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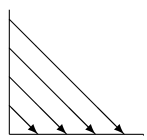
Enumerable sets and Hilbert's hotel

The great mathematician Hilbert once invented a hotel with an infinite number of rooms numbered $0, 1, 2, \ldots$. Of course this hotel fits all elements in N, the i:th element just takes room nr i. This hotel is very flexible: if the hotel is full and one more guest is coming, we just ask each person to move to the next room, leaving the first empty. If there are more guests coming, we just repeat the process. So, the hotel fits all elements in N + Bool, the elements coming from Bool takes room nr 0 and 1 and the elements from N take the remaining rooms, guest nr i takes room $i + 2$.

We can also fit the set N + N! The guest with the name $\mathrm{inl}(i)$ takes the room number $2 * i$ and the guest $\mathrm{inr}(j)$ takes the room $2 * j + 1$.

We can even fit the set N × N, we start to fill the first room with guest nr $(0, 0)$, then we fill the next two rooms with the two pairs $(0, 1)$ and $(1, 0)$ (the sum of the two components is 1), then the next three with the three pairs whose components have the sum 2. We continue this process, filling n rooms with the pairs whose sum is $n - 1$. In this way we sweep the entire quadrant taking one diagonal at a time (see figure). [1]



Another way to fit the set N × N is to let the guest (i, j) take the room with number $2^i * 3^j$. [2] This method leaves almost all rooms empty, but we have plenty of them!

We can ask ourselves if it is possible to fit *all* sets in this hotel? We have to be more precise. What does it mean that a set A fits the hotel? There are two requirement which must be fulfilled:

- Each element in A is assigned (at least) one room
- Each room has at most one element in A

We can express this mathematically in two alternative ways, one alternative is equivalent to the other.

The first alternative is to say that there exists a *surjective* function $g \in \mathrm{N} \to A$ mapping a room number to its guest, so $g(i)$ is the guest in the ith room. It is clear that the first property holds since the function is surjective. And the fact that g is a

[1] Try to write this function as a closed formula!
[2] How can we be sure that there will be at most one guest in each room?

function expresses that each i gives a unique function value, which is the second requirement.

The second alternative is to say that there exists a *total* and *injective* function $n \in A \to N$. The value $n(a)$ is the room number assigned to the guest a. The totality of the function means that it is defined for all its arguments, i.e. that all elements in the set A is assigned a room number. And the fact that the function is injective implies that different guests cannot get the same room number, i.e. each room has atmost one guest.

We can summarize this by the following definitions:

Definition 1 (enumerable) *A set* A *is* enumerable *if there exists a surjective function* $g \in N \to A$.

Definition 2 (enumerable) *A set* A *is* enumerable *if there exists a total and injective function* $n \in A \to N$.

More enumberable sets

It is easy to see that ListN is an enumerable set: we do this by constructing a total and injective function $f \in \text{ListN} \to N$ by

$$f(a_1.a_2.\dots.a_n) = 2^{a_1+1} * 3^{a_2+1} * \cdots * p_n{}^{a_n+1}$$

where p_n is the nth prime number. This function is injective: If

$$2^{a_1+1} * 3^{a_2+1} * \cdots * p_n{}^{a_n+1} = 2^{b_1+1} * 3^{b_2+1} * \cdots * p_n{}^{b_n+1}$$

then we must have that $a_i = b_i$ for all i by the unique factorization theorem (which says that there is only one way to factor a number into prime factors). [3]

A set which is not enumerable

The set $N \to \text{Bool}$ is not enumerable. This is proven by a so called *diagonalization argument*: Suppose we have an enumeration $f_0, f_1, \dots, f_i, \dots$ of all elements in the set. Then we construct a function d which cannot be an element in this enumeration by defining

$$f(i) = \neg f_i(i)$$

This function cannot be equal to any of the functions in the enumeration. If it were equal to for instance f_i then we would have that $f_i(j) = d(j)$ for all j. But $f_i(i) \neq d(i)$ by the construction of d.

[3]Why are the exponents on the form $a_i + 1$ instead of the simpler a_i?

To summarize the proof: We show that any enumeration of the set N $\rightarrow$ Bool fails to enumerate the entire set by constructing a function which is not in the enumeration.

As an example, suppose that the enumeration would be:

$$f_0 = \underline{\text{true}}, \text{false}, \text{false}, \text{false}, \text{false}, \text{true}, \text{true}, \text{false}, \ldots$$
$$f_1 = \text{true}, \underline{\text{false}}, \text{true}, \text{false}, \text{false}, \text{false}, \text{true}, \text{false}, \ldots$$
$$f_2 = \text{false}, \text{true}, \underline{\text{true}}, \text{true}, \text{false}, \text{false}, \text{false}, \text{true}, \ldots$$
$$f_3 = \text{false}, \text{false}, \text{true}, \underline{\text{true}}, \text{true}, \text{false}, \text{false}, \text{false}, \ldots$$

where we have underlined the diagonal elements $f_1(1), f_2(2), f_3(3), \ldots$.

To construct a function d which is different from the function f_0 we make sure it is different for the argument 0. And to make d different from f_1 we make it different for the argument 1. And so on.

Using a similar diagonalization argument, we can prove that the following sets are not enumerable:

- The set of real numbers.
- The set $P(\text{N})$ of all subsets of N.
- The set of all partial functions from N to $\{1\}$.