Lecture
Models of Computation
(DIT310, TDA184)

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Today

- Repetition (mainly). Please interrupt if you want to discuss something in more detail.
- Course evaluation.

Models of computation

- Actual hardware or programming languages: Lots of (irrelevant?) details.
- In this course: Idealised models of computation.
- PRF, RF.
- X.
- Turing machines.

The Church-Turing thesis

- The thesis:
  Every effectively calculable function on the positive integers can be computed using a Turing machine.
- Widely believed to be true.
- Many models are Turing-complete.

Comparing sets' sizes

- Injections, surjections, bijections.
- Countable (injection to $\mathbb{N}$), uncountable.
- Diagonalisation.
- Not every function is computable.

Inductively defined sets

An inductively defined set:

$$\frac{}{\textsf{nil} \in List\ A} \qquad \frac{x \in A \qquad xs \in List\ A}{\textsf{cons}\ x\ xs \in List\ A}$$

Primitive recursion:

$$\begin{aligned}
&listrec \in B \to (A \to List\ A \to B \to B) \to \\
&\qquad\quad List\ A \to B \\
&listrec\ n\ c\ \textsf{nil} \qquad\qquad\;\; = n \\
&listrec\ n\ c\ (\textsf{cons}\ x\ xs) = c\ x\ xs\ (listrec\ n\ c\ xs)
\end{aligned}$$

Inductively defined sets

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$$\frac{}{\mathsf{nil} \in \mathit{List}\ A} \qquad \frac{x \in A \qquad xs \in \mathit{List}\ A}{\mathsf{cons}\ x\ xs \in \mathit{List}\ A}$$

Pattern (with recursive constructor arguments last):

$$\begin{array}{l}
\mathit{drec} \in \text{One assumption per constructor} \to D \to A \\
\mathit{drec}\ f_1 \ldots f_k\ (c_1\ x_1 \ldots x_{n_1}) = \\
\quad f_1\ x_1 \ldots x_{n_1}\ (\mathit{drec}\ f_1 \ldots f_k\ x_{i_1}) \ldots (\mathit{drec}\ f_1 \ldots f_k\ x_{n_1}) \\
\vdots \\
\mathit{drec}\ f_1 \ldots f_k\ (c_k\ x_1 \ldots x_{n_k}) = \\
\quad f_k\ x_1 \ldots x_{n_k}\ (\mathit{drec}\ f_1 \ldots f_k\ x_{i_k}) \ldots (\mathit{drec}\ f_1 \ldots f_k\ x_{n_k})
\end{array}$$

Inductively defined sets

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$$\frac{}{\mathsf{nil} \in \mathit{List}\ A} \qquad \frac{x \in A \qquad xs \in \mathit{List}\ A}{\mathsf{cons}\ x\ xs \in \mathit{List}\ A}$$

Structural induction (P: a predicate on $\mathit{List}\ A$):

$$\frac{\begin{array}{c} P\ \mathsf{nil} \\ \forall x \in A.\ \forall\ xs \in \mathit{List}\ A.\ P\ xs \Rightarrow P\ (\mathsf{cons}\ x\ xs) \end{array}}{\forall xs \in \mathit{List}\ A.\ P\ xs}$$

Quiz

Write down the "type" of one of the higher-order primitive recursion schemes for the following inductively defined set:

$$\frac{n \in \mathbb{N}}{\textsf{leaf}\ n \in \mathit{Tree}} \qquad \frac{l, r \in \mathit{Tree}}{\textsf{node}\ l\ r \in \mathit{Tree}}$$

PRF

Sketch:

$$f\ () = \mathsf{zero}$$

$$f\ (x) = \mathsf{suc}\ x$$

$$f\ (x_1, ..., x_k, ..., x_n) = x_k$$

$$f\ (x_1, ..., x_n) = g\ (h_1\ (x_1, ..., x_n), ..., h_k\ (x_1, ..., x_n))$$

$$f\ (x_1, ..., x_n, \mathsf{zero}) \quad = g\ (x_1, ..., x_n)$$

$$f\ (x_1, ..., x_n, \mathsf{suc}\ x) = h\ (x_1, ..., x_n, f\ (x_1, ..., x_n, x), x)$$

