Algorithms: Lecture 6

Chalmers University of Technology

Recap

- Greedy & Dynamic Programming
  - ➢extend solutions from smaller sub-instances incrementally to larger sub-instances, up to the full instance.
- Divide & Conquer
  - ➢follows the pattern of reducing a given problem to smaller instances of itself

  BUT

  - ➢it makes jumps rather than incremental steps.

Recap

- **Divide-and-conquer**
  - Split problem instance into a few significantly smaller sub-instances.
  - Sub-instances are solved, independently, in the same way (recursion).
  - Combine partial solutions to sub-instances into an overall solution.

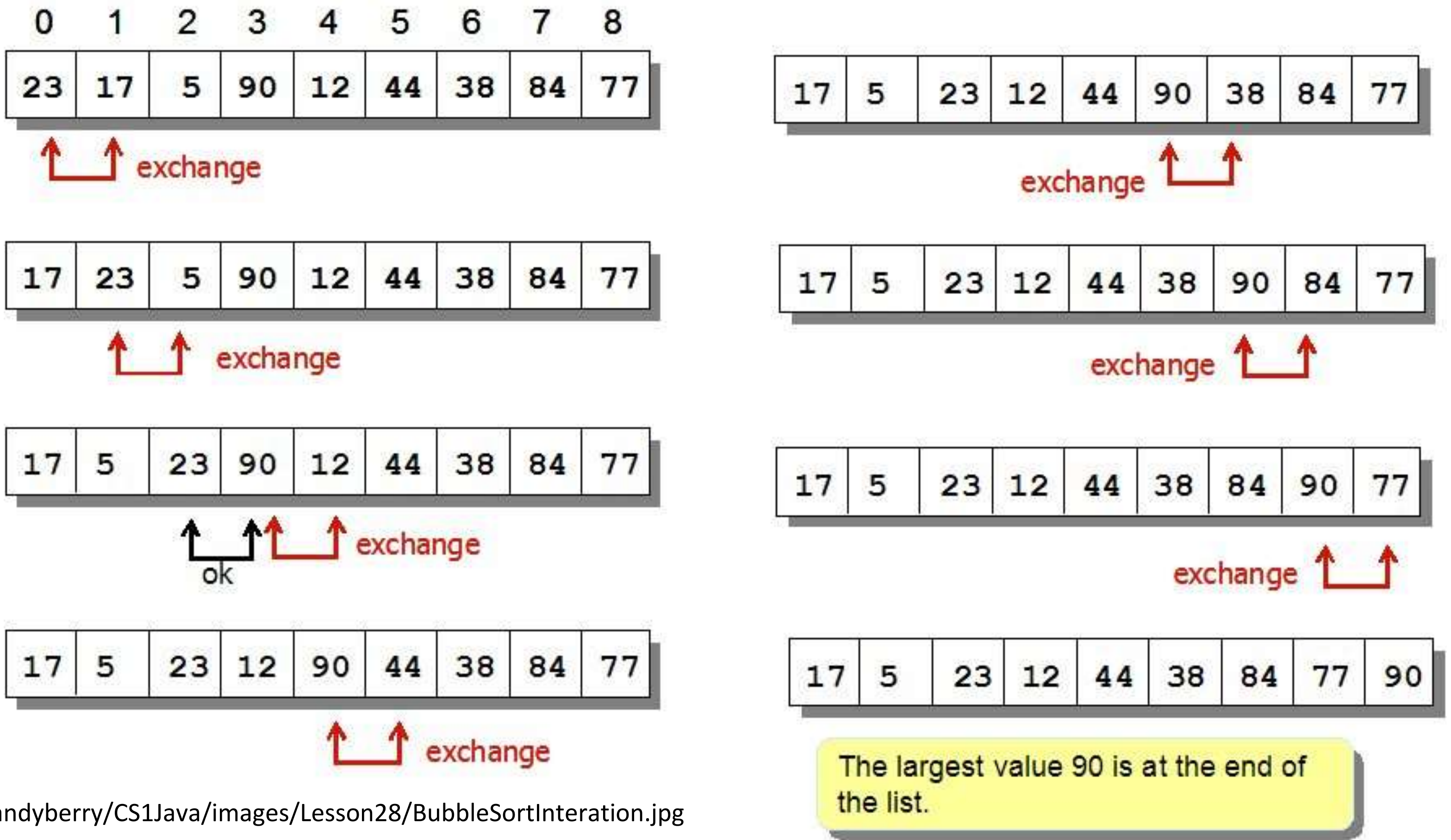
- **Most common usage**
  - Break up problem instance of size n into two equal parts of size ½n.
  - Solve two parts recursively.
  - Combine two solutions into overall solution in linear time.

