_5 & \leq & 5 \\
& -5x_1 + 2x_2 - 4x_3 - 4x_4 - 2x_5 & \leq & -5 \\
& 2x_1 - 2x_2 + 5x_3 + 3x_4 + \ \ x_5 & \leq & 4 \\
& \vec{x} & \geq & \vec{0}
\end{array}
$$

(b) (0.5 points) Write down the dual of this LP directly, without using the canonical form.

Solution: For each constraint in the primal, we introduce a dual variable y_j. We use the coefficients of the primal and the translation scheme given in the lecture notes to derive the constraints of the dual:

$$
\begin{aligned}
\min \quad & 6y_1 - y_2 + 5y_3 + 4y_4 \\
\text{s.t.} \quad & 2y_1 + 2y_2 + 5y_3 + 2y_4 \geq 4 \\
& 2y_1 + y_2 - 2y_3 - 2y_4 \geq -2 \\
& -4y_1 - 2y_2 + 4y_3 + 5y_4 \geq 5 \\
& 4y_1 - y_2 + 4y_3 + 3y_4 \geq 6 \\
& 8y_1 - 3y_2 + 2y_3 + y_4 \geq 7 \\
& y_1 \geq 0 \\
& y_2 \leq 0 \\
& y_4 \geq 0
\end{aligned}
$$

2. (a) (0.5 points) Use CVX to solve the primal problem above. Submit your code (as MATLAB script), and write down the solution vector and objective value.

Solution: An optimal solution x^* is

$$(x_1^* = \frac{37}{59}, x_2^* = \frac{166}{59}, x_3^* = \frac{91}{59}, x_4^* = 0, x_5^* = \frac{39}{59})$$

Its value is $\frac{544}{59} \approx 9.22$

(b) (0.5 points) Do the same for the dual.

Solution: An optimal solution y^* to the dual problem is

$$(y_1^* = \frac{31}{59}, y_2^* = -\frac{22}{59}, y_3^* = \frac{20}{59}, y_4^* = 1)$$

Its value is $\frac{544}{59} \approx 9.22$

(c) (0.5 points) What do you notice when comparing the solutions for primal and dual? Give an explanation.

Solution: Both optimal solutions have the same value, in other words the duality gap is 0. This is simply a consequence of the strong duality of LP.

(d) (0.5 points) Check whether all complementary slackness conditions hold between primal and dual solution.

Solution: The complementary slackness conditions for solutions x^* and y^* are as follows:

- *If the i-th primal constraint has slack in x^*, the i-th variable of y^* is 0.*

  Let us compute the binding constraints in x^*:

$$\begin{aligned}
2x_1^* + 2x_2^* - 4x_3^* + 4x_4^* + 8x_5^* &= 6 \\
2x_1^* + x_2^* - 2x_3^* - x_4^* - 3x_5^* &= -1 \\
5x_1^* - 2x_2^* + 4x_3^* + 4x_4^* + 2x_5^* &= 5 \\
2x_1^* - 2x_2^* + 5x_3^* + 3x_4^* + x_5^* &= 4
\end{aligned}$$

  All the constraints in the primal are binding under x^*, therefore this condition is true.

- *If the j-th dual constraint has slack in y^*, the j-th variable of x^* is 0.*

  Let us compute the binding constraints in y^*:

$$\begin{aligned}
2y_1^* + 2y_2^* + 5y_3^* + 2y_4^* &= 4 \\
2y_1^* + y_2^* - 2y_3^* - 2y_4^* &= -2 \\
-4y_1^* - 2y_2^* + 4y_3^* + 5y_4^* &= 5 \\
4y_1 - y_2 + 4y_3 + 3y_4 &= \frac{403}{59} > 6 \\
8y_1 - 3y_2 + 2y_3 + y_4 &= 7
\end{aligned}$$

  Only the fourth dual constraint has slack. Since $x_4^* = 0$, this condition is true.

- *If the j-th variable of x^* is non-zero, the j-th dual constraint is binding in y^*.*

  x_1^*, x_2^* and x_4^* are non-zero. Correspondingly, the first, second and fourth dual constraints are binding under y^*, therefore the condition is true.

- *If the i-th variable of y^* is non-zero, the i-th primal constraint is binding in x^*.*

  All variable of y^* are non-zero. Correspondingly, all the primal constraints are binding under x^*, therefore the condition is true.

All complementary slackness conditions are respected for x^* and y^*, further confirming that they are optimal solutions of the primal and dual.

Deadline is Wednesday, 2018-01-31 10:00.