Formal Methods for Software Development

Temporal Model Checking (part 2) + First-Order Logic

Wolfgang Ahrendt

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Part I

Finishing Temporal Model Checking

Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system $\mathcal{T}$ (e.g., derived from a PROMELA program).

Verification task: is the LTL formula ϕ satisfied in all traces of $\mathcal{T}$, i.e.,

$$\mathcal{T} \models \phi \quad ?$$

LTL Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Construct generalised Büchi automaton $\mathcal{GB}_{\neg\phi}$ for negation of ϕ
2. Construct an equivalent normal Büchi automaton $\mathcal{B}_{\neg\phi}$, i.e.,

$$\mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) = \mathcal{L}^\omega(\mathcal{GB}_{\neg\phi})$$

3. Construct product $\mathcal{T} \otimes \mathcal{B}_{\neg\phi}$
4. Analyse whether $\mathcal{T} \otimes \mathcal{B}_{\neg\phi}$ has a

   path π looping through an 'accepting node'

5. If such a π is found, then

   $$\mathcal{T} \not\models \phi$$

   and

   σ_π is a counter example.

   If no such π is found, then

   $$\mathcal{T} \models \phi$$

What Remains?

last lecture

3.–5. product of transition system and Büchi automaton (construction and analysis)

this lecture

2. generalised Büchi automata and their normalisation
1. translating LTL into generalised Büchi automata

Generalised Büchi Automata $\mathcal{GB}$ and Translation to (normal) Büchi Automata $\mathcal{B}$

Generalised Büchi Automata

A generalised Büchi automaton is defined as:

$$\mathcal{GB} = (Q, \delta, Q_0, \mathcal{F})$$

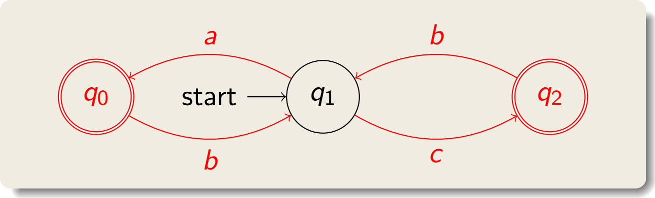
Q, δ, Q_0 as for standard Büchi automata

$\mathcal{F} = \{F_1, \ldots, F_k\}$ is a set of sets of accepting locations
($F_i = \{f_{i1}, \ldots, f_{im_i}\} \subseteq Q$)

Definition (Acceptance for generalised Büchi automata)

A generalised Büchi automaton accepts an ω-word $w \in \Sigma^\omega$ iff
for every $i \in \{1, \ldots, k\}$ at least one $q \in F_i$ is visited infinitely often.

Generalised vs. Normal Büchi Automata: Example



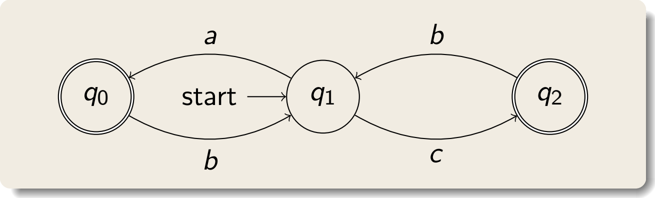
$\mathcal{GB}$ with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$ different from normal $\mathcal{B}$ with $F = \{q_0, q_2\}$

Are the following ω-words accepted?

ω-word	$\mathcal{B}$	$\mathcal{GB}$
$(ab)^\omega$	✔	✘
$(abcb)^\omega$	✔	✔

Translate Generalised to Normal Büchi Automata

$$\mathcal{GB} \text{ with } \mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}:$$

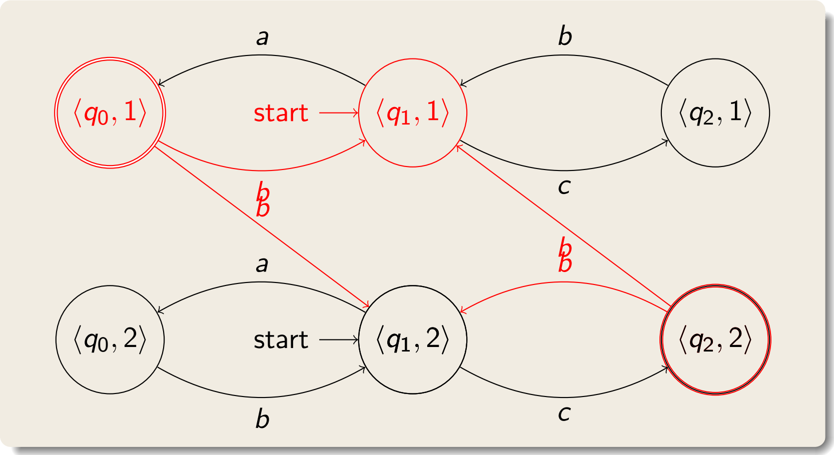


Construct $\mathcal{B}$ (different from last slide) which accepts the same words:

$$\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{GB})$$

Translate Generalised to Normal Büchi Automata

Construct $\mathcal{B}$ for $\mathcal{GB}$ with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



One clone for each $F_i \in \mathcal{F}$ Every transition from "F_1" is redirected to "clone 2" Every transition from "F_2" is redirected to

Translate Generalised to Normal Büchi Automata (formal)

Given generalised Büchi automaton
$\mathcal{GB} = (Q, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$

