

Formal Methods for Software Development

Temporal Model Checking (part 2) + First-Order Logic

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Part I

Finishing Temporal Model Checking

Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system \mathcal{T} (e.g., derived from a PROMELA program).

Verification task: is the LTL formula ϕ satisfied in all traces of \mathcal{T} , i.e.,

$$\mathcal{T} \models \phi \quad ?$$

LTL Model Checking—Overview

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and

σ_π is a counter example.

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If no such π is found, then

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What Remains?

last lecture

- 3.–5. product of transition system and Büchi automaton (construction and analysis)

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this lecture

2. generalised Büchi automata and their normalisation
1. translating LTL into generalised Büchi automata

Generalised Büchi Automata \mathcal{GB} and Translation to (normal) Büchi Automata \mathcal{B}

Generalised Büchi Automata

A **generalised** Büchi automaton is defined as:

$$\mathcal{GB} = (Q, \delta, Q_0, \mathcal{F})$$

Q, δ, Q_0 as for standard Büchi automata

$\mathcal{F} = \{F_1, \dots, F_k\}$ is a **set of sets of accepting locations**

($F_i = \{f_{i1}, \dots, f_{im_i}\} \subseteq Q$)

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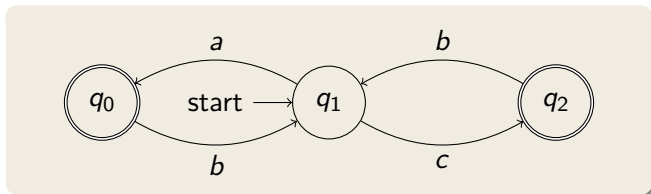
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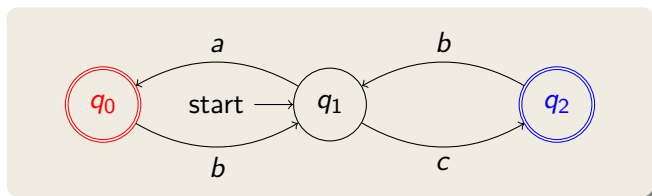
Definition (Acceptance for generalised Büchi automata)

A generalised Büchi automaton **accepts** an ω -word $w \in \Sigma^\omega$ iff **for every** $i \in \{1, \dots, k\}$ **at least one** $q \in F_i$ is visited infinitely often.

Generalised vs. Normal Büchi Automata: Example

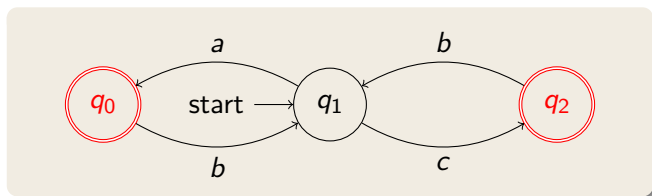


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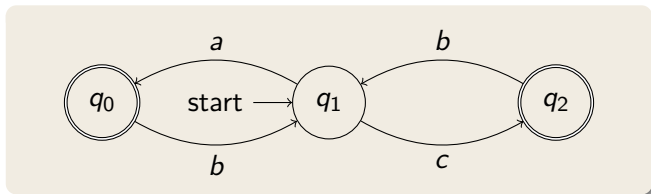
\mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$

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\mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$ different from normal \mathcal{B} with $F = \{q_0, q_2\}$

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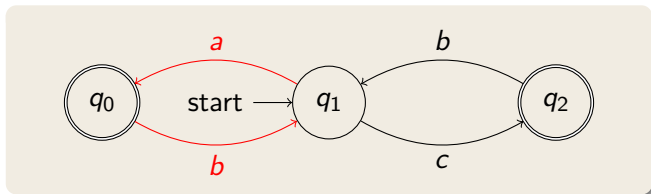


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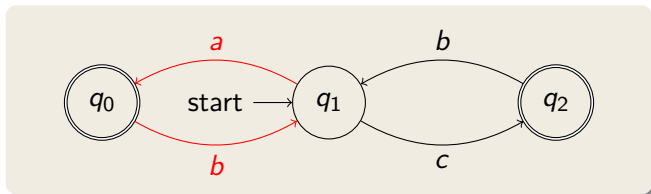


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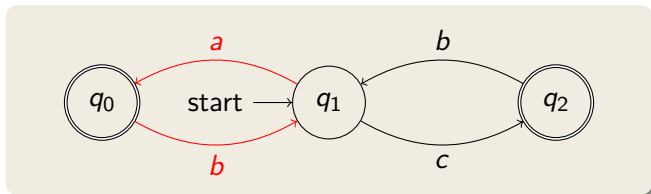


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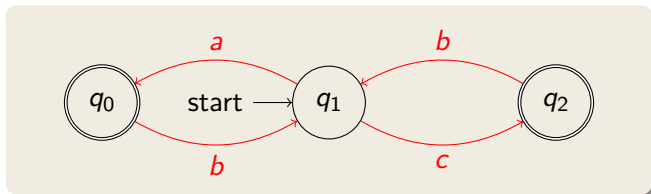


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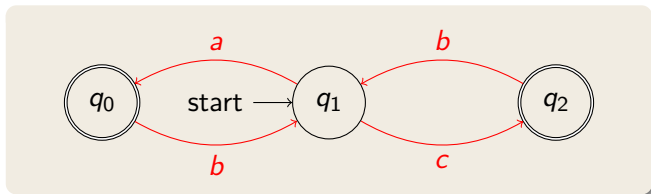


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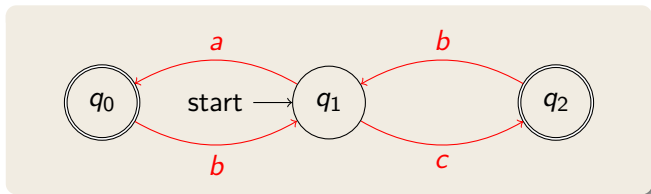