PRF

- Abstract syntax (PRF_n).
- Denotational semantics:

$$[\![\_]\!] \in PRF_n \to (\mathbb{N}^n \to \mathbb{N})$$

- Big-step operational semantics:

$$f\ [\rho] \Downarrow\ n$$

PRF

- Strictly weaker than χ/Turing machines.
- Some χ-computable *total* functions
  are not PRF-computable,
  for instance the PRF semantics.

RF

- PRF $+$ minimisation.
- For $f \in \mathbb{N} \rightharpoonup \mathbb{N}$:
  f is RF-computable $\Leftrightarrow$
  f is χ-computable $\Leftrightarrow$
  f is Turing-computable.

X

$$
\begin{aligned}
e ::= &\ x \\
| &\ (e_1\ e_2) \\
| &\ \lambda x.\, e \\
| &\ \mathsf{C}(e_1, \ldots, e_n) \\
| &\ \mathbf{case}\ e\ \mathbf{of}\ \{\, \mathsf{C}_1(x_1, \ldots, x_n) \rightarrow e_1; \ldots \} \\
| &\ \mathbf{rec}\ x = e
\end{aligned}
$$

- Untyped, strict.
- $\mathbf{rec}\ x = e \ \approx\ \mathbf{let}\ x = e\ \mathbf{in}\ x$.

X

- Abstract syntax.
- Substitution of closed expressions.
- Big-step operational semantics, not total.
- The semantics as a partial function:

$$[\![\_]\!] \in CExp \rightharpoonup CExp$$

- Representation of inductively defined sets.

Representing expressions

Coding function:

$$
\begin{aligned}
&\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp} \\
&\ulcorner x \urcorner \quad\quad = \mathsf{Var}(\ulcorner x \urcorner) \\
&\ulcorner e_1\ e_2 \urcorner = \mathsf{Apply}(\ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner) \\
&\ulcorner \lambda x.\, e \urcorner = \mathsf{Lambda}(\ulcorner x \urcorner, \ulcorner e \urcorner) \\
&\vdots
\end{aligned}
$$

Representing expressions

Coding function:

$$
\begin{aligned}
&\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp} \\
&\ulcorner \mathsf{var}\ x \urcorner = \mathsf{const}\ \ulcorner \mathsf{Var} \urcorner\ (\mathsf{cons}\ \ulcorner x \urcorner\ \mathsf{nil}) \\
&\ulcorner \mathsf{apply}\ e_1\ e_2 \urcorner = \mathsf{const}\ \ulcorner \mathsf{Apply} \urcorner \\
&\qquad\qquad (\mathsf{cons}\ \ulcorner e_1 \urcorner\ (\mathsf{cons}\ \ulcorner e_2 \urcorner\ \mathsf{nil})) \\
&\ulcorner \mathsf{lambda}\ x\ e \urcorner = \mathsf{const}\ \ulcorner \mathsf{Lambda} \urcorner \\
&\qquad\qquad (\mathsf{cons}\ \ulcorner x \urcorner\ (\mathsf{cons}\ \ulcorner e \urcorner\ \mathsf{nil})) \\
&\vdots
\end{aligned}
$$