Equivalent normal Büchi automaton
$\mathcal{B} = (Q', \delta', Q'_0, F')$ with

- $Q' = Q \times \{1, ..., k\}$
- $\delta'(\langle q, i \rangle, \sigma) = \begin{cases} \{\langle q', i \rangle \mid q' \in \delta(q, \sigma)\} & \text{if } q \notin F_i \\ \{\langle q', (i \bmod k) + 1 \rangle \mid q' \in \delta(q, \sigma)\} & \text{if } q \in F_i \end{cases}$
- $Q'_0 = \{\langle q, 1 \rangle | q \in Q_0\}$
- $F' = \{\langle q, 1 \rangle | q \in F_1\}$

Construction of a Generalised Büchi Automaton $\mathcal{GB}_\phi$ for an LTL-Formula ϕ

Focus on $\Box$-free and $\Diamond$-free LTL

- Following construction assumes formulas without $\Box$ and $\Diamond$.
- Only temporal modality is $\mathcal{U}$.
- $\Box$ can be removed using

$$\Box\phi \quad \equiv \quad \neg\Diamond\neg\phi$$

- $\Diamond$ can be removed using

$$\Diamond\phi \quad \equiv \quad \mathrm{true}\,\mathcal{U}\phi$$

Theory and Example at Once

We introduce the general consruction togher with example.

Task:

construct

$$\mathcal{GB}_\phi$$

for

$$\phi \equiv r\,\mathcal{U}s$$

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

$$FL(\phi) = \{\varphi \mid \varphi \text{ is subformula or negated subformula of } \phi\}$$

($\neg\neg\varphi$ is identified with φ)

Example

$$FL(r\,\mathcal{U}s) = \{r, \neg r, s, \neg s, r\,\mathcal{U}s, \neg(r\,\mathcal{U}s)\}$$

$\mathcal{GB}_\phi$-Construction: Locations

Locations of $\mathcal{GB}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total
- $\psi \in FL(\phi)$ then exactly one of ψ and $\neg\psi$ in q

Downward Closed
- $\psi_1 \wedge \psi_2 \in q$ then $\psi_1 \in q$ and $\psi_2 \in q$
- $\psi_1 \vee \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$
- $\psi_1 \rightarrow \psi_2 \in q$ then $\psi_1 \notin q$ or $\psi_2 \in q$

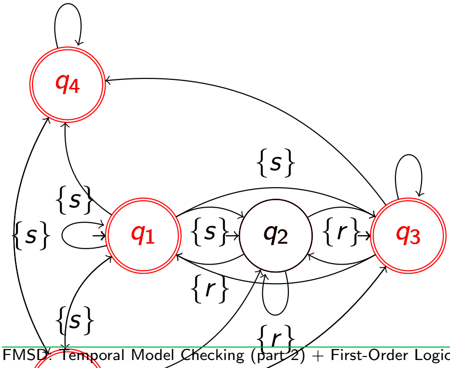
Until Consistent
- $\psi_2 \in q$ then $\psi_1 \mathcal{U} \psi_2 \in q$
- $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \in q$

$$FL(r\mathcal{U}s) = \{r, \neg r, s, \neg s, r\mathcal{U}s, \neg(r\mathcal{U}s)\}$$

	$\in Q$
$\{r\mathcal{U}s, \neg r, s\}$	✔
$\{r\mathcal{U}s, \neg r, \neg s\}$	✘
$\{\neg(r\mathcal{U}s), r, s\}$	✘
$\{\neg(r\mathcal{U}s), r, \neg s\}$	✔

$\mathcal{GB}_\phi$-Construction: Transitions

$$\underbrace{\{r\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



Transitions $(q, \alpha, q') \in \delta_\phi$:

$\alpha = q \cap AP$

AP set of propositional variables outgoing edges of q_1 labeled $\{s\}$, of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \mathcal{U} \psi_2 \in q'$
2. If $\psi_1 \mathcal{U} \psi_2 \in q'$ and $\psi_1 \in q$ then $\psi_1 \mathcal{U} \psi_2 \in q$

Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

Accepting locations

Remarks on Generalized Büchi Automata

- Construction always gives exponential number of states in $|\phi|$
- Satisfiability checking of LTL is PSPACE-complete
- There exist (more complex) constructions that minimize number of required states
  - One of these is used in Spin, which moreover computes the states lazily

Part II

Starting First-order Logic

Motivation for Introducing First-Order Logic

1) We specify JAVA programs with Java Modeling Language (JML)

JML combines

- JAVA expressions
- First-Order Logic (FOL)

2) We verify JAVA programs using Dynamic Logic

Dynamic Logic combines

- First-Order Logic (FOL)
- JAVA programs

FOL: Language and Calculus

We introduce:

- FOL as a language
- Sequent calculus for proving FOL formulas
- KeY system as propositional, and first-order, prover (for now)
- Formal semantics

First-Order Logic: Signature

Signature

A first-order signature Σ consists of

- a set T_Σ of types
- a set F_Σ of function symbols
- a set P_Σ of predicate symbols
- a typing α_Σ

Intuitively, the typing α_Σ determines

- for each function and predicate symbol:
  - its arity, i.e., number of arguments
  - its argument types
- for each function symbol its result type.

Formally:

- $\alpha_\Sigma(p) \in {T_\Sigma}^*$ for all $p \in P_\Sigma$ (arity of p is $|\alpha_\Sigma(p)|$)
- $\alpha_\Sigma(f) \in {T_\Sigma}^* \times T_\Sigma$ for all $f \in F_\Sigma$ (arity of f is $|\alpha_\Sigma(f)| - 1$)

Example Signature Σ_1 + Constants

$T_{\Sigma_1} = \{\mathtt{int}\}$,
$F_{\Sigma_1} = \{\mathtt{+}, \mathtt{-}\} \cup \{\ldots, \mathtt{-2}, \mathtt{-1}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots\}$,
$P_{\Sigma_1} = \{\mathtt{<}\}$

$\alpha_{\Sigma_1}(\mathtt{<}) = (\mathtt{int},\mathtt{int})$
$\alpha_{\Sigma_1}(\mathtt{+}) = \alpha_{\Sigma_1}(\mathtt{-}) = (\mathtt{int},\mathtt{int},\mathtt{int})$
$\alpha_{\Sigma_1}(\mathtt{0}) = \alpha_{\Sigma_1}(\mathtt{1}) = \alpha_{\Sigma_1}(\mathtt{-1}) = \ldots = (\mathtt{int})$

Constant Symbols

A function symbol `f` with $|\alpha_{\Sigma_1}(f)| = 1$ (i.e., with arity 0) is called *constant symbol*.

Here, the constant symbols are: $\ldots, \mathtt{-2}, \mathtt{-1}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots$

Syntax of First-Order Logic: Signature Cont'd

Type declaration of signature symbols

- Write $\tau\ x$; to declare variable x of type τ
- Write $p(\tau_1, \ldots, \tau_r)$; for $\alpha(p) = (\tau_1, \ldots, \tau_r)$
- Write $\tau\ f(\tau_1, \ldots, \tau_r)$; for $\alpha(f) = (\tau_1, \ldots, \tau_r, \tau)$

$r = 0$ is allowed, then write f instead of $f()$.

Example

Variables	`integerArray a; int i;`
Predicate Symbols	`isEmpty(List); alertOn;`
Function Symbols	`int arrayLookup(int); Object o;`

Example Signature Σ_1 + Notation

Typing of Signature:

$\alpha_{\Sigma_1}(\texttt{<}) = (\texttt{int},\texttt{int})$
$\alpha_{\Sigma_1}(\texttt{+}) = \alpha_{\Sigma_1}(\texttt{-}) = (\texttt{int},\texttt{int},\texttt{int})$
$\alpha_{\Sigma_1}(\texttt{0}) = \alpha_{\Sigma_1}(\texttt{1}) = \alpha_{\Sigma_1}(\texttt{-1}) = \ldots = (\texttt{int})$

can alternatively be written as:

```
<(int,int);
int +(int,int);
int 0;  int 1;  int -1;  ...
```

First-Order Terms

We assume a set V of variables ($V \cap (F_\Sigma \cup P_\Sigma) = \emptyset$).
Each $v \in V$ has a unique type $\alpha_\Sigma(v) \in T_\Sigma$.

Terms are defined recursively:

Terms

A first-order term of type $\tau \in T_\Sigma$

- is either a variable of type τ, or
- has the form $f(t_1, \ldots, t_n)$,
  where $f \in F_\Sigma$ has result type τ, and each t_i is term of the correct type, following the typing α_Σ of f.

If f is a constant symbol, the term is written f, instead of $f()$.

Terms over Signature Σ_1

Example terms over Σ_1:
(assume variables `int` v_1; `int` v_2;)

- `-7`
- `+(-2, 99)`
- `-(7, 8)`
- `+(-(7, 8), 1)`
- $+(-(v_1, 8), v_2)$

Our variant of FOL allows infix notation for common functions:

- `-2 + 99`
- `7 - 8`
- `(7 - 8) + 1`
- $(v_1 - 8) + v_2$

Atomic Formulas

Atomic Formulas

Given a signature Σ.
An atomic formula has either of the forms

- *true*
- *false*
- $t_1 = t_2$ *("equality")*,
  where t_1 and t_2 are first-order terms of the same type.
- $p(t_1, \ldots, t_n)$ *("predicate")*,
  where $p \in P_\Sigma$, and each t_i is term of the correct type, following the typing α_Σ of p.

Atomic Formulas over Signature Σ_1

Example formulas over Σ_1:
(assume variable `int` *v*;)

- $7 = 8$
- `<(7, 8)`
- `<(-2 - v, 99)`
- `<(v, v + 1)`

Our variant of FOL allows infix notation for common predicates:

- `7 < 8`
- `-2 - v < 99`
- `v < v + 1`

First-Order Formulas

Formulas

- each atomic formula is a formula
- with ϕ and ψ formulas, x a variable, and τ a type, the following are also formulas:
  - $\neg\phi$ (*"not ϕ"*)
  - $\phi \wedge \psi$ (*"ϕ and ψ"*)
  - $\phi \vee \psi$ (*"ϕ or ψ"*)
  - $\phi \rightarrow \psi$ (*"ϕ implies ψ"*)
  - $\phi \leftrightarrow \psi$ (*"ϕ is equivalent to ψ"*)
  - $\forall\ \tau\ x;\ \phi$ (*"for all x of type τ holds ϕ"*)
  - $\exists\ \tau\ x;\ \phi$ (*"there exists an x of type τ such that ϕ"*)

In $\forall\ \tau\ x;\ \phi$ and $\exists\ \tau\ x;\ \phi$ the variable x is 'bound' (i.e., 'not free').

Formulas with no free variable are 'closed'.