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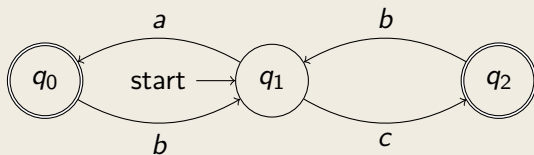
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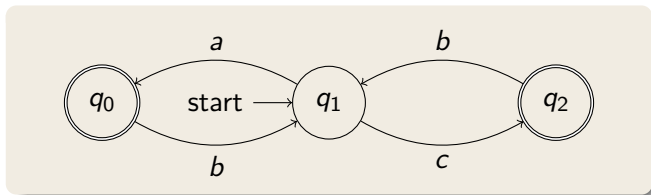
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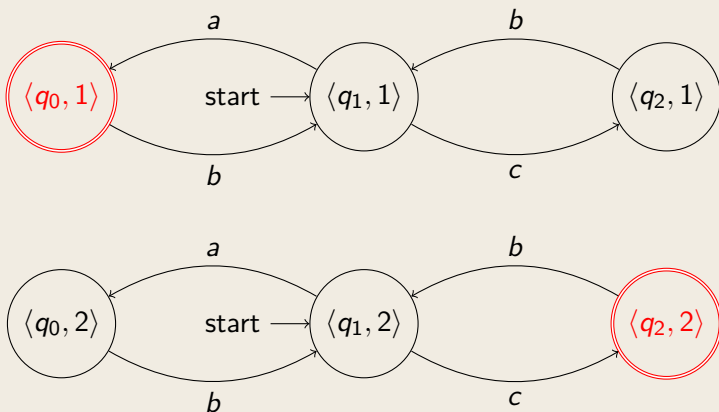


Construct \mathcal{B} (different from last slide) which accepts the same words:

$$\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{GB})$$

Translate Generalised to Normal Büchi Automata

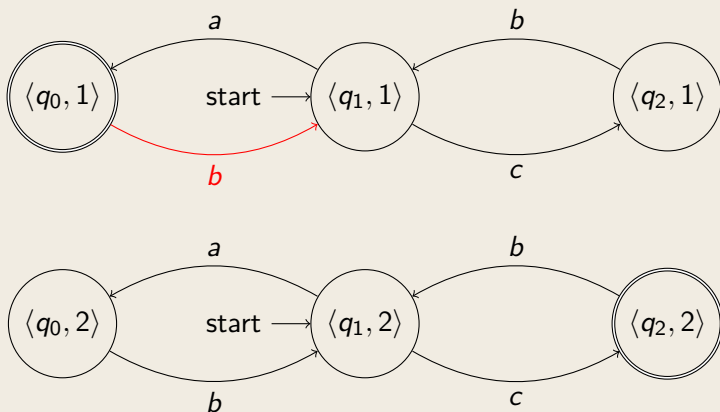
Construct \mathcal{B} for \mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



One clone for each $F_i \in \mathcal{F}$

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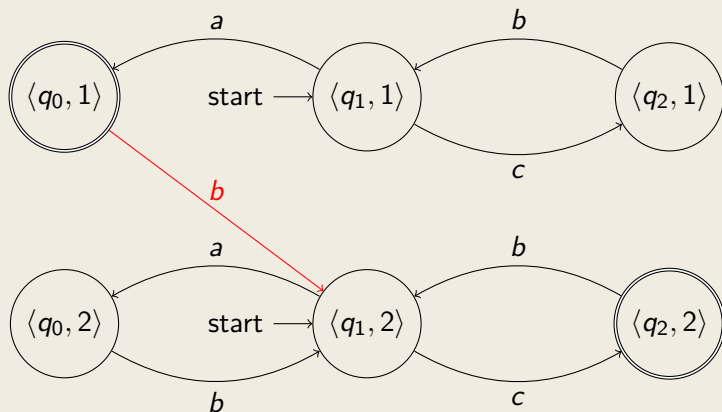
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Every **transition from “ F_1 ”** is redirected to “clone 2”

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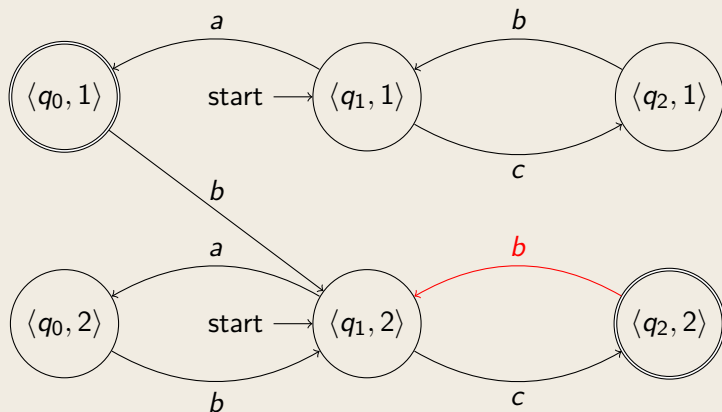
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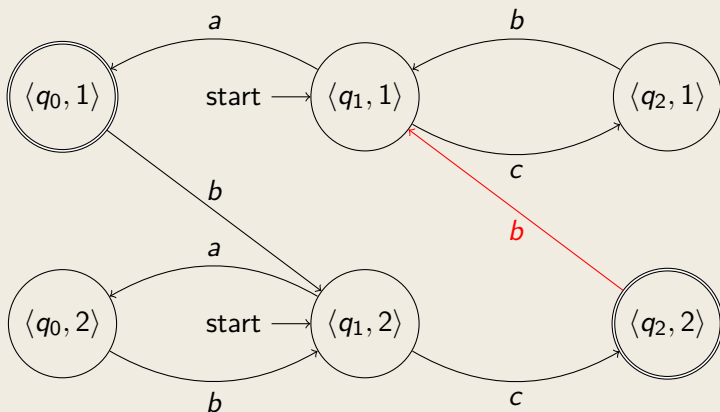
Construct \mathcal{B} for \mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



Every **transition from "F₂"** is redirected to "clone 1"

Translate Generalised to Normal Büchi Automata

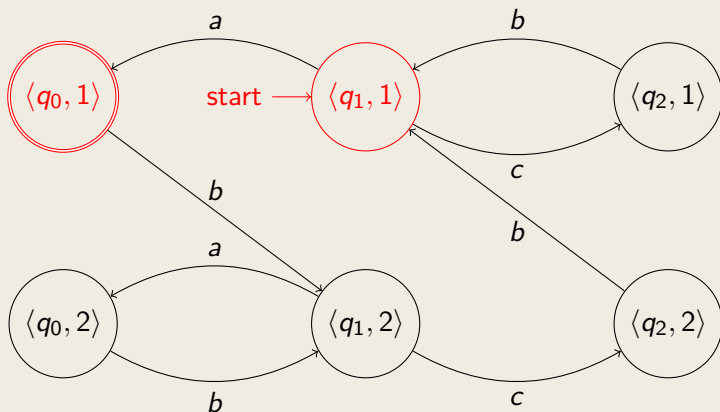
Construct \mathcal{B} for \mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



