Sample problem

A student wants to know how much food from Mcdonald's he/she should eat on a daily basis in order to spend the least amount of money but maintaining at least the minimum recommended level of nutrition.

The student has the option to buy some combination of Big Mac, Caesar salad, French fries, Soda and Orange juice. The nutritional content and prices of each of them is given by the following table:

Food	Calories (kcal)	Carbs (g)	Protein (g)	Price (kr)
Big Mac	550	42	26	41
Caesar Salad	350	26	23	50
French fries	380	48	4	20
Soda	210	58	0	15
Orange juice	150	12	2	15
Recommended	2,000	130	56	

1. Formulate this problem as an ILP

Solution:

If we denote by B the number of units of Big Mac we decide to buy, C the number of Caesar Salad, F number of (boxes of) french fries, S glasses of soda and J orange juice boxes. Then we can write the daily cost of food as

$$\text{cost}(B, C, F, S, J) = 41 * B + 50 * C + 20 * F + 15 * S + 15 * J$$

In the interest of being compact we can extract the price information into a single cost vector $c = [41, 50, 20, 15, 15]$ and likewise write our variables as a vector $x = [B, C, F, S, J]$. Now the cost function is just the **dot** product of the—transposed—cost vector and the variable vector.

$$\text{cost} = c^T x$$

Now, for each type of food we get a specific amount of each nutrient, and the sum of those nutrients needs to be bigger than the minimum recommendation. Hence, we can write out a table of the constraints by looking at the columns of our table above.

$$
\begin{aligned}
550 * B + 350 * C + 380 * F + 210 * S + 150 * J &\geq 2,000 \qquad (\textit{Calories}) \\
42 * B + 26 * C + 48 * F + 58 * S + 12 * J &\geq 130 \qquad (\textit{Carbs}) \\
26 * B + 23 * C + 4 * F + 0 * S + 2 * J &\geq 56 \qquad (\textit{Protein})
\end{aligned}
$$

In the same way that we extracted the cost data into a vector to separate it from the variables, we can extract all of the nutrient data into a matrix A, and the recommended minimums into a vector b.

$$
A = \begin{bmatrix} 550 & 350 & 380 & 210 & 150 \\ 42 & 26 & 48 & 58 & 12 \\ 26 & 23 & 4 & 0 & 2 \end{bmatrix}
$$

$$
b = \begin{bmatrix} 2,000 \\ 130 \\ 56 \end{bmatrix}
$$

And now the constraint is that $Ax \geq b$, where $\geq$ means "greater or equal in every coordinate". So we can write down the more general form of the problem for our specific matrices and vectors. That is, to minimize the cost $c^T x$ subject to the constraint $Ax \geq b$ where $x \in \mathbb{Z}^5$ and $x \geq \vec{0}$

$$
\begin{aligned}
\min \quad & z = c^T x \\
\text{s.t.} \quad & Ax \geq b \\
& x \in \mathbb{Z}^5 \geq \vec{0}
\end{aligned}
$$

2. Solve it using CVX (jointly with a solver that allows integer optimization e.g., mosek).

Solution:

Optimal solution:

$$\{B = 2,\ C = 0,\ F = 2,\ S = 1,\ J = 0\}$$

Optimal value: 137 kr

The relaxed problem has as optimal solution

$$\{B = 1.7,\ C = 0,\ F = 2.7,\ S = 0,\ J = 0\}$$

An optimal value: 126.104 kr

3. Formulate and describe the variables and constraints of the dual ILP of this problem

Solution: