Advanced Algorithms 2017
Exam Solutions (Jan. 2018)

1.1. Consider the moment just before the algorithm assigns job j to some machine. The load of this machine is $T - t_j$. Since the algorithm chooses a machine with minimum load, all machines have a load at least $T - t_j$. Summing up the loads yields at least $m(T - t_j)$. But since the job j is added as well, we can add t_j to this lower bound.

1.2. For the optimal makesepan T^* we know that $T^* \geq \sum_{k=1}^{m} T_k/m \geq T - t_j + t_j/m$, hence $T \leq T^* + t_j(1 - 1/m) \leq (2 - 1/m)T^*$.

2. A step in the known algorithm for center selection was to produce k disks of radius 2 (rather than 1) that cover the given set. We simply take the solution and replace every disk with 7 disks of half radius.[1]

3. Simply assign capacities q_{ij} to the edges (i, j), instead of infinite capacities. Now the cost of every violated constraint contributes positively to the cut capacity, i.e., it makes the cut capacity by q_{ij} units larger, and thus the profit by q_{ij} units smaller.

4.1. If any two adjacent nodes were in the solution, then each of them would be black, but the other node would be white, an obvious contradiction. Hence the solution is always valid (an independent set). The failure probability is 0.

4.2. Every node with degree d fulfills the selection criterion with probability $(1/\Delta)(1 - 1/\Delta)^d \geq (1/\Delta)(1 - 1/\Delta)^\Delta$, because: a node is white with probability $1 - 1/\Delta$, the black/white choices are independent, and $d \leq \Delta$.

[1] Small correction: The exercise text claimed 6 rather than 7, but this does not affect the "essence" of the problem.

4.3. Since $\lim_{\Delta\to\infty}(1-1/\Delta)^{\Delta} = 1/e$, the expression in 4.2 is at least c/Δ for some constant c. By linearity of expectation it follows that the expected number of nodes in I is at least cn/Δ.

4.4. The maximum size of a solution is n. Together with 4.3 this already yields the assertion.

5.1. Since n and k are fixed, the assertion is equivalent to $E[t] = sk/n$. For $k = 1$, the expected value of the randomly selected number is s/n, by the definition of expectation. Due to linearity of expectation, the sum of k randomly selected numbers has the expected value ks/n. So, yes, the assertion is true.
The exercise did not ask for the role of k; this remark is added here for the sake of completeness only: The expected value does not depend on k, but the variance of the estimate decreases for growing k.

5.2. We can define k random variables for the outcomes of the random choices of numbers from R. These random variables are 0/1-valued and independent, and our estimate is their sum (up to a fixed scaling facor). So, yes, all prerequisites for Chernoff bounds are fulfilled.