Every transition from " F_2 " is **redirected to "clone 1"**

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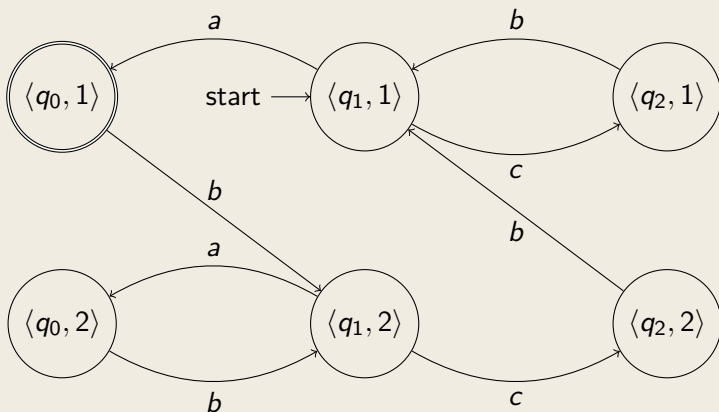
Construct \mathcal{B} for \mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



Keep only **initial and final locations of "clone 1"**

Translate Generalised to Normal Büchi Automata

Construct \mathcal{B} for \mathcal{GB} with $\mathcal{F} = \{\overbrace{\{q_0\}}^{F_1}, \overbrace{\{q_2\}}^{F_2}\}$:



resulting *normal* Büchi automaton \mathcal{B} with $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{GB})$

Translate Generalised to Normal Büchi Automata (formal)

Given **generalised** Büchi automaton

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Equivalent **normal** Büchi automaton

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Construction of a
Generalised Büchi Automaton
 \mathcal{GB}_ϕ
for an
LTL-Formula ϕ

Focus on \square -free and \diamond -free LTL

- ▶ Following construction assumes formulas without \square and \diamond .
- ▶ Only temporal modality is \mathcal{U} .
- ▶ \square can be removed using

$$\square\phi \equiv \neg\diamond\neg\phi$$

- ▶ \diamond can be removed using

$$\diamond\phi \equiv \text{true } \mathcal{U}\phi$$

Theory and Example at Once

We introduce the general construction together with example.

Task:

construct

\mathcal{GB}_ϕ

for

$\phi \equiv rUs$

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

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Example

$$FL(rUs) = \{r, \neg r, s, \neg s, rUs, \neg(rUs)\}$$

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$\in Q$

$$\{rUs, \neg r, s\}$$

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$$\frac{\{rUs, \neg r, s\}}{\in Q} \quad \checkmark$$

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Transitions $(q, \alpha, q') \in \delta_\phi$:

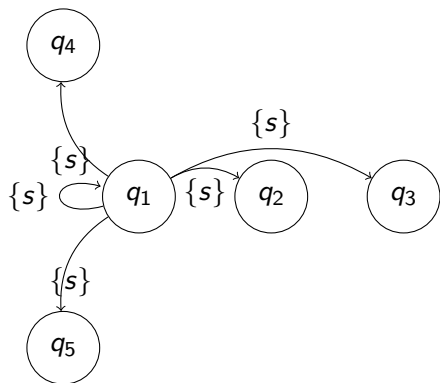
$$\alpha = q \cap AP$$

AP set of propositional variables
outgoing edges of q_1 labeled $\{s\}$,
of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$
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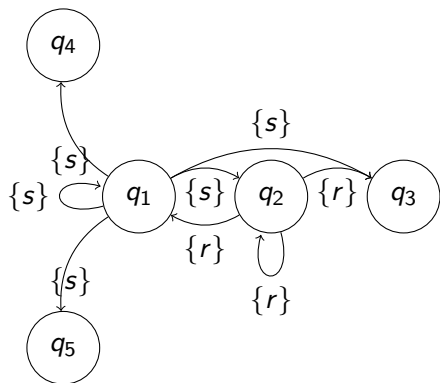
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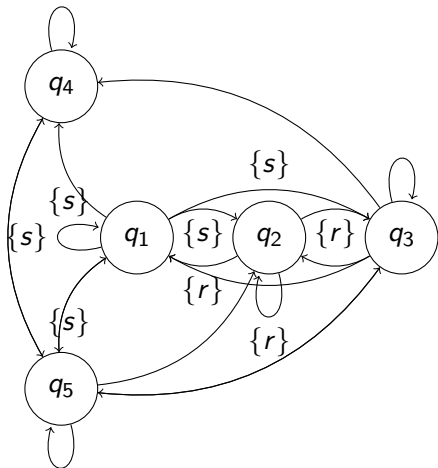
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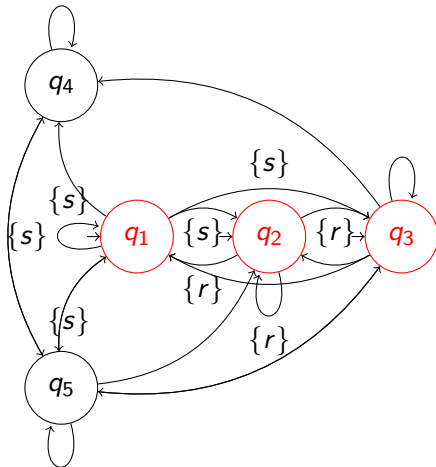
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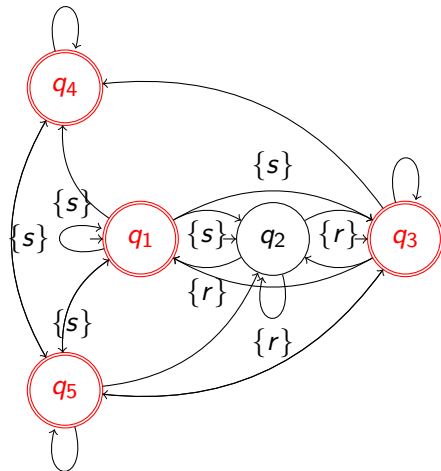
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Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

Accepting locations

$$\mathcal{F} = \{F_1, \dots, F_n\}$$

- ▶ One F_i for each $\psi_{i1} \mathcal{U} \psi_{i2} \in FL(\phi)$;
here: $\mathcal{F} = \{F_1\}$
- ▶ F_i set of locations that do *not* contain $\psi_{i1} \mathcal{U} \psi_{i2}$ **or** that contain ψ_{i2}
here: $F_1 = \{q_1, q_3, q_4, q_5\}$

Remarks on Generalized Büchi Automata

- ▶ Construction **always** gives exponential number of states in $|\phi|$
- ▶ Satisfiability checking of LTL is PSPACE-complete
- ▶ There exist (more complex) constructions that minimize number of required states
 - ▶ One of these is used in SPIN, which moreover computes the states lazily