First-order Formulas: Examples

(signatures/types left out here)

Example (There are at least two elements)

$\exists x, y; \neg(x = y)$

Example (Strict partial order)

Irreflexivity	$\forall x; \neg(x < x)$
Asymmetry	$\forall x; \forall y; (x < y \rightarrow \neg(y < x))$
Transitivity	$\forall x; \forall y; \forall z;$ $(x < y \wedge y < z \rightarrow x < z)$

(Is any of the three formulas redundant?)

Semantics (briefly here, more thorough later)

Domain

A domain $\mathcal{D}$ is a set of elements which are (potentially) the *meaning* of terms and variables.

Interpretation

An interpretation $\mathcal{I}$ (over $\mathcal{D}$) assigns *meaning* to the symbols in $F_\Sigma \cup P_\Sigma$ (assigning functions to function symbols, relations to predicate symbols).

Valuation

In a given $\mathcal{D}$ and $\mathcal{I}$, a closed formula evaluates to either T or F.

Validity

A closed formula is valid if it evaluates to T in all $\mathcal{D}$ and $\mathcal{I}$.

In the context of specification/verification of programs:
each $(\mathcal{D}, \mathcal{I})$ is called a 'state'.

Useful Valid Formulas

Let ϕ and ψ be arbitrary, closed formulas (whether valid or not).

The following formulas are valid:

- $\neg(\phi \land \psi) \leftrightarrow \neg\phi \lor \neg\psi$
- $\neg(\phi \lor \psi) \leftrightarrow \neg\phi \land \neg\psi$
- $(\mathit{true} \land \phi) \leftrightarrow \phi$
- $(\mathit{false} \lor \phi) \leftrightarrow \phi$
- $\mathit{true} \lor \phi$
- $\neg(\mathit{false} \land \phi)$
- $(\phi \rightarrow \psi) \leftrightarrow (\neg\phi \lor \psi)$
- $\phi \rightarrow \mathit{true}$
- $\mathit{false} \rightarrow \phi$
- $(\mathit{true} \rightarrow \phi) \leftrightarrow \phi$
- $(\phi \rightarrow \mathit{false}) \leftrightarrow \neg\phi$

Useful Valid Formulas

Assume that x is the only variable which may appear freely in ϕ or ψ.

The following formulas are valid:

- $\neg(\exists\ \tau\ x;\ \phi) \leftrightarrow \forall\ \tau\ x;\ \neg\phi$
- $\neg(\forall\ \tau\ x;\ \phi) \leftrightarrow \exists\ \tau\ x;\ \neg\phi$
- $(\forall\ \tau\ x;\ (\phi \land \psi)) \leftrightarrow (\forall\ \tau\ x;\ \phi) \land (\forall\ \tau\ x;\ \psi)$
- $(\exists\ \tau\ x;\ (\phi \lor \psi)) \leftrightarrow (\exists\ \tau\ x;\ \phi) \lor (\exists\ \tau\ x;\ \psi)$

Are the following formulas also valid?

- $(\forall\ \tau\ x;\ (\phi \lor \psi)) \leftrightarrow (\forall\ \tau\ x;\ \phi) \lor (\forall\ \tau\ x;\ \psi)$
- $(\exists\ \tau\ x;\ (\phi \land \psi)) \leftrightarrow (\exists\ \tau\ x;\ \phi) \land (\exists\ \tau\ x;\ \psi)$

Remark on Concrete Syntax

	Text book	Spin	KeY
Negation	$\neg$	!	!
Conjunction	$\wedge$	&&	&
Disjunction	$\vee$	||		
Implication	$\rightarrow$, $\supset$	->	->
Equivalence	$\leftrightarrow$	<->	<->
Universal Quantifier	$\forall\, x;\ \phi$	n/a	`\forall` $\tau\ x;\ \phi$
Existential Quantifier	$\exists\, x;\ \phi$	n/a	`\exists` $\tau\ x;\ \phi$
Value equality	$=$	`==`	`=`

Motivation for a Sequent Calculus

How to show a formula valid in propositional logic?
$\rightarrow$ use a semantic truth table.

How about FOL? Formula: $\texttt{isEven}(x) \lor \texttt{isOdd}(x)$

x	$\texttt{isEven}(x)$	$\texttt{isOdd}(x)$	$\texttt{isEven}(x) \lor \texttt{isOdd}(x)$
1	F	T	T
2	T	F	T
...	...	...	...

Checking validity via **semantics** does not work.

Instead...

Reasoning by Syntactic Transformation

Prove validity of ϕ by **syntactic** transformation of ϕ

Logic Calculus: Sequent Calculus based on notion of sequent:

$$\underbrace{\psi_1, \ldots, \psi_m}_{\text{antecedent}} \quad \Longrightarrow \quad \underbrace{\phi_1, \ldots, \phi_n}_{\text{succedent}}$$

has same meaning as

$$(\psi_1 \land \cdots \land \psi_m) \quad \rightarrow \quad (\phi_1 \lor \cdots \lor \phi_n)$$

which has (for closed formulas ψ_i, ϕ_i) same meaning as

$$\{\psi_1, \ldots, \psi_m\} \quad \models \quad \phi_1 \lor \cdots \lor \phi_n$$

Notation for Sequents

$$\psi_1, \ldots, \psi_m \quad \Longrightarrow \quad \phi_1, \ldots, \phi_n$$

Consider antecedent/succedent as sets of formulas, may be empty

Schema Variables

$\phi, \psi, \ldots$ match formulas, $\Gamma, \Delta, \ldots$ match sets of formulas
Characterize infinitely many sequents with single schematic sequent, e.g.,

$$\Gamma \quad \Longrightarrow \quad \phi \wedge \psi, \Delta$$

matches any sequent with occurrence of conjunction in succedent

Here, we call $\phi \wedge \psi$ main formula and Γ, Δ side formulas of sequent

Sequent Calculus Rules

Write syntactic transformation schema for sequents that reflects semantics of connectives

$$\text{RuleName}\ \frac{\overbrace{\Gamma_1 \Longrightarrow \Delta_1 \quad \cdots \quad \Gamma_r \Longrightarrow \Delta_r}^{\text{premisses}}}{\underbrace{\Gamma \Longrightarrow \Delta}_{\text{conclusion}}}$$

Meaning: For proving the conclusion, it suffices to prove all premisses.

