Advanced Algorithms. Assignment 1

In the following exercises, Δ denotes the largest degree of nodes in a given graph. That is, every node is assumed to have at most Δ neighbors.

Exercise 1.
An independent set in a graph is a subset X of nodes such that no edges exist between any two nodes of X. Consider the following rather intutitive greedy algorithm for the Maximum Independent Set problem (finding an independent set of maximum size): We take any node, put it in the solution, and remove this node and all its neighbors from the graph. We iterate this step until the graph is empty.

Prove that this algorithm always returns an independent set of size at least $1/\Delta$ of the maximum possible size.

Secondly, show that $1/\Delta$ is the best possible approximation ratio for this algorithm: Give for every fixed Δ an example of a graph where the ratio can actually be equal to $1/\Delta$.

Some remarks and advice (skip this paragraph if you want):
Specifically you must prove: If X is the greedy solution, and Y is an arbitrary independent set, then $|X| \geq |Y|/\Delta$. It should be rather easy to show $|X| \geq |Y|/(\Delta + 1)$, which is acceptable as a "weak" submission. A hint may be needed to prove the stronger ratio $|X| \geq |Y|/\Delta$: Assign every node not being in X arbitrarily to *one* neighbor in X, and then study where the nodes of Y may be located. Make sure that your Y is really an arbitrary independent set; you may not assume that Y is obtained in any special way.

Exercise 2.
A dominating set in a graph is a subset D of nodes such that every node is in D or has at least one neighbor in D. The Dominating Set problem asks to find a dominating set with a minimum number of nodes in a given graph.

Propose an approximation algorithm that returns a dominating set being only $O(\log \Delta)$ times larger than a minimum solution. But do not create a new algorithm from scratch. Instead make use of an already known approximability result and apply it properly. But be precise and specific when you describe the connection.