Part II

Starting First-order Logic

Motivation for Introducing First-Order Logic

1) We specify JAVA programs with **Java Modeling Language (JML)**

JML combines

- ▶ JAVA expressions
- ▶ **First-Order Logic (FOL)**

2) We verify JAVA programs using **Dynamic Logic**

Dynamic Logic combines

- ▶ **First-Order Logic (FOL)**
- ▶ JAVA programs

We introduce:

- ▶ FOL as a language
- ▶ Sequent calculus for proving FOL formulas
- ▶ KeY system as propositional, and first-order, prover (for now)
- ▶ Formal semantics

First-Order Logic: Signature

Signature

A first-order signature Σ consists of

- ▶ a set T_Σ of types
- ▶ a set F_Σ of function symbols
- ▶ a set P_Σ of predicate symbols
- ▶ a typing α_Σ

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Intuitively, the typing α_Σ determines

- ▶ for each function and predicate symbol:
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Formally:

- ▶ $\alpha_\Sigma(p) \in T_\Sigma^*$ for all $p \in P_\Sigma$ (arity of p is $|\alpha_\Sigma(p)|$)
- ▶ $\alpha_\Sigma(f) \in T_\Sigma^* \times T_\Sigma$ for all $f \in F_\Sigma$ (arity of f is $|\alpha_\Sigma(f)| - 1$)

Example Signature $\Sigma_1 + \text{Constants}$

$$T_{\Sigma_1} = \{\text{int}\},$$

$$F_{\Sigma_1} = \{+, -\} \cup \{\dots, -2, -1, 0, 1, 2, \dots\},$$

$$P_{\Sigma_1} = \{<\}$$

$$\alpha_{\Sigma_1}(<) = (\text{int}, \text{int})$$

$$\alpha_{\Sigma_1}(+) = \alpha_{\Sigma_1}(-) = (\text{int}, \text{int}, \text{int})$$

$$\alpha_{\Sigma_1}(0) = \alpha_{\Sigma_1}(1) = \alpha_{\Sigma_1}(-1) = \dots = (\text{int})$$

Constant Symbols

A function symbol f with $|\alpha_{\Sigma_1}(f)| = 1$ (i.e., with arity 0) is called *constant symbol*.

Here, the constant symbols are: $\dots, -2, -1, 0, 1, 2, \dots$

Syntax of First-Order Logic: Signature Cont'd

Type declaration of signature symbols

- ▶ Write τx ; to declare variable x of type τ
- ▶ Write $p(\tau_1, \dots, \tau_r)$; for $\alpha(p) = (\tau_1, \dots, \tau_r)$
- ▶ Write $\tau f(\tau_1, \dots, \tau_r)$; for $\alpha(f) = (\tau_1, \dots, \tau_r, \tau)$

$r = 0$ is allowed, then write f instead of $f()$.

Example

Variables `integerArray a; int i;`

Predicate Symbols `isEmpty(List); alertOn;`

Function Symbols `int arrayLookup(int); Object o;`

Example Signature Σ_1 + Notation

Typing of Signature:

$$\alpha_{\Sigma_1}(<) = (\text{int}, \text{int})$$

$$\alpha_{\Sigma_1}(+) = \alpha_{\Sigma_1}(-) = (\text{int}, \text{int}, \text{int})$$

$$\alpha_{\Sigma_1}(0) = \alpha_{\Sigma_1}(1) = \alpha_{\Sigma_1}(-1) = \dots = (\text{int})$$

can alternatively be written as:

```
<(int, int);
```

```
int +(int, int);
```

```
int 0; int 1; int -1; ...
```

First-Order Terms

We assume a set V of variables ($V \cap (F_\Sigma \cup P_\Sigma) = \emptyset$).
Each $v \in V$ has a unique type $\alpha_\Sigma(v) \in T_\Sigma$.

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Terms are defined recursively:

Terms

A first-order term of type $\tau \in T_\Sigma$

- ▶ is either a variable of type τ , or
- ▶ has the form $f(t_1, \dots, t_n)$,
where $f \in F_\Sigma$ has result type τ , and each t_i is term of the correct type, following the typing α_Σ of f .

If f is a constant symbol, the term is written f , instead of $f()$.

Terms over Signature Σ_1

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- ▶ $+(-2, 99)$
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Our variant of FOL allows infix notation for common functions:

- ▶ $-2 + 99$
- ▶ $7 - 8$
- ▶ $(7 - 8) + 1$
- ▶ $(v_1 - 8) + v_2$

Atomic Formulas

Given a signature Σ .

An atomic formula has either of the forms

- ▶ *true*
- ▶ *false*
- ▶ $t_1 = t_2$ (“equality”),
where t_1 and t_2 are first-order terms of the same type.
- ▶ $p(t_1, \dots, t_n)$ (“predicate”),
where $p \in P_\Sigma$, and each t_i is term of the correct type,
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Atomic Formulas over Signature Σ_1

Example formulas over Σ_1 :
(assume variable `int v`;))

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Our variant of FOL allows infix notation for common predicates:

- ▶ $7 < 8$
- ▶ $-2 - v < 99$
- ▶ $v < v + 1$

Formulas

- ▶ each atomic formula is a formula
- ▶ with ϕ and ψ formulas, x a variable, and τ a type, the following are also formulas:
 - ▶ $\neg\phi$ (“not ϕ ”)
 - ▶ $\phi \wedge \psi$ (“ ϕ and ψ ”)
 - ▶ $\phi \vee \psi$ (“ ϕ or ψ ”)
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Formulas with no free variable are ‘closed’.

First-order Formulas: Examples

(signatures/types left out here)

Example (There are at least two elements)

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Example (There are at least two elements)

$$\exists x, y; \neg(x = y)$$

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Example (Strict partial order)



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Example (Strict partial order)

Irreflexivity $\forall x; \neg(x < x)$

Asymmetry $\forall x; \forall y; (x < y \rightarrow \neg(y < x))$

Transitivity $\forall x; \forall y; \forall z;$
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(Is any of the three formulas redundant?)

Semantics (briefly here, more thorough later)

Domain

A domain \mathcal{D} is a set of elements which are (potentially) the *meaning* of terms and variables.

Interpretation

An interpretation \mathcal{I} (over \mathcal{D}) assigns *meaning* to the symbols in $F_\Sigma \cup P_\Sigma$ (assigning functions to function symbols, relations to predicate symbols).

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In a given \mathcal{D} and \mathcal{I} , a closed formula evaluates to either T or F .

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A closed formula is **valid** if it evaluates to T in **all** \mathcal{D} and \mathcal{I} .

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In the context of specification/verification of programs:
each $(\mathcal{D}, \mathcal{I})$ is called a **'state'**.

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Remark on Concrete Syntax

	Text book	SPIN	KeY
Negation	\neg	!	!
Conjunction	\wedge	&&	&
Disjunction	\vee		
Implication	\rightarrow, \supset	\rightarrow	\rightarrow
Equivalence	\leftrightarrow	$\langle \leftrightarrow \rangle$	$\langle \leftrightarrow \rangle$
Universal Quantifier	$\forall x; \phi$	n/a	<code>\forall x; ϕ</code>
Existential Quantifier	$\exists x; \phi$	n/a	<code>\exists x; ϕ</code>
Value equality	=	==	=

Motivation for a Sequent Calculus

How to show a formula valid in propositional logic?

→ use a semantic truth table.

How about FOL? Formula: $\text{isEven}(x) \vee \text{isOdd}(x)$

x	$\text{isEven}(x)$	$\text{isOdd}(x)$	$\text{isEven}(x) \vee \text{isOdd}(x)$
1	F	T	T
2	T	F	T
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Checking validity via **semantics** does not work.

Instead...

Reasoning by Syntactic Transformation

Prove validity of ϕ by **syntactic** transformation of ϕ

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Logic Calculus: **Sequent Calculus** based on notion of **sequent**:

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which has (for closed formulas ψ_i, ϕ_i) same meaning as

$$\{\psi_1, \dots, \psi_m\} \models \phi_1 \vee \dots \vee \phi_n$$

Notation for Sequents

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Consider antecedent/succedent as sets of formulas, may be empty

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Schema Variables

ϕ, ψ, \dots match formulas, Γ, Δ, \dots match sets of formulas

Characterize infinitely many sequents with single schematic sequent, e.g.,

$$\Gamma \Rightarrow \phi \wedge \psi, \Delta$$

matches any sequent with occurrence of conjunction in succedent

Here, we call $\phi \wedge \psi$ **main formula** and Γ, Δ **side formulas** of sequent

Sequent Calculus Rules

Write syntactic transformation schema for sequents that reflects semantics of connectives

$$\text{RuleName} \frac{\overbrace{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_r \Rightarrow \Delta_r}^{\text{premises}}}{\underbrace{\Gamma \Rightarrow \Delta}_{\text{conclusion}}}$$

Meaning: For proving the conclusion, it suffices to prove all premisses.

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Admissible to have no premisses (iff conclusion is valid, e.g., axiom)

A rule is **sound** (correct) iff the validity of its premisses implies the validity of its conclusion.

'Propositional' Sequent Calculus Rules

close $\frac{}{\Gamma, \phi \Rightarrow \phi, \Delta}$

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	left side (antecedent)	right side (succedent)
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Sequent Calculus Proofs

Goal to prove: $\mathcal{G} = \psi_1, \dots, \psi_m \Rightarrow \phi_1, \dots, \phi_n$

- ▶ find rule \mathcal{R} whose conclusion **matches** \mathcal{G}
- ▶ instantiate \mathcal{R} such that its conclusion is **identical** to \mathcal{G}
- ▶ apply that instantiation to all premisses of \mathcal{R} , resulting in new goals $\mathcal{G}_1, \dots, \mathcal{G}_r$
- ▶ recursively find proofs for $\mathcal{G}_1, \dots, \mathcal{G}_r$
- ▶ tree structure with goal as root
- ▶ **close** proof branch when rule without premiss encountered

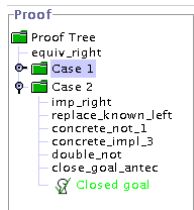
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Goal-directed proof search

- ▶ Paper proofs: root at bottom, grow upwards
- ▶ KeY tool proofs: root at top, grow downwards



A Simple Proof

$$\frac{\frac{}{p} \quad \frac{}{p \rightarrow q}}{p \wedge (p \rightarrow q)}}{\Rightarrow (p \wedge (p \rightarrow q)) \rightarrow q}$$

A Simple Proof

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A Simple Proof

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$$\frac{\frac{\frac{}{p \Rightarrow p, q}}{} \quad \frac{}{p, q \Rightarrow q}}{p, (p \rightarrow q) \Rightarrow q}}{p \wedge (p \rightarrow q) \Rightarrow q} \\ \Rightarrow (p \wedge (p \rightarrow q)) \rightarrow q$$

A Simple Proof

$$\frac{\text{CLOSE} \frac{*}{p \Rightarrow p, q} \quad \frac{*}{p, q \Rightarrow q} \text{CLOSE}}{p, (p \rightarrow q) \Rightarrow q}$$
$$\frac{p \wedge (p \rightarrow q) \Rightarrow q}{\Rightarrow (p \wedge (p \rightarrow q)) \rightarrow q}$$

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A proof is **closed** iff all its branches are closed

Demo

prop.key

Proving Validity of First-Order Formulas

Proving a universally quantified formula

Claim: $\forall x; \phi$ is true

How is such a claim proved in mathematics?

All even numbers are divisible by 2 $\forall \text{int } x; (\text{even}(x) \rightarrow \text{divByTwo}(x))$

Let c be an arbitrary number Declare “unused” constant `int c`

The even number c is divisible by 2 prove `even(c) \rightarrow divByTwo(c)`

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The even number c is divisible by 2 prove $\text{even}(c) \rightarrow \text{divByTwo}(c)$

Sequent rule \forall -right

$$\text{forallRight} \frac{\Gamma \Rightarrow [x/c] \phi, \Delta}{\Gamma \Rightarrow \forall \tau x; \phi, \Delta}$$

- ▶ $[x/c] \phi$ is result of replacing each occurrence of x in ϕ with c
- ▶ c **new** constant of type τ

Proving Validity of First-Order Formulas Cont'd

Proving an existentially quantified formula

Claim: $\exists x; \phi$ is true

How is such a claim proved in mathematics?

There is at least one prime number $\exists \text{int } x; \text{prime}(x)$

Provide any "witness", say, 7 Use variable-free term `int 7`

7 is a prime number `prime(7)`

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Sequent rule \exists -right

$$\text{existsRight} \frac{\Gamma \Rightarrow [x/t] \phi, \exists \tau x; \phi, \Delta}{\Gamma \Rightarrow \exists \tau x; \phi, \Delta}$$

- ▶ t any variable-free term of type τ
- ▶ We might need other instances besides t ! Keep $\exists \tau x; \phi$

Proving Validity of First-Order Formulas Cont'd

Using a universally quantified formula

We assume $\forall x; \phi$ is true

How is such a fact **used** in a mathematical proof?