Today’s Lecture

- **Important technique for Searching & Sorting**
  - Binary Search O(log n) (last lecture)
  - Brute force Sorting, e.g., Bubble sort : $O(n^2)$.
  - Divide-and-conquer: O(n log n).

Bubble Sort

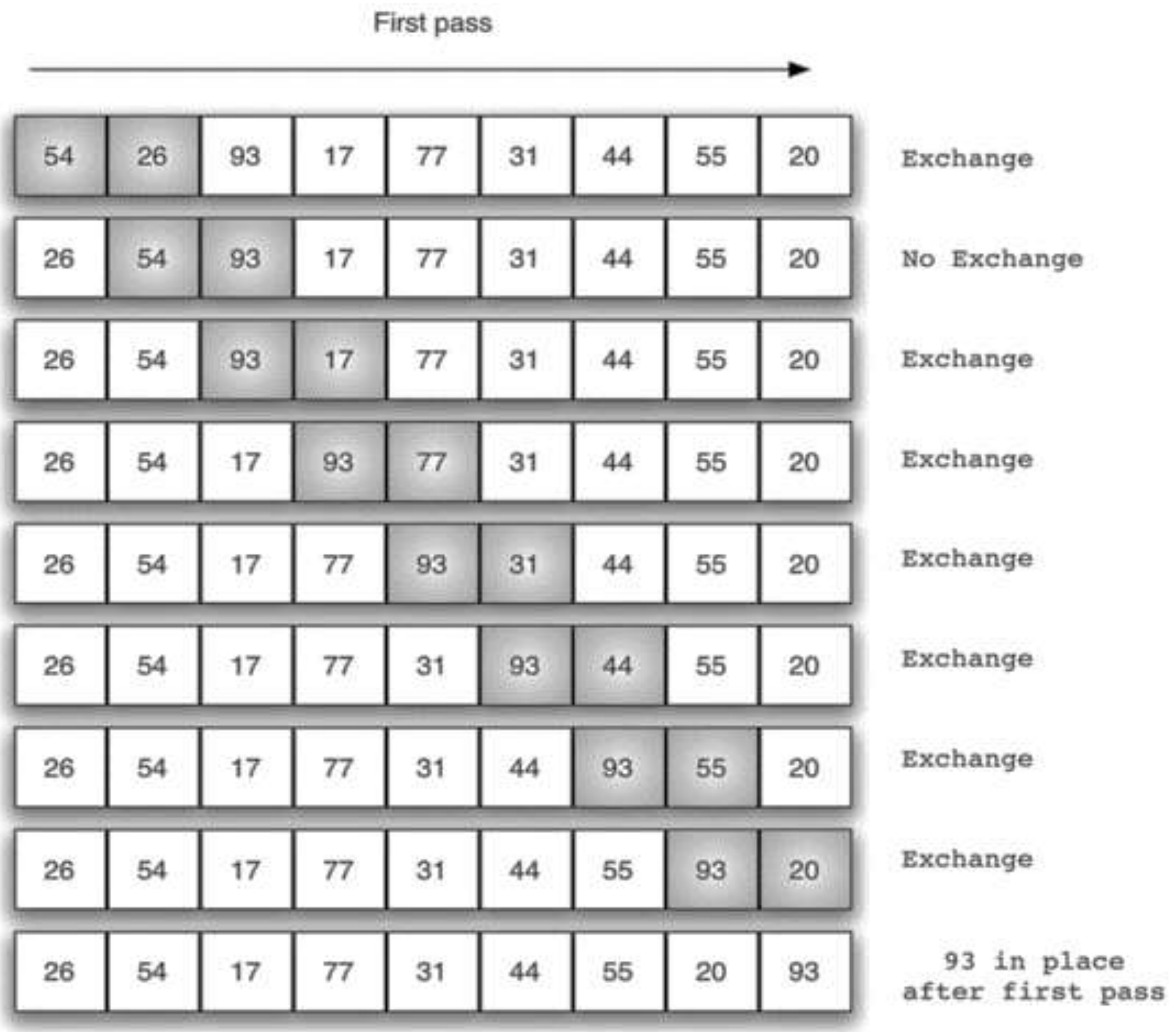
- Scan the list of elements from left to right
  - ➢whenever two neighbored elements are in the wrong order, swap them.