Example

$$\text{andRight}\ \frac{\Gamma \Longrightarrow \phi, \Delta \qquad \Gamma \Longrightarrow \psi, \Delta}{\Gamma \Longrightarrow \phi \land \psi, \Delta}$$

Admissible to have no premisses (iff conclusion is valid, e.g., axiom)

A rule is sound (correct) iff the validity of its premisses implies the validity of its conclusion.

‘Propositional’ Sequent Calculus Rules

close $\dfrac{}{\Gamma, \phi \Longrightarrow \phi, \Delta}$ true $\dfrac{}{\Gamma \Longrightarrow \mathrm{true}, \Delta}$ false $\dfrac{}{\Gamma, \mathrm{false} \Longrightarrow \Delta}$

	left side (antecedent)	right side (succedent)
not	$\dfrac{\Gamma \Longrightarrow \phi, \Delta}{\Gamma, \neg\phi \Longrightarrow \Delta}$	$\dfrac{\Gamma, \phi \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg\phi, \Delta}$
and	$\dfrac{\Gamma, \phi, \psi \Longrightarrow \Delta}{\Gamma, \phi \wedge \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow \phi, \Delta \qquad \Gamma \Longrightarrow \psi, \Delta}{\Gamma \Longrightarrow \phi \wedge \psi, \Delta}$
or	$\dfrac{\Gamma, \phi \Longrightarrow \Delta \qquad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \vee \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow \phi, \psi, \Delta}{\Gamma \Longrightarrow \phi \vee \psi, \Delta}$
imp	$\dfrac{\Gamma \Longrightarrow \phi, \Delta \qquad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \to \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma, \phi \Longrightarrow \psi, \Delta}{\Gamma \Longrightarrow \phi \to \psi, \Delta}$

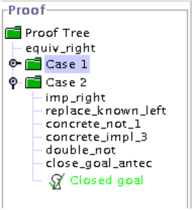
Sequent Calculus Proofs

Goal to prove: $\mathcal{G} = \quad \psi_1, \ldots, \psi_m \implies \phi_1, \ldots, \phi_n$

- find rule $\mathcal{R}$ whose conclusion matches $\mathcal{G}$
- instantiate $\mathcal{R}$ such that its conclusion is identical to $\mathcal{G}$
- apply that instantiation to all premisses of $\mathcal{R}$, resulting in new goals $\mathcal{G}_1, \ldots, \mathcal{G}_r$
- recursively find proofs for $\mathcal{G}_1, \ldots, \mathcal{G}_r$
- tree structure with goal as root
- close proof branch when rule without premiss encountered

Goal-directed proof search

- Paper proofs: root at bottom, grow upwards
- KeY tool proofs: root at top, grow downwards



A Simple Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{p \Longrightarrow p,\ q}\ \textsc{Close} \qquad \dfrac{*}{p,\ q \Longrightarrow q}\ \textsc{Close}}{p,\ (p \to q) \Longrightarrow q}}{p \wedge (p \to q) \Longrightarrow q}}{\Longrightarrow (p \wedge (p \to q)) \to q}}{}$$

A proof is closed iff all its branches are closed

Demo

`prop.key`

Proving Validity of First-Order Formulas

Proving a universally quantified formula

Claim: $\forall \tau\, x;\ \phi$ is true

How is such a claim proved in mathematics?

All even numbers are divisible by 2	$\forall\, \mathbf{int}\ x;\ (\mathrm{even}(x) \rightarrow \mathrm{divByTwo}(x))$
Let c be an arbitrary number	Declare "unused" constant $\mathbf{int}\ \mathtt{c}$
The even number c is divisible by 2	prove $\mathrm{even}(c) \rightarrow \mathrm{divByTwo}(c)$

Sequent rule $\forall$-right

$$\text{forallRight}\ \frac{\Gamma \Longrightarrow [x/c]\,\phi, \Delta}{\Gamma \Longrightarrow \forall \tau\, x;\ \phi, \Delta}$$

- $[x/c]\,\phi$ is result of replacing each occurrence of x in ϕ with c
- c **new** constant of type τ

Proving Validity of First-Order Formulas Cont'd

Proving an existentially quantified formula

Claim: $\exists\, \tau\, x;\ \phi$ is true

How is such a claim proved in mathematics?

There is at least one prime number	$\exists\, \mathbf{int}\, x;\ \mathrm{prime}(x)$
Provide any "witness", say, 7	Use variable-free term $\mathbf{int}\ 7$
7 is a prime number	$\mathrm{prime}(7)$

Sequent rule $\exists$-right

$$\text{existsRight}\ \frac{\Gamma \Longrightarrow [x/t]\,\phi,\ \exists\,\tau\,x;\ \phi, \Delta}{\Gamma \Longrightarrow \exists\,\tau\,x;\ \phi, \Delta}$$

- t any variable-free term of type τ
- We might need other instances besides t! Keep $\exists\,\tau\,x;\ \phi$

Proving Validity of First-Order Formulas Cont'd

Using a universally quantified formula

We assume $\forall \tau\, x;\ \phi$ is true

How is such a fact used in a mathematical proof?