We know that all primes are odd $\forall \text{int } x; (\text{prime}(x) \rightarrow \text{odd}(x))$

In particular, this holds for 17 Use variable-free term `int 17`

We know: if 17 is prime it is odd $\text{prime}(17) \rightarrow \text{odd}(17)$

Proving Validity of First-Order Formulas Cont'd

Using a universally quantified formula

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Sequent rule \forall -left

$$\text{forallLeft} \frac{\Gamma, \forall \tau x; \phi, [x/t] \phi \Rightarrow \Delta}{\Gamma, \forall \tau x; \phi \Rightarrow \Delta}$$

- ▶ t any variable-free term of type τ
- ▶ We might need other instances besides t ! Keep $\forall \tau x; \phi$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

We assume $\exists x; \phi$ is true

How is such a fact **used** in a mathematical proof?

We know such an element exists. Let's give it a new name for future reference.

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

We assume $\exists \tau x; \phi$ is true

How is such a fact **used** in a mathematical proof?

We know such an element exists. Let's give it a new name for future reference.

Sequent rule \exists -left

$$\text{existsLeft} \frac{\Gamma, [x/c] \phi \Rightarrow \Delta}{\Gamma, \exists \tau x; \phi \Rightarrow \Delta}$$

- ▶ c **new** constant of type τ

Proving Validity of First-Order Formulas Cont'd

Using an equation between terms

We assume $t = t'$ is true

How is such a fact used in a mathematical proof?

$$x = y - 1 \Rightarrow 1 = x + 1/y$$

Use $x = y - 1$ to modify $x + 1/y$:

replace x in succedent with right-hand side of antecedent

Proving Validity of First-Order Formulas Cont'd

Using an equation between terms

We assume $t = t'$ is true

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Using an equation between terms

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$$x = y-1 \Rightarrow 1 = y-1+1/y$$

Sequent rule =-left

$$\text{applyEqL} \frac{\Gamma, t = t', [t/t'] \phi \Rightarrow \Delta}{\Gamma, t = t', \phi \Rightarrow \Delta} \quad \text{applyEqR} \frac{\Gamma, t = t' \Rightarrow [t/t'] \phi, \Delta}{\Gamma, t = t' \Rightarrow \phi, \Delta}$$

- ▶ Always replace left- with right-hand side (use **eqSymm** if necessary)
- ▶ t, t' variable-free terms of the same type

Proving Validity of First-Order Formulas Cont'd

Closing a subgoal in a proof

- ▶ We derived a sequent that is obviously valid

$$\text{close } \frac{}{\Gamma, \phi \Rightarrow \phi, \Delta} \quad \text{true } \frac{}{\Gamma \Rightarrow \text{true}, \Delta} \quad \text{false } \frac{}{\Gamma, \text{false} \Rightarrow \Delta}$$

- ▶ We derived an **equation** that is obviously valid

$$\text{eqClose } \frac{}{\Gamma \Rightarrow t = t, \Delta}$$

Sequent Calculus for FOL at One Glance

	left side, antecedent	right side, succedent
\forall	$\frac{\Gamma, \forall \tau x; \phi, [x/t'] \phi \Rightarrow \Delta}{\Gamma, \forall \tau x; \phi \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow [x/c] \phi, \Delta}{\Gamma \Rightarrow \forall \tau x; \phi, \Delta}$
\exists	$\frac{\Gamma, [x/c] \phi \Rightarrow \Delta}{\Gamma, \exists \tau x; \phi \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow [x/t'] \phi, \exists \tau x; \phi, \Delta}{\Gamma \Rightarrow \exists \tau x; \phi, \Delta}$
$=$	$\frac{\Gamma, t = t' \Rightarrow [t/t'] \phi, \Delta}{\Gamma, t = t' \Rightarrow \phi, \Delta}$	$\frac{}{\Gamma \Rightarrow t = t, \Delta}$

(+ application rule on left side)

- ▶ $[t/t'] \phi$ is result of replacing each occurrence of t in ϕ with t'
- ▶ t, t' variable-free terms of type τ
- ▶ c **new** constant of type τ (occurs not on current proof branch)
- ▶ Equations can be reversed by commutativity

Recap: 'Propositional' Sequent Calculus Rules

main	left side (antecedent)	right side (succedent)
not	$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta}$	$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}$
and	$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}$
or	$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$
imp	$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \rightarrow \psi \Rightarrow \Delta}$	$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}$
close	$\frac{}{\Gamma, \phi \Rightarrow \phi, \Delta}$	true $\frac{}{\Gamma \Rightarrow \text{true}, \Delta}$ false $\frac{}{\Gamma, \text{false} \Rightarrow \Delta}$

Example (A simple theorem about binary relations)

$$\exists x; \forall y; p(x, y) \implies \forall y; \exists x; p(x, y)$$

Untyped logic: let static type of x and y be \top

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\frac{\forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y)}{\exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y)}$$

\exists -left: substitute **new** constant c of type \top for x

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\frac{\frac{\forall y; p(c, y) \Rightarrow \exists x; p(x, d)}{\forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y)}}{\exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y)}$$

\forall -right: substitute **new** constant d of type \top for y

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\frac{\frac{\frac{p(c, d), \forall y; p(c, y) \Rightarrow \exists x; p(x, d)}{\forall y; p(c, y) \Rightarrow \exists x; p(x, d)}}{\forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y)}}{\exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y)}$$

\forall -left: free to substitute **any** term of type \top for y , choose d

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\frac{\frac{\frac{p(c, d), \forall y; p(c, y) \Rightarrow p(c, d), \exists x; p(x, d)}{p(c, d), \forall y; p(c, y) \Rightarrow \exists x; p(x, d)}}{\forall y; p(c, y) \Rightarrow \exists x; p(x, d)}}{\forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y)}}{\exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y)}$$

\exists -right: free to substitute **any** term of type \top for x , choose c

Proving Validity of First-Order Formulas Cont'd

Example (A simple theorem about binary relations)

$$\begin{array}{c} * \\ \hline p(c, d), \forall y; p(c, y) \Rightarrow p(c, d), \exists x; p(x, d) \\ \hline p(c, d), \forall y; p(c, y) \Rightarrow \exists x; p(x, d) \\ \hline \forall y; p(c, y) \Rightarrow \exists x; p(x, d) \\ \hline \forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y) \\ \hline \exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y) \end{array}$$

Close

Example (A simple theorem about binary relations)

$$\begin{array}{c} * \\ \hline p(c, d), \forall y; p(c, y) \Rightarrow p(c, d), \exists x; p(x, d) \\ \hline p(c, d), \forall y; p(c, y) \Rightarrow \exists x; p(x, d) \\ \hline \forall y; p(c, y) \Rightarrow \exists x; p(x, d) \\ \hline \forall y; p(c, y) \Rightarrow \forall y; \exists x; p(x, y) \\ \hline \exists x; \forall y; p(x, y) \Rightarrow \forall y; \exists x; p(x, y) \end{array}$$

Demo

relSimple.key

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero.

$$\neg(x = 0), \neg(y = 0) \quad \Rightarrow$$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y .

$$\neg(x = 0), \neg(y = 0), \exists \text{int } k; y = k * x \implies$$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y .