http://www.csit.parkland.edu/~mbrandyberry/CS1Java/images/Lesson28/BubbleSortInteration.jpg

Bubble Sort

- Every pass puts one elements to its proper place & reduces the instance size by 1
  - **n(n-1)/2**
    - **$O(n^2)$**



Bubble Sort

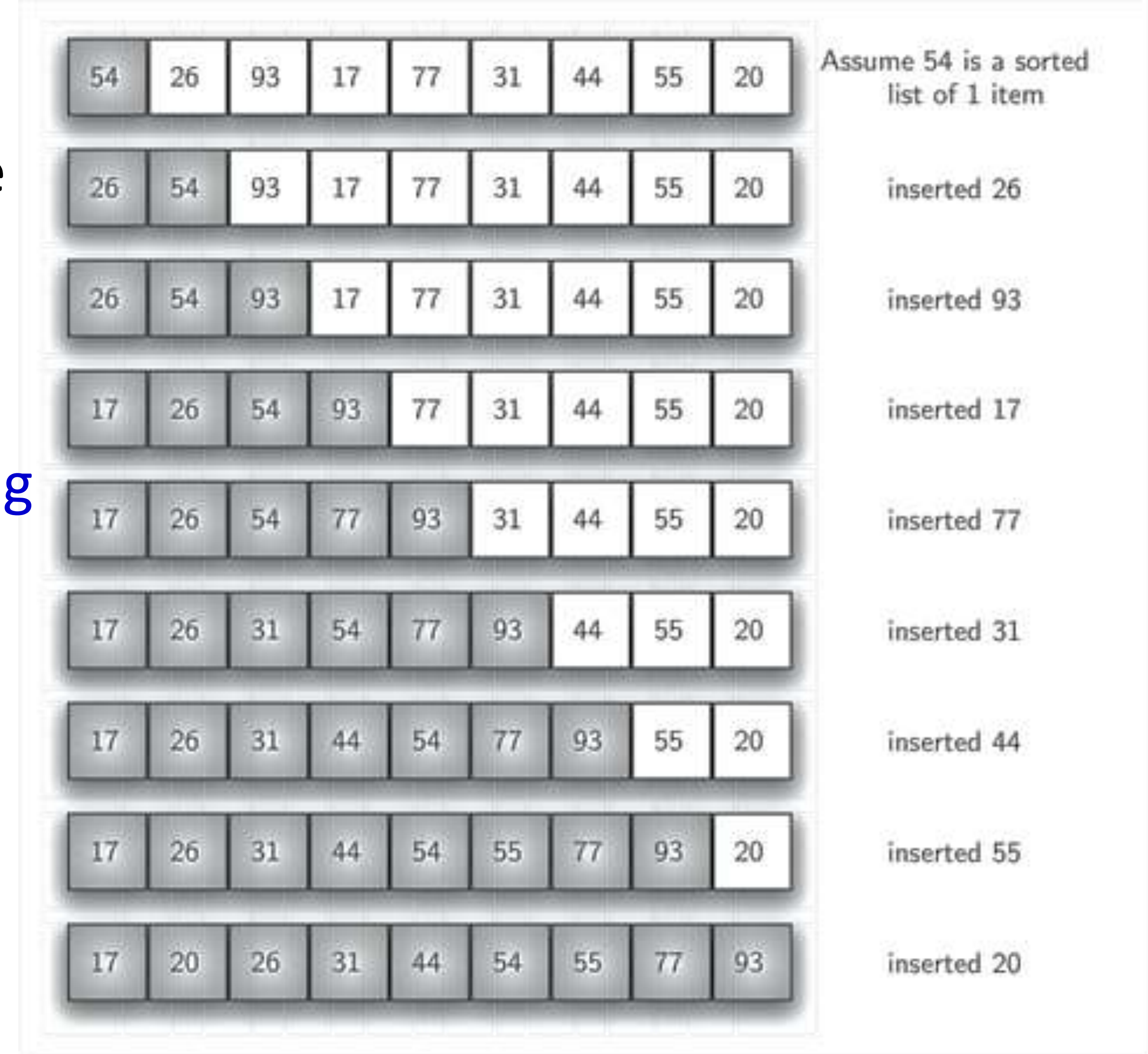
- **In place:** Needs only one array of size n for everything, except, possibly a few memory units.

- **Best:** In the first pass, if we don't have to make any swaps, that means that the array is sorted already.