Representing expressions

Coding function:

$$
\begin{aligned}
&\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp} \\
&\ulcorner \mathsf{var}\ x \urcorner = \mathsf{const}\ \ulcorner \mathsf{Var} \urcorner\ (\mathsf{cons}\ \ulcorner x \urcorner\ \mathsf{nil}) \\
&\ulcorner \mathsf{apply}\ e_1\ e_2 \urcorner = \mathsf{const}\ \ulcorner \mathsf{Apply} \urcorner \\
&\qquad (\mathsf{cons}\ \ulcorner e_1 \urcorner\ (\mathsf{cons}\ \ulcorner e_2 \urcorner\ \mathsf{nil})) \\
&\ulcorner \mathsf{lambda}\ x\ e \urcorner = \mathsf{const}\ \ulcorner \mathsf{Lambda} \urcorner \\
&\qquad (\mathsf{cons}\ \ulcorner x \urcorner\ (\mathsf{cons}\ \ulcorner e \urcorner\ \mathsf{nil})) \\
&\vdots
\end{aligned}
$$

Alternative “type”:

$$\ulcorner \_ \urcorner \in \mathit{Exp}\ A \to \mathit{CExp}\ (\mathit{Rep}\ A)$$

$\mathit{Rep}\ A$: Representations of programs of type A.

Computability

- $f \in A \rightharpoonup B$ is χ-computable if

$$\exists\, e \in CExp.\ \forall\, a \in A.\, [\![e \ulcorner a \urcorner]\!] = \ulcorner f\ a \urcorner.$$

- Use reasonable coding functions:
  - Injective.
  - Computable. But how is this defined?
- X-decidable: $f \in A \to Bool$.
- X-semi-decidable:
  If $f\ a =$ false then $[\![e \ulcorner a \urcorner]\!]$ is undefined.

Some computable partial functions

- The semantics $[\![\_]\!] \in CExp \rightharpoonup CExp$:

$$\forall\, e \in CExp.\, [\![eval \ulcorner e \urcorner]\!] = \ulcorner [\![e]\!] \urcorner.$$

- The coding function $\ulcorner \_ \urcorner \in Exp \to CExp$:

$$\forall\, e \in Exp.\, [\![code \ulcorner e \urcorner]\!] = \ulcorner \ulcorner e \urcorner \urcorner.$$

- The "Terminates in n steps?" function $terminates\text{-}in \in CExp \times \mathbb{N} \to Bool$:

$$\forall\, p \in CExp \times \mathbb{N}.\\ [\![\underline{terminates\text{-}in} \ulcorner p \urcorner]\!] = \ulcorner terminates\text{-}in\ p \urcorner.$$

Some non-computable functions

The halting problem with self-application,

$$\begin{aligned}&\mathit{halts\text{-}self} \in \mathit{CExp} \rightarrow \mathit{Bool}\\&\mathit{halts\text{-}self}\ p =\\&\quad \mathbf{if}\ p \ulcorner p \urcorner\ \text{terminates}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false},\end{aligned}$$

can be reduced to the halting problem,

$$\begin{aligned}&\mathit{halts} \in \mathit{CExp} \rightarrow \mathit{Bool}\\&\mathit{halts}\ p = \mathbf{if}\ p\ \text{terminates}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false}.\end{aligned}$$

Some non-computable functions

Proof sketch:

- Assume that $\underline{halts}$ implements $halts$.
- Define $\underline{halts\text{-}self}$ in the following way:

$$\underline{halts\text{-}self} = \lambda\, p.\, \underline{halts}\ \mathsf{Apply}(p, code\ p)$$

- $\underline{halts\text{-}self}$ implements $halts\text{-}self$,

$$\forall\ e \in CExp.$$
$$\llbracket \underline{halts\text{-}self}\ \ulcorner e \urcorner \rrbracket = \ulcorner halts\text{-}self\ \ e \urcorner,$$

  because $\mathsf{Apply}(\ulcorner e \urcorner, code\ \ulcorner e \urcorner) \Downarrow \ulcorner e\ \ulcorner e \urcorner \urcorner$.