We know that all primes are odd	$\forall\, \mathbf{int}\, x;\ (\mathrm{prime}(x) \rightarrow \mathrm{odd}(x))$
In particular, this holds for 17	Use variable-free term $\mathbf{int}\ 17$
We know: if 17 is prime it is odd	$\mathrm{prime}(17) \rightarrow \mathrm{odd}(17)$

Sequent rule $\forall$-left

$$\text{forallLeft}\ \frac{\Gamma, \forall \tau\, x;\ \phi,\ [x/t]\,\phi \Longrightarrow \Delta}{\Gamma, \forall \tau\, x;\ \phi \Longrightarrow \Delta}$$

- t any variable-free term of type τ
- We might need other instances besides t! Keep $\forall \tau\, x;\ \phi$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

We assume $\exists \tau\, x;\ \phi$ is true

How is such a fact used in a mathematical proof?

We know such an element exists. Let's give it a new name for future reference.

Sequent rule $\exists$-left

$$\text{existsLeft}\ \frac{\Gamma, [x/c]\,\phi \Longrightarrow \Delta}{\Gamma, \exists \tau\, x;\ \phi \Longrightarrow \Delta}$$

- c **new** constant of type τ

Proving Validity of First-Order Formulas Cont'd

Using an equation between terms

We assume $t = t'$ is true

How is such a fact used in a mathematical proof?

$x = y-1 \Longrightarrow 1 = x+1/y$

Use $x = y-1$ to modify $x+1/y$:
replace x in succedent with right-hand side of antecedent

$x = y-1 \Longrightarrow 1 = y-1+1/y$

Sequent rule $=$-left

$$\text{applyEqL}\ \frac{\Gamma, t = t', [t/t']\,\phi \Longrightarrow \Delta}{\Gamma, t = t', \phi \Longrightarrow \Delta} \qquad \text{applyEqR}\ \frac{\Gamma, t = t' \Longrightarrow [t/t']\,\phi, \Delta}{\Gamma, t = t' \Longrightarrow \phi, \Delta}$$

- Always replace left- with right-hand side (use eqSymm if necessary)
- t, t' variable-free terms of the same type

Proving Validity of First-Order Formulas Cont'd

Closing a subgoal in a proof

- We derived a sequent that is obviously valid

$$\text{close}\ \frac{}{\Gamma, \phi \Longrightarrow \phi, \Delta} \qquad \text{true}\ \frac{}{\Gamma \Longrightarrow \mathrm{true}, \Delta} \qquad \text{false}\ \frac{}{\Gamma, \mathrm{false} \Longrightarrow \Delta}$$

- We derived an equation that is obviously valid

$$\text{eqClose}\ \frac{}{\Gamma \Longrightarrow t = t, \Delta}$$

Sequent Calculus for FOL at One Glance

	left side, antecedent	right side, succedent
$\forall$	$\dfrac{\Gamma, \forall \tau\, x;\ \phi,\ [x/t']\,\phi \Longrightarrow \Delta}{\Gamma, \forall \tau\, x;\ \phi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow [x/c]\,\phi, \Delta}{\Gamma \Longrightarrow \forall \tau\, x;\ \phi, \Delta}$
$\exists$	$\dfrac{\Gamma, [x/c]\,\phi \Longrightarrow \Delta}{\Gamma, \exists \tau\, x;\ \phi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow [x/t']\,\phi,\ \exists \tau\, x;\ \phi, \Delta}{\Gamma \Longrightarrow \exists \tau\, x;\ \phi, \Delta}$
$=$	$\dfrac{\Gamma, t = t' \Longrightarrow [t/t']\,\phi, \Delta}{\Gamma, t = t' \Longrightarrow \phi, \Delta}$ (+ application rule on left side)	$\dfrac{}{\Gamma \Longrightarrow t = t, \Delta}$

- $[t/t']\,\phi$ is result of replacing each occurrence of t in ϕ with t'
- t,t' variable-free terms of type τ
- c **new** constant of type τ (occurs not on current proof branch)
- Equations can be reversed by commutativity

Recap: 'Propositional' Sequent Calculus Rules

main	left side (antecedent)	right side (succedent)
not	$\dfrac{\Gamma \Longrightarrow \phi, \Delta}{\Gamma, \neg\phi \Longrightarrow \Delta}$	$\dfrac{\Gamma, \phi \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg\phi, \Delta}$
and	$\dfrac{\Gamma, \phi, \psi \Longrightarrow \Delta}{\Gamma, \phi \wedge \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow \phi, \Delta \qquad \Gamma \Longrightarrow \psi, \Delta}{\Gamma \Longrightarrow \phi \wedge \psi, \Delta}$
or	$\dfrac{\Gamma, \phi \Longrightarrow \Delta \qquad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \vee \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma \Longrightarrow \phi, \psi, \Delta}{\Gamma \Longrightarrow \phi \vee \psi, \Delta}$
imp	$\dfrac{\Gamma \Longrightarrow \phi, \Delta \qquad \Gamma, \psi \Longrightarrow \Delta}{\Gamma, \phi \rightarrow \psi \Longrightarrow \Delta}$	$\dfrac{\Gamma, \phi \Longrightarrow \psi, \Delta}{\Gamma \Longrightarrow \phi \rightarrow \psi, \Delta}$

close $\dfrac{}{\Gamma, \phi \Longrightarrow \phi, \Delta}$ true $\dfrac{}{\Gamma \Longrightarrow \mathrm{true}, \Delta}$ false $\dfrac{}{\Gamma, \mathrm{false} \Longrightarrow \Delta}$