- **Worst:** if many elements are far from their proper places(reverse order), because the algorithm moves them only step by step.
  - **Insertion Sort to overcome**

Insertion Sort

- After k rounds of Insertion Sort, the first k elements (k = 1,…,n) are sorted.
- To insert the (k + 1)st element we search for the correct position, using binary search.
  - ➢ **O(n log n)?**
  - ➢ we may be forced to move O(k) elements in the k-th round, giving again an overall time complexity of **$O(n^2)$.**



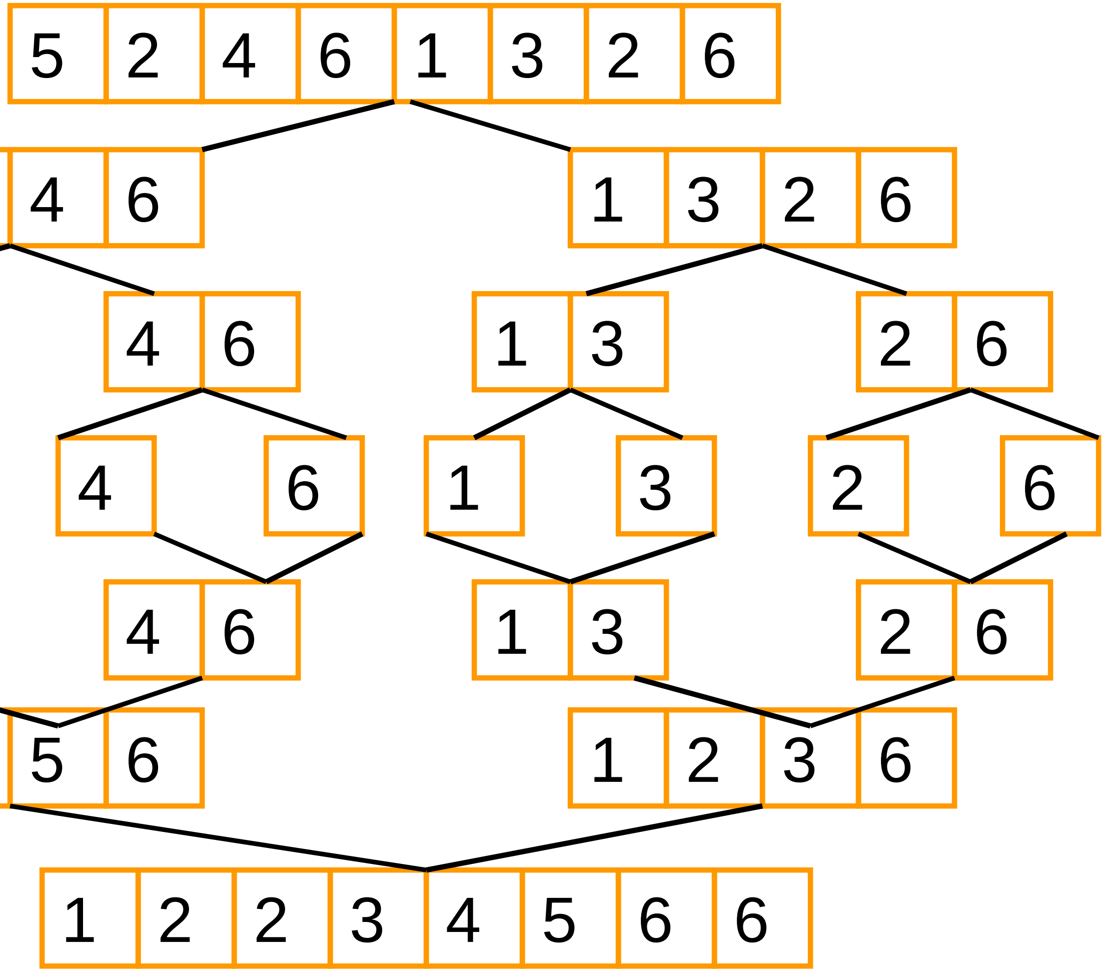
Insertion Sort

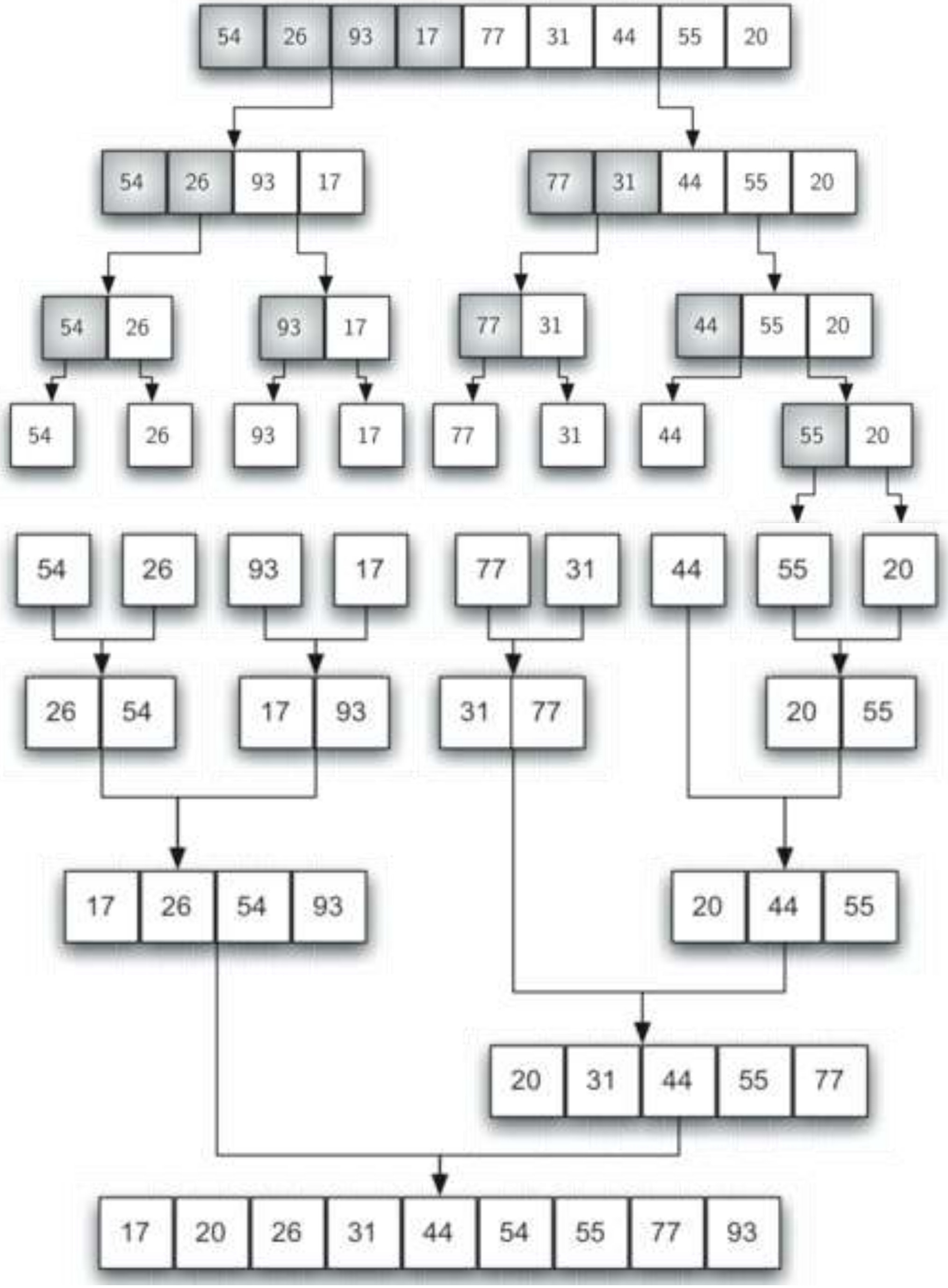
- **Idea:** We can avoid moving the elements
  - ➢ Insert an element in $O(1)$ time at a desired position using doubly linked list.
- But, how do we apply Binary Search without indices?
  - ➢ We have to apply linear search, and once again: **$O(n^2)$** for all n rounds.
- However, $O(n \log n)$ sorting algorithms are known, as we already know
  - ➢ **Divide-and-conquer**

Mergesort

- **How it works:**
  - arbitrary split the set into two halves
  - recursively sort the two halves separately
  - merge the two sorted halves
- Merging the two sorted halves involves *comparing* the elements to each other
  - ➢ scan both ordered sequences simultaneously and always move the currently smallest element to the next position in the result sequence, **implies O(n)**

Mergesort Example




54 26 93 17 77 31 44 55 20
54 26 93 17
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17 26 54 93
20 44 55
20 31 44 55 77
17 20 26 31 44 54 55 77 93

Time Complexity for Mergesort

Recurrence Relation:

