On the definition of the $\hat{\delta}$ function

We write $q.a$ instead of $\delta(q,a)$ and $q.x$ instead of $\hat{\delta}(q,x)$.

The text book presents the following recursive definition

$$(1) \qquad q.\epsilon = q \qquad q.(xa) = \delta(q.x, a)$$

We presented the definition

$$(2) \qquad q.\epsilon = q \qquad q.(ax) = \delta(q,a).x$$

The goal of this note is to explain why the definitions are equivalent.

Analysis of the definition (1)

We are going to prove from (1)

$$q.(xy) = (q.x).y$$

by *induction on* y. We write $\psi(y)$ for $q.(xy) = (q.x).y$ and we prove

$$\frac{}{\psi(\epsilon)} \qquad \frac{\psi(y)}{\psi(ya)}$$

Indeed $\psi(\epsilon)$ is

$$q.(x\epsilon) = q.x = (q.x).\epsilon$$

and if we assume $\psi(y)$ that is $q.(xy) = (q.x).y$ we have

$$q.(x(ya)) = q.((xy)a) = (q.(xy)).a = ((q.x).y).a = (q.x).(ya)$$

which is $\psi(ya)$.

Analysis of the definition (2)

We are going to prove from (1)

$$\forall q. \quad q.(xy) = (q.x).y$$

by *induction on* x. We write $\phi(x)$ for $\forall q.\ \ q.(xy) = (q.x).y$ and we prove

$$\frac{}{\psi(\epsilon)} \qquad \frac{\phi(x)}{\phi(ax)}$$

Indeed $\psi(\epsilon)$ is

$$\forall q. \quad q.(x\epsilon) = q.x = (q.x).\epsilon$$

and if we assume $\phi(x)$ that is $\forall q.\ \ q.(xy) = (q.x).y$ we have

$$\forall q. \quad q.((ax)y) = q.(a.(xy)) = (q.a).(xy) = ((q.a).x).y = (q.(ax)).y$$

which is $\phi(ax)$.

Notice that in this proof using the definition (2) we need to *quantify over all states*

Conclusion

We have show that the definition (1) satisfies $q.\epsilon = q$ and $q.(xy) = (q.x).y$. In particular it satisfies $q.(ax) = (q.a).x$.

Thus if we consider the two functions $f_1(q, x) = q.x$ following the recursion (1) and $f_2(q, x) = q.x$ following the recursion (2), both functions satisfy the same recursive equations

$$f_1(q, \epsilon) = f_2(q, \epsilon) = q \qquad f_1(q, ax) = f_1(q.a, x) \qquad f_2(q, ax) = f_2(q.a, x)$$

For f_2 it is by definition, and for f_1 it is by what we have just proved.

It follows from this that we have by induction on x

$$f_1(q, x) = f_2(q, x)$$

so the two definitions are equivalent.