Show: $(y/x) * x = y$ ('/' is division on integers, i.e., the equation is not always true, e.g. $x = 2, y = 1$)

$$\neg(x = 0), \neg(y = 0), \exists \text{int } k; y = k * x \implies (y/x) * x = y$$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y .

Show: $(y/x) * x = y$ ('/' is division on integers, i.e., the equation is not always true, e.g. $x = 2, y = 1$)

Proof: We know x divides y , i.e. there exists a k such that $y = k * x$.

Let now c denote such a k .

$$\frac{\neg(x = 0), \neg(y = 0), y = c * x \implies (y/x) * x = y}{\neg(x = 0), \neg(y = 0), \exists \text{int } k; y = k * x \implies (y/x) * x = y}$$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y .

Show: $(y/x) * x = y$ ('/' is division on integers, i.e., the equation is not always true, e.g. $x = 2, y = 1$)

Proof: We know x divides y , i.e. there exists a k such that $y = k * x$. Let now c denote such a k . Hence we can replace y by $c * x$ on the right side.

$$\frac{\frac{\neg(x = 0), \neg(y = 0), y = c * x \implies ((c * x)/x) * x = y}{\neg(x = 0), \neg(y = 0), y = c * x \implies (y/x) * x = y}}{\neg(x = 0), \neg(y = 0), \exists \text{int } k; y = k * x \implies (y/x) * x = y}$$

Proving Validity of First-Order Formulas Cont'd

Using an existentially quantified formula

Let x, y denote integer constants, both are not zero. We know further that x divides y .

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*

⋮

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$$\neg(x = 0), \neg(y = 0), \exists \text{int } k; y = k * x \implies (y/x) * x = y$$

Demo

`rel.key, twoInstances.key`

Feature List

- ▶ Can work on multiple proofs simultaneously (task list)
- ▶ Point-and-click navigation within proof
- ▶ Undo proof steps, prune proof trees
- ▶ Pop-up menu with proof rules applicable in pointer focus
- ▶ Preview of rule effect as tool tip
- ▶ Quantifier instantiation and equality rules by drag-and-drop
- ▶ Possible to hide (and unhide) parts of a sequent
- ▶ Saving and loading of proofs

Literature for this Lecture

KeYbook *W. Ahrendt, B. Beckert, R. Bubel, R. Hähnle, P. Schmitt, M. Ulbrich, editors.*

Deductive Software Verification - The KeY Book

Vol 10001 of *LNCS*, Springer, 2016

(E-book at link.springer.com)

- ▶ *W. Ahrendt, S. Grebing, Using the KeY Prover*
Chapter 15 in [KeYbook]

further reading:

- ▶ P.H. Schmitt, **First-Order Logic**,
Chapter 2 in [KeYbook]

Part III

First-Order Semantics

First-Order Semantics

From propositional to first-order semantics

- ▶ In prop. logic, an interpretation of variables with $\{T, F\}$ sufficed
- ▶ In first-order logic we must assign meaning to:
 - ▶ function symbols
 - ▶ predicate symbols
 - ▶ variables bound in quantifiers
- ▶ Respect typing: `int i`, `List l` **must** denote different items

First-Order Semantics

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 - ▶ predicate symbols
 - ▶ variables bound in quantifiers
- ▶ Respect typing: `int i`, `List l` **must** denote different items

What we need (to interpret a first-order formula)

1. A **typed domain of items**
2. A mapping from **function symbols** to **functions on items**
3. A mapping from **predicate symbols** to **relation on items**
4. A mapping from **variables** to **items**

First-Order Domains

1. A **typed domain of items**:

Definition (Typed Domain)

A non-empty set \mathcal{D} of items is a **domain**.

A **typing** of \mathcal{D} wrt. signature Σ is a mapping $\delta : \mathcal{D} \rightarrow T_\Sigma$

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We require from \mathcal{D} and δ that **no type is empty**:
for each $\tau \in T_\Sigma$, there is a $d \in \mathcal{D}$ with $\delta(d) = \tau$

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We require from \mathcal{D} and δ that **no type is empty**:
for each $\tau \in T_\Sigma$, there is a $d \in \mathcal{D}$ with $\delta(d) = \tau$

- ▶ If $\delta(d) = \tau$, we say **d has type τ** .
- ▶ $\mathcal{D}^\tau = \{d \in \mathcal{D} \mid \delta(d) = \tau\}$ is called **subdomain of type τ** .
- ▶ It follows that $\mathcal{D}^\tau \neq \emptyset$ for each $\tau \in T_\Sigma$.

First-Order States

2. A mapping from **function symbol** to **functions on items**
3. A mapping from **predicate symbol** to **relation on items**

Definition (Interpretation, First-Order State)

Let \mathcal{D} be a domain with typing δ .

First-Order States

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Let \mathcal{D} be a domain with typing δ .

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Let \mathcal{I} be a mapping, called **interpretation**, from **function** and **predicate symbols** to **functions** and **relations on items**, respectively, such that

$$\begin{aligned} \mathcal{I}(f) &: \mathcal{D}^{\tau_1} \times \dots \times \mathcal{D}^{\tau_r} \rightarrow \mathcal{D}^{\tau} && \text{when } \alpha_{\Sigma}(f) = (\tau_1, \dots, \tau_r, \tau) \\ \mathcal{I}(p) &\subseteq \mathcal{D}^{\tau_1} \times \dots \times \mathcal{D}^{\tau_r} && \text{when } \alpha_{\Sigma}(p) = (\tau_1, \dots, \tau_r) \end{aligned}$$

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Then $\mathcal{S} = (\mathcal{D}, \delta, \mathcal{I})$ is a **first-order state**.

First-Order States Cont'd

Example

Signature: `int i; short j; int f(int); Object obj; <(int,int);`
 $\mathcal{D} = \{17, 2, o\}$ where all numbers are short

$$\mathcal{I}(i) = 17$$

$$\mathcal{I}(j) = 17$$

$$\mathcal{I}(\text{obj}) = o$$

\mathcal{D}^{int}	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\text{int}} \times \mathcal{D}^{\text{int}}$	in $\mathcal{I}(<)$?
(2, 2)	<i>F</i>
(2, 17)	<i>T</i>
(17, 2)	<i>F</i>
(17, 17)	<i>F</i>

One of uncountably many possible first-order states!