- Let $T(n)$ be worst case time on a sequence of n elements
- If $n = 1$, then $T(n) = O(1)$ (constant)
- If $n > 1$, then $T(n) = 2\ T(n/2) + O(n)$
  - two sub-problems of size $n/2$ each that are solved recursively
  - $O(n)$ time to do the merge
- Solving the recurrence gives $T(n) = O(n \log n)$
- Remember general result from the Master Theorem
  - ➢ **$T(n) = aT(n/b) + cn^k$, and for $a = b^k$ it gives $O(n^k \log n)$**
  - ➢ For Mergesort, we have **$a=2$, $b=2$ and $k= 1$.**

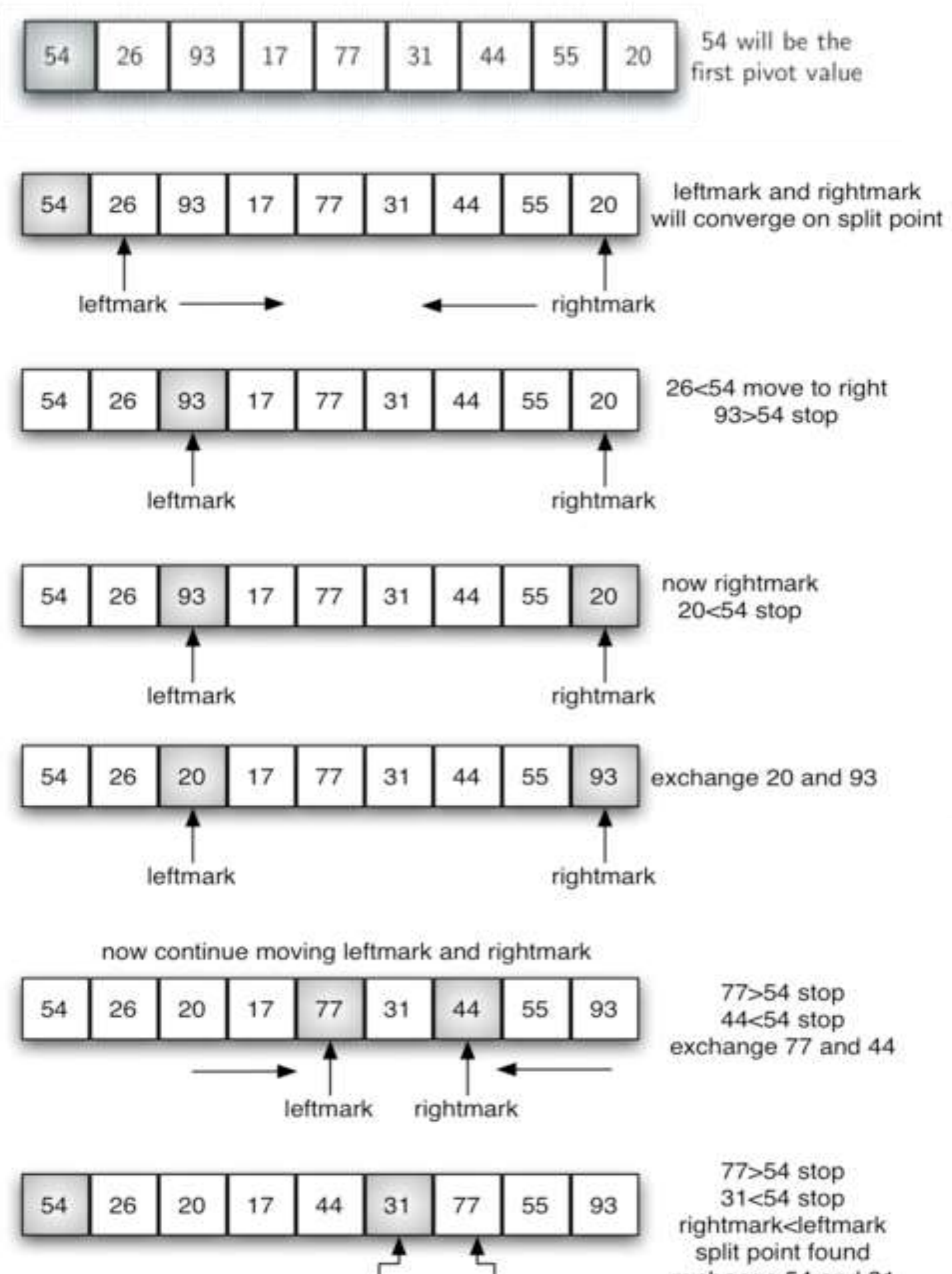
Caveat

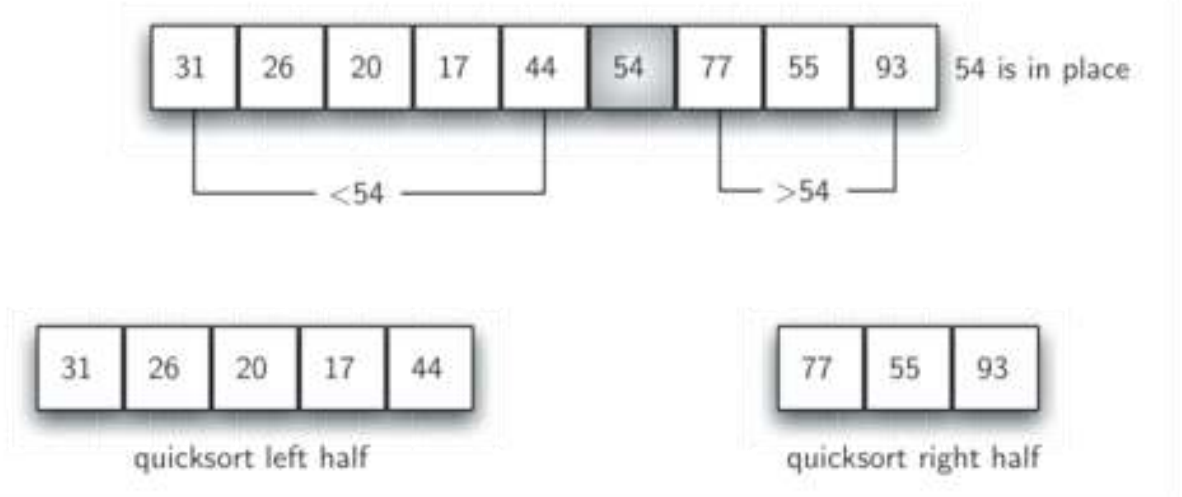
- **Simple Structure but** not the fastest sorting algorithm in practice
  - Too many copy operations (In every merging phase on every recursion level we have to move all elements of the merged subsets into a new array.)
- NOT *in place*
  - Additional memory required, while n could be very large in practice.
- Other alternatives with O(n log n) time
  - Different hidden constants factors
  - Hard to analyze theoretically
  - runtime experiments can figure out what is really faster.

- Remark: our Skyline algorithm from the previous lecture implicitly uses Mergesort to sort all endpoints of the rectangles.

Quicksort

- How it works:
  - choose one element to be the *pivot/ splitter,* called *p*
  - put all the elements < *p*, and those > *p* in two different subsets
  - recursively sort the two subsets and concatenate putting *p* in between
- *In place*
- Conquer phase trivial
- Implementation of Divide makes quick sort - quick


54 26 93 17 77 31 44 55 20
