Relations

Everybody knows what a relation is. The mathematical definition of a relation is a way to formalize this concept so that it can be used in different—and much wider—contexts than we do in everyday language. To be a relative of somebody, or somebody's child, or to be in the same class as somebody, are examples of relations. But we even need to study such relations as being older than somebody, or for two numbers to leave the same remainder when divided by 12, or for a binary string $\mathbf{x}$ to have 1's in all the places where a string $\mathbf{y}$ has 1's.

A relation is defined on a set (which can consist of people, or numbers, or whatever) and it can be described by saying which pairs of elements are related to each other. The set of all ordered pairs of elements in a set is called the *cross product*, or *Cartesian product* of the set with itself, and the formal definition of a relation is as follows:

Definition 1 A *relation* $\mathcal{R}$ on a set M is a subset of the cross product $M \times M$.

Observe that the cross product consists of all *ordered* pairs of elements from M. This means that the pair (a, b) is *not* the same as (b, a). In other words, there is a "direction" in a relation $\mathcal{R}$ because the occurrence of (a, b) in $\mathcal{R}$—which means that a stands in the relation $\mathcal{R}$ to b—does *not* imply that b stands in the relation $\mathcal{R}$ to a.

Let $\mathcal{R}$ be the relation "less than" on the set $M = \{1, 2, 3\}$. Then $\mathcal{R}$ consists of the three pairs $(1, 2), (1, 3)$ and $(2, 3)$, which all are elements of $M \times M$. We can thus write

$$\mathcal{R} = \{(1, 2), (1, 3), (2, 3)\}.$$

Another way of indicating that 1 stands in the relation $\mathcal{R}$ to 2 is to write $1\mathcal{R}2$.

We mainly study four properties of relations:

Definition 2 A relation $\mathcal{R}$ on a set M is

1. *reflexive* if we have: *$x\mathcal{R}x$ for every $x \in M$*,
2. *symmetric* if for every pair $x, y \in M$ we have: *if $x\mathcal{R}y$ then $y\mathcal{R}x$*,
3. *antisymmetric* if for every pair $x, y \in M$ we have: *if $x\mathcal{R}y$ and $y\mathcal{R}x$ then $x = y$*,
4. *transitive* if for every triple $x, y, z \in M$ we have: *if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$*.

Observe that only reflexivity demands that any element in M must be related to some element in M. The empty relation, that is, the relation $\mathcal{R}$ where $x\mathcal{R}y$ doesn't hold for any pair (x, y), is symmetric, antisymmetric and transitive (but not reflexive, unless $M = \emptyset$). In particular, a relation can be neither symmetric nor antisymmetric, and a relation can be both.

The relation "=" has all four properties, and this is the only relation that does (prove this!).

Definition 3 A relation that is reflexive, symmetric and transitive is called an *equivalence relation*.

The most important equivalence relation for us will be the one used for modular arithmetic, that is, the relation that says two integers are equivalent if they leave the same rest when divided by a given integer.

Definition 4 Let n be a positive integer. The relation "congruence modulo n," denoted by $\equiv_n$, is defined on the set of all integers by

$$x \equiv_n y \iff x - y \text{ is divisible by } n.$$

We normally use the symbol $|$ to denote divisibility. More precisely, "$a|b$" is read as "a divides b." That a divides b means that b is an integer multiple of a.

Definition 5 Let $\sim$ be an equivalence relation on M and suppose $a \in M$. Then the *equivalence class of a* (with respect to M) is the set $[a] = \{x \in M \mid a \sim x\}$.

Definition 6 Let M be a set and $A_1, A_2, \ldots, A_n$ subsets of M. Then $\{A_1, A_2, \ldots, A_n\}$ is a *partition of M* if the following two conditions are satisfied:

(i) $A_1 \cup A_2 \cup \cdots \cup A_n = M$,

(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$.

A partition of a set M is thus a set of non-empty subsets of M so that each element in M belongs to precisely one of the subsets.

Theorem 7 *The equivalence classes of an equivalence relation on a set M form a partition of M.*

In other words, if R is an equivalence relation on a set M then each element of M belongs to precisely one equivalence class with respect to R.

For many equivalence relations it is clear what the equivalence classes are. If the relation is "being born in the same year," then each equivalence class consists of all those who are born in a certain year. For the equivalence relation "congruence modulo 4" (on $\mathbb{N}$) there are precisely four equivalence classes, namely $\{0, 4, 8, \ldots\}$, $\{1, 5, 9, \ldots\}$, $\{2, 6, 10, \ldots\}$, $\{3, 7, 11, \ldots\}$. It is often useful to choose a *representative* for each equivalence class. For congruence modulo 4 we would typically pick the numbers 0, 1, 2, 3, each of which is the smallest in its class. More generally, for the relation "congruence modulo n," we usually pick the numbers $0, 1, 2, \ldots, n-1$ as representatives for the equivalence classes. This is what most programming languages do in their modular arithmetic.

Definition 8 Let $\sim$ be an equivalence relation on a set M. A set $R \subseteq M$ is a *set of representatives* for $\sim$ if R contains precisely one element from each equivalence class.

Definition 9 A relation that is reflexive, antisymmetric and transitive is called a *partial order*.