Semantics of Equality

Definition

Interpretation is fixed as $\mathcal{I}(=) = \{(d, d) \mid d \in \mathcal{D}\}$

Exercise: write down the predicate table for example domain

Signature Symbols vs. Domain Elements

- ▶ Domain elements different from the terms representing them
- ▶ First-order formulas and terms have **no access** to domain

Example

Signature: Object obj1, obj2;

Domain: $\mathcal{D} = \{o\}$

In this state, necessarily $\mathcal{I}(\text{obj1}) = \mathcal{I}(\text{obj2}) = o$

Variable Assignments

4. A mapping from **variables** to **items**

Think of variable assignment as environment for storage of local variables

Definition (Variable Assignment)

A **variable assignment** β maps variables to domain elements.
It respects the variable type, i.e., if x has type τ then $\beta(x) \in \mathcal{D}^\tau$

Definition (Modified Variable Assignment)

Let y be variable of type τ , β variable assignment, $d \in \mathcal{D}^\tau$:

$$\beta_y^d(x) := \begin{cases} \beta(x) & x \neq y \\ d & x = y \end{cases}$$

Semantic Evaluation of Terms

Given a first-order state \mathcal{S} and a variable assignment β it is possible to evaluate first-order terms under \mathcal{S} and β

Definition (Valuation of Terms)

$val_{\mathcal{S},\beta} : \text{Term} \rightarrow \mathcal{D}$ such that $val_{\mathcal{S},\beta}(t) \in \mathcal{D}^\tau$ for $t \in \text{Term}_\tau$:

- ▶ $val_{\mathcal{S},\beta}(x) = \beta(x)$
- ▶ $val_{\mathcal{S},\beta}(f(t_1, \dots, t_r)) = \mathcal{I}(f)(val_{\mathcal{S},\beta}(t_1), \dots, val_{\mathcal{S},\beta}(t_r))$

Semantic Evaluation of Terms Cont'd

Example

Signature: `int i; short j; int f(int);`

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

Variables: Object `obj`; `int x`;

$$\mathcal{I}(i) = 17$$

$$\mathcal{I}(j) = 17$$

\mathcal{D}^{int}	$\mathcal{I}(f)$
2	17
17	2

Var	β
obj	<i>o</i>
x	17

► $val_{\mathcal{S},\beta}(f(f(i)))$?

► $val_{\mathcal{S},\beta}(x)$?

Definition (Valuation of Formulas)

$val_{S,\beta}(\phi)$ for $\phi \in For$

- ▶ $val_{S,\beta}(p(t_1, \dots, t_r)) = T$ iff $(val_{S,\beta}(t_1), \dots, val_{S,\beta}(t_r)) \in \mathcal{I}(p)$

Definition (Valuation of Formulas)

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- ▶ $val_{S,\beta}(\phi \wedge \psi) = T$ iff $val_{S,\beta}(\phi) = T$ and $val_{S,\beta}(\psi) = T$

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- ▶ $\neg, \vee, \rightarrow, \leftrightarrow$ as in propositional logic

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- ▶ $\neg, \vee, \rightarrow, \leftrightarrow$ as in propositional logic
- ▶ $val_{S,\beta}(\forall \tau x; \phi) = T$ iff $val_{S,\beta_x^d}(\phi) = T$ for all $d \in \mathcal{D}^\tau$

Semantic Evaluation of Formulas

Definition (Valuation of Formulas)

$val_{S,\beta}(\phi)$ for $\phi \in For$

- ▶ $val_{S,\beta}(p(t_1, \dots, t_r)) = T$ iff $(val_{S,\beta}(t_1), \dots, val_{S,\beta}(t_r)) \in \mathcal{I}(p)$
- ▶ $val_{S,\beta}(\phi \wedge \psi) = T$ iff $val_{S,\beta}(\phi) = T$ and $val_{S,\beta}(\psi) = T$
- ▶ $\neg, \vee, \rightarrow, \leftrightarrow$ as in propositional logic
- ▶ $val_{S,\beta}(\forall \tau x; \phi) = T$ iff $val_{S,\beta_x^d}(\phi) = T$ for all $d \in \mathcal{D}^\tau$
- ▶ $val_{S,\beta}(\exists \tau x; \phi) = T$ iff $val_{S,\beta_x^d}(\phi) = T$ for at least one $d \in \mathcal{D}^\tau$

Semantic Evaluation of Formulas Cont'd

Example

Signature: short j ; int $f(\text{int})$; Object obj ; $\langle \text{int}, \text{int} \rangle$;

$\mathcal{D} = \{17, 2, o\}$ where all numbers are short

$$\begin{aligned} \mathcal{I}(j) &= 17 \\ \mathcal{I}(\text{obj}) &= o \end{aligned}$$

\mathcal{D}^{int}	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\text{int}} \times \mathcal{D}^{\text{int}}$	in $\mathcal{I}(\langle \rangle)$?
(2, 2)	F
(2, 17)	T
(17, 2)	F
(17, 17)	F

- ▶ $\text{val}_{S,\beta}(f(j) < j)$?
- ▶ $\text{val}_{S,\beta}(\exists \text{int } x; f(x) = x)$?
- ▶ $\text{val}_{S,\beta}(\forall \text{Object } o1; \forall \text{Object } o2; o1 = o2)$?

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$val_{S,\beta}(\phi) = T$		(ϕ is satisfiable)
$S \models \phi$	iff for all $\beta : val_{S,\beta}(\phi) = T$	(ϕ is true in S)
$\models \phi$	iff for all $S : S \models \phi$	(ϕ is valid)

Closed formulas that are satisfiable are also true: one top-level notion

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$$\begin{array}{lll} \text{val}_{\mathcal{S},\beta}(\phi) = T & & (\phi \text{ is } \mathbf{satisfiable}) \\ \mathcal{S} \models \phi & \text{iff for all } \beta : \text{val}_{\mathcal{S},\beta}(\phi) = T & (\phi \text{ is } \mathbf{true} \text{ in } \mathcal{S}) \\ \models \phi & \text{iff for all } \mathcal{S} : \mathcal{S} \models \phi & (\phi \text{ is } \mathbf{valid}) \end{array}$$

Closed formulas that are satisfiable are also true: one top-level notion

Example

- ▶ $f(j) < j$ is true in \mathcal{S}
- ▶ $\exists \text{int } x; i = x$ is valid
- ▶ $\exists \text{int } x; \neg(x = x)$ is not satisfiable