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{p(c,d),\ \forall\, y;\ p(c,y) \Longrightarrow p(c,d),\ \exists\, x;\ p(x,d)}}{p(c,d),\ \forall\, y;\ p(c,y) \Longrightarrow \exists\, x;\ p(x,d)}}{\forall\, y;\ p(c,y) \Longrightarrow \exists\, x;\ p(x,d)}}{\forall\, y;\ p(c,y) \Longrightarrow \forall\, y;\ \exists\, x;\ p(x,y)}}{\exists\, x;\ \forall\, y;\ p(x,y) \Longrightarrow \forall\, y;\ \exists\, x;\ p(x,y)}$$

Untyped logic: let static type of x and y be $\top$

$\exists$-left: substitute new constant c of type $\top$ for x

$\forall$-right: substitute new constant d of type $\top$ for y

$\forall$-left: free to substitute any term of type $\top$ for y, choose d

$\exists$-right: free to substitute any term of type $\top$ for x, choose c

Close

Demo

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y.

Show: $(y/x) * x = y$ ($'/'$ is division on integers, i.e., the equation is not always true, e.g. $x = 2, y = 1$)

Proof: We know x divides y, i.e. there exists a k such that $y = k * x$. Let now c denote such a k. Hence we can replace y by $c * x$ on the right side. ... □

$$\cfrac{\cfrac{\cfrac{\cfrac{*}{\vdots}}{\neg(x = 0), \neg(y = 0), y = c * x \Longrightarrow ((c * x)/x) * x = y}}{\neg(x = 0), \neg(y = 0), y = c * x \Longrightarrow (y/x) * x = y}}{\neg(x = 0), \neg(y = 0), \exists\, \texttt{int}\; k;\; y = k * x \Longrightarrow (y/x) * x = y}$$

Features of the KeY Theorem Prover

Demo

`rel.key`, `twoInstances.key`

Feature List

- Can work on multiple proofs simultaneously (task list)
- Point-and-click navigation within proof
- Undo proof steps, prune proof trees
- Pop-up menu with proof rules applicable in pointer focus
- Preview of rule effect as tool tip
- Quantifier instantiation and equality rules by drag-and-drop
- Possible to hide (and unhide) parts of a sequent
- Saving and loading of proofs

Literature for this Lecture

KeYbook *W. Ahrendt*, B. Beckert, R. Bubel, R. Hähnle, P. Schmitt, M. Ulbrich, editors.
Deductive Software Verification - The KeY Book
Vol 10001 of *LNCS*, Springer, 2016
(E-book at `link.springer.com`)

- *W. Ahrendt*, S. Grebing, Using the KeY Prover
  Chapter 15 in [KeYbook]

further reading:

- P.H. Schmitt, First-Order Logic,
  Chapter 2 in [KeYbook]

Part III

First-Order Semantics

First-Order Semantics

From propositional to first-order semantics

- In prop. logic, an interpretation of variables with $\{T, F\}$ sufficed
- In first-order logic we must assign meaning to:
  - function symbols
  - predicate symbols
  - variables bound in quantifiers
- Respect typing: `int i`, `List l` must denote different items

What we need (to interpret a first-order formula)

1. A typed domain of items
2. A mapping from function symbols to functions on items
3. A mapping from predicate symbols to relation on items
4. A mapping from variables to items

First-Order Domains

1. A typed domain of items:

Definition (Typed Domain)

A non-empty set $\mathcal{D}$ of items is a domain.
A typing of $\mathcal{D}$ wrt. signature Σ is a mapping $\delta : \mathcal{D} \to T_\Sigma$

We require from $\mathcal{D}$ and δ that no type is empty:
for each $\tau \in T_\Sigma$, there is a $d \in \mathcal{D}$ with $\delta(d) = \tau$

- If $\delta(d) = \tau$, we say d has type τ.
- $\mathcal{D}^\tau = \{d \in \mathcal{D} \mid \delta(d) = \tau\}$ is called subdomain of type τ.
- It follows that $\mathcal{D}^\tau \neq \emptyset$ for each $\tau \in T_\Sigma$.

First-Order States

2. A mapping from function symbol to functions on items
3. A mapping from predicate symbol to relation on items

Definition (Interpretation, First-Order State)

Let $\mathcal{D}$ be a domain with typing δ.

Let $\mathcal{I}$ be a mapping, called interpretation, from function and predicate symbols to functions and relations on items, respectively, such that

$$\mathcal{I}(f) : \mathcal{D}^{\tau_1} \times \cdots \times \mathcal{D}^{\tau_r} \to \mathcal{D}^{\tau} \quad \text{when } \alpha_{\Sigma}(f) = (\tau_1, \ldots, \tau_r, \tau)$$
$$\mathcal{I}(p) \subseteq \mathcal{D}^{\tau_1} \times \cdots \times \mathcal{D}^{\tau_r} \quad \text{when } \alpha_{\Sigma}(p) = (\tau_1, \ldots, \tau_r)$$

Then $\mathcal{S} = (\mathcal{D}, \delta, \mathcal{I})$ is a first-order state.

