# Software Engineering using Formal Methods

## Propositional and (Linear) Temporal Logic

Wolfgang Ahrendt

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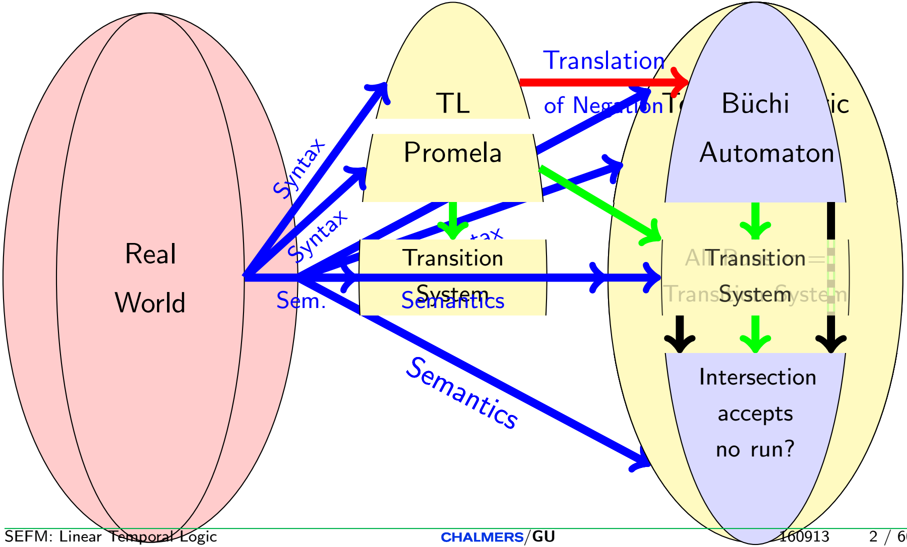
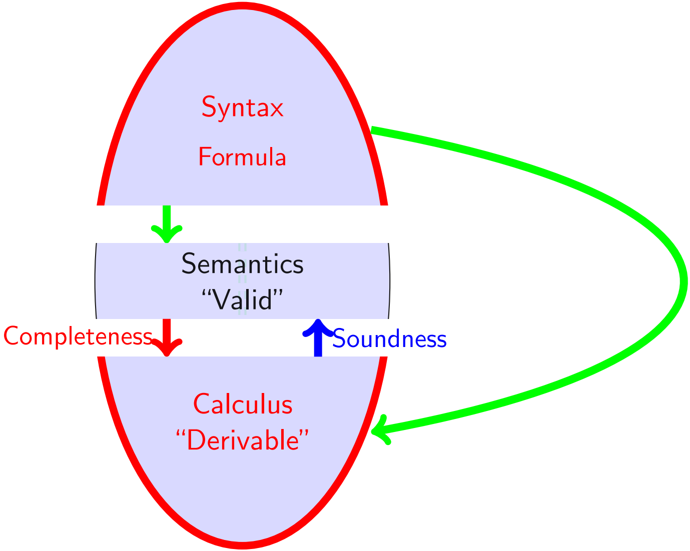