54 will be the first pivot value
leftmark and rightmark will converge on split point
leftmark
rightmark
26<54 move to right
93>54 stop
now rightmark
20<54 stop
54 26 20 17 77 31 44 55 93
exchange 20 and 93
now continue moving leftmark and rightmark
77>54 stop
44<54 stop
exchange 77 and 44
54 26 20 17 44 31 77 55 93
77>54 stop
31<54 stop
rightmark<leftmark
split point found


31 26 20 17 44 54 77 55 93
54 is in place
<54
>54
31 26 20 17 44
quicksort left half
77 55 93
quicksort right half

Time Complexity for Quicksort

- **Worst case:** the splitter is always the minimum or maximum element of the set, $O(n^2)$ is needed.

- Only **careful selection** of the splitter can guarantee the better bound.
- If the splitters would exactly halve the sets on every recursion level, we have our standard recursion:
  - $T(n) = 2\,T(n/2) + O(n)$
  - With solution: $T(n) = O(n \log n)$

Ideal Splitter for Quicksort

- **Rank of an element:** the position of this element if the set were already sorted.
- **Median:** Element with rank **n/2**

- **Computing Median?**
  - Sort and read off the element of rank n/2
  - **Stupid idea…** sorting is the actual problem for which we need to find out Median.
- **A splitter is selected at random!**
  - the worst case (rank nearly 1 or n) is very unlikely.
  - The splitters will mostly have ranks in the middle.
    - ➢ reasonably balanced partitions in two sets.
    - ➢ **O(n log n) time is needed on expectation.**
- In practice, chose three random elements and take their median as the splitter.