First-Order States Cont'd

Example

Signature: `int i; short j; int f(int); Object obj; <(int,int);`
$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

$$\mathcal{I}(i) = 17$$
$$\mathcal{I}(j) = 17$$
$$\mathcal{I}(\texttt{obj}) = o$$

$\mathcal{D}^{\text{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\text{int}} \times \mathcal{D}^{\text{int}}$	in $\mathcal{I}(<)$?
$(2, 2)$	F
$(2, 17)$	T
$(17, 2)$	F
$(17, 17)$	F

One of uncountably many possible first-order states!

Semantics of Equality

Definition

Interpretation is fixed as $\mathcal{I}(=) = \{(d, d) \mid d \in \mathcal{D}\}$

Exercise: write down the predicate table for example domain

Signature Symbols vs. Domain Elements

- Domain elements different from the terms representing them
- First-order formulas and terms have no access to domain

Example

Signature: `Object obj1, obj2;`
Domain: $\mathcal{D} = \{o\}$

In this state, necessarily $\mathcal{I}(\texttt{obj1}) = \mathcal{I}(\texttt{obj2}) = o$

Variable Assignments

4. A mapping from variables to items

Think of variable assignment as environment for storage of local variables

Definition (Variable Assignment)

A variable assignment β maps variables to domain elements.
It respects the variable type, i.e., if x has type τ then $\beta(x) \in \mathcal{D}^\tau$

Definition (Modified Variable Assignment)

Let y be variable of type τ, β variable assignment, $d \in \mathcal{D}^\tau$:

$$\beta_y^d(x) := \begin{cases} \beta(x) & x \neq y \\ d & x = y \end{cases}$$

Semantic Evaluation of Terms

Given a first-order state $\mathcal{S}$ and a variable assignment β
it is possible to evaluate first-order terms under $\mathcal{S}$ and β

Definition (Valuation of Terms)

$val_{\mathcal{S},\beta}$: Term $\rightarrow \mathcal{D}$ such that $val_{\mathcal{S},\beta}(t) \in \mathcal{D}^{\tau}$ for $t \in \text{Term}_{\tau}$:

- $val_{\mathcal{S},\beta}(x) = \beta(x)$
- $val_{\mathcal{S},\beta}(f(t_1, \ldots, t_r)) = \mathcal{I}(f)(val_{\mathcal{S},\beta}(t_1), \ldots, val_{\mathcal{S},\beta}(t_r))$

Semantic Evaluation of Terms Cont'd

Example

Signature: **int** `i`; **short** `j`; **int** `f`(**int**);

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

Variables: `Object obj`; **int** `x`;

$\mathcal{I}(\mathtt{i}) = 17$
$\mathcal{I}(\mathtt{j}) = 17$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(\mathtt{f})$
2	17
17	2

Var	β
obj	o
x	17

- $val_{\mathcal{S},\beta}(\mathtt{f}(\mathtt{f}(\mathtt{i})))$?
- $val_{\mathcal{S},\beta}(x)$?

Semantic Evaluation of Formulas

Definition (Valuation of Formulas)

$val_{\mathcal{S},\beta}(\phi)$ for $\phi \in For$

- $val_{\mathcal{S},\beta}(p(t_1, \ldots, t_r)) = T$ iff $(val_{\mathcal{S},\beta}(t_1), \ldots, val_{\mathcal{S},\beta}(t_r)) \in \mathcal{I}(p)$
- $val_{\mathcal{S},\beta}(\phi \wedge \psi) = T$ iff $val_{\mathcal{S},\beta}(\phi) = T$ and $val_{\mathcal{S},\beta}(\psi) = T$
- $\neg, \vee, \rightarrow, \leftrightarrow$ as in propositional logic
- $val_{\mathcal{S},\beta}(\forall\, \tau\; x;\, \phi) = T$ iff $val_{\mathcal{S},\beta_x^d}(\phi) = T$ for all $d \in \mathcal{D}^\tau$
- $val_{\mathcal{S},\beta}(\exists\, \tau\; x;\, \phi) = T$ iff $val_{\mathcal{S},\beta_x^d}(\phi) = T$ for at least one $d \in \mathcal{D}^\tau$

Semantic Evaluation of Formulas Cont'd

Example

Signature: `short j; int f(int); Object obj; <(int,int);`

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

$\mathcal{I}(j) = 17$

$\mathcal{I}(\texttt{obj}) = o$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\mathbf{int}} \times \mathcal{D}^{\mathbf{int}}$	in $\mathcal{I}(<)$?
$(2, 2)$	F
$(2, 17)$	T
$(17, 2)$	F
$(17, 17)$	F

- $val_{\mathcal{S},\beta}(f(j) < j)$?
- $val_{\mathcal{S},\beta}(\exists\, \mathbf{int}\ x;\ f(x) = x)$?
- $val_{\mathcal{S},\beta}(\forall\, \texttt{Object}\ o1;\ \forall\, \texttt{Object}\ o2;\ o1 = o2)$?

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$val_{\mathcal{S},\beta}(\phi) = T$			(ϕ is **satisfiable**)
$\mathcal{S} \models \phi$	iff	for all β : $val_{\mathcal{S},\beta}(\phi) = T$	(ϕ is **true** in $\mathcal{S}$)
$\models \phi$	iff	for all $\mathcal{S}$: $\mathcal{S} \models \phi$	(ϕ is **valid**)

Closed formulas that are satisfiable are also true: one top-level notion

Example

- $f(j) < j$ is true in $\mathcal{S}$
- $\exists\,\mathbf{int}\ x;\ i = x$ is valid
- $\exists\,\mathbf{int}\ x;\ \neg(x = x)$ is not satisfiable