Some non-computable functions

The halting problem can be reduced to:

- Semantic equality:

$$\begin{aligned}&equal \in CExp \times CExp \to Bool\\&equal\ (e_1, e_2) =\\&\quad \textbf{if}\ [\![e_1]\!] = [\![e_2]\!]\ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false}\end{aligned}$$

- Pointwise equality of elements in $Fun = \{ (f, e) \mid f \in \mathbb{N} \to Bool, e \in Exp,$ e implements $f \}$:

$$\begin{aligned}&pointwise\text{-}equal \in Fun \times Fun \to Bool\\&pointwise\text{-}equal\ ((f, \_), (g, \_)) =\\&\quad \textbf{if}\ \forall\ n \in \mathbb{N}.\ f\ n = g\ n\ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false}\end{aligned}$$

Quiz

What is wrong with the following reduction of the halting problem to $\textit{pointwise-equal}$?

$$\underline{\textit{halts}} = \lambda\, p.\, \underline{\textit{not}}\ (\underline{\textit{pointwise-equal}}$$
$$\quad \mathsf{Lambda}(\ulcorner n \urcorner,$$
$$\qquad \mathsf{Apply}(\ulcorner \underline{\textit{terminates-in}} \urcorner,$$
$$\qquad\quad \mathsf{Const}(\ulcorner \mathsf{Pair} \urcorner,$$
$$\qquad\qquad \mathsf{Cons}(p, \mathsf{Cons}(\mathsf{Var}(\ulcorner n \urcorner), \mathsf{Nil}()))))$$
$$\quad \ulcorner \lambda\, \_.\, \mathsf{False}() \urcorner)$$

Bonus question: How can the problem be fixed?

Some non-computable functions

The halting problem can be reduced to:

- An optimal optimiser:

$$\begin{aligned}&\mathit{optimise} \in \mathit{CExp} \rightarrow \mathit{CExp}\\&\mathit{optimise}\ e =\\&\quad \text{some optimally small expression with}\\&\quad \text{the same semantics as } e\end{aligned}$$

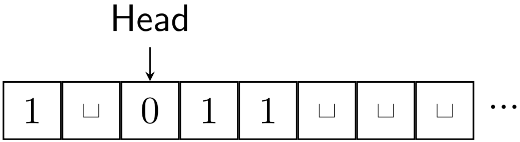
- Is a computable real number equal to zero?

$$\begin{aligned}&\mathit{is\text{-}zero} \in \mathit{Interval} \rightarrow \mathit{Bool}\\&\mathit{is\text{-}zero}\ x = \mathbf{if}\ [\![x]\!] = 0\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false}\end{aligned}$$

- Many other functions, see Rice's theorem.

Turing machines

- A tape with a head:



- A state.
- Rules.

Turing machines

- Abstract syntax.
- Small-step operational semantics.
- The semantics as a family of partial functions:

$$[\![\_]\!] \in \forall\, tm \in TM.\, List\ \Sigma_{tm} \rightharpoonup List\ \Gamma_{tm}$$

- Several variants:
  - Accepting states.
  - Possibility to stay put.
  - A tape without a left end.
  - Multiple tapes.
  - Only two symbols (plus $\sqcup$).

Turing-computability

- Representing inductively defined sets.
- Turing-computable partial functions.
- Turing-decidable languages.
- Turing-recognisable languages.

Some computable partial functions

- The semantics (uncurried):

  $$\{ (tm, xs) \mid tm \in TM, xs \in List\ \Sigma_{tm} \} \rightharpoonup List\ \Gamma_{tm}$$

  Self-interpreter/universal TM.

  (The definition of computability can be generalised so that it applies to dependent partial functions.)
- The χ semantics.

Some non-computable functions

- The Post correspondence problem
  (seen as a function to $Bool$).
- Is a context-free grammar ambiguous?

Equivalence

- The Turing machine semantics is also χ-computable.
- Partial functions $f \in \mathbb{N} \rightharpoonup \mathbb{N}$ are Turing-computable iff they are χ-computable.

Finally

- We have studied the concept of “computation”.
- How can “computation” be formalised?
  - To simplify our work: Idealised models.
  - The Church-Turing thesis.
- We have explored the limits of computation:
  - Programs that can run arbitrary programs.
  - A number of non-computable functions.

Good luck!