Center of a Point Set on the Line

Given: n points $x_1, \ldots, x_n$ on the real line.

Goal: Compute a point x so that the sum of distances to all given points $\sum_{i=1}^{n} |x - x_i|$ is minimized.

Distance: Walking or driving distance along the street, not the Euclidean distance.

Median of the given coordinates, not the **average**.

Selection and Median Finding

- **Given:** A set of n elements, where an order relation is defined, and an integer k.
- **Goal:** Output the element of rank k, that is, the kth smallest element.
- **Median:** Special case in Selection problem, **k := n/2**
  - often better suited as a **“typical"** value than the average, because it is robust against outliers.
- Wealth in a population
  - **Mean vs. Median**

Algorithm for Selection and Median Finding

- Choose: a random splitter s and compare all elements to *s* in *O(n)* time to get *rank r* of *s*.
- Decide:
  - If $r > k$ then throw out *s* and all elements *larger than s*. REPEAT
  - If $r < k$ then throw out s and all elements *smaller than s*, and set k := k-r REPEAT
  - **If r = k then return s.** STOP
- Time Complexity
  - Given the splitters are always in the middle: $T(n) = T(n/2) + O(n)$
  - ➢ ***$T(n) = aT(n/b) + cn^k$* , and for a < b^k it gives *$O(n^k)$***
  - ➢ We have **a=1, b=2** and **k= 1,** therefore we get: ***O(n)***
  - ➢ ***O(n) is expected time, worst case could still be $O(n^2)$***
  - ➢ **Fast Algorithm:** Intuition is that Selection needs much less information than Sorting.

Algorithm for Selection and Median Finding

- A deterministic divide-and-conquer, with ***O(n)*** time exists
  - Complicated
  - More importantly, the hidden constant in ***O(n)*** is large
  - Practically, random splitter algorithm is better

Information Flow and Optimal Time Bounds

- One of our primary goals is to make algorithms as fast as possible.
  - ➢How good are our time bounds for sorting and searching algorithms?
- **Searching:**
  - Find a specific element in an ordered set of size n
  - Comparisons counted as the elementary operations
- **Binary Search:** $\log_2 n$ comparisons of elements
- **Claim:** No other algorithm with comparisons as elementary operations can have a better worst-case bound.
  - ➢Claim holds due to the **information-theoretic** argument

Information Flow and Optimal Time Bounds

- **Binary Search:** $\log_2 n$ comparisons of elements
- **Claim:** No other algorithm with comparisons as elementary operations can have a better worst-case bound.
  - ➢ Claim holds due to the **information-theoretic** argument
  - How much information do we gain from our elementary operation?
    - ➢ Binary Answer ("smaller" or "larger"), splitting the set of possible results in two subsets for which either of the answers is true.
    - ➢ worst case: the answer is true for the larger subset, always
    - ➢ candidate solutions are reduced by a factor at most 2
    - ➢ n possible solutions in the beginning, any algorithm needs at least $\log_2 n$ comparisons in the worst case.
  - such arguments are used to define the lower bound on the execution of a computation based on the rate at which information can be accumulated.

Information Flow and Optimal Time Bounds

- **Sorting:** We have $O(n \log_2 n)$ algorithms.
- **Claim:** No other algorithm with comparisons as elementary operations can have a better worst-case bound.
  - The n elements can be ordered in n! possible ways, and only one of them is the correct order
  - Claim holds due to a similar reasoning as for Searching
    - ➢ Any sorting algorithm can be forced to use $\log_2 n!$ comparisons
    - ➢ Calculation shows that $\log_2 n!$ is $n \log_2 n$ subject to a constant factor

$$\log_2 n! = \sum_{k=1}^{n} \log_2 k \geq (n/2) \log_2(n/2).$$

Information Flow and Optimal Time Bounds

- **Selection Problem:** O(n)
- **Reasoning for Searching does not apply here**
  - O ($\log_2$ n) would be a very poor lower bound
- **O(n) is optimal**
  - No order known before hand, ALL the n elements needs to be read
  - Every change in the instance can change the result

Information Flow and Optimal Time Bounds

- **Faster Algorithms for special cases:**
- **Bucket Sort: O(m + n)**
  - ➢ n elements come from a fixed range of m different numbers.
- **O(n) sorting in lexicographic order**
  - Words defined over a fixed alphabet
  - Total length of the given words: n
- Do these two results contradict?
  - **NO!**
  - **?**
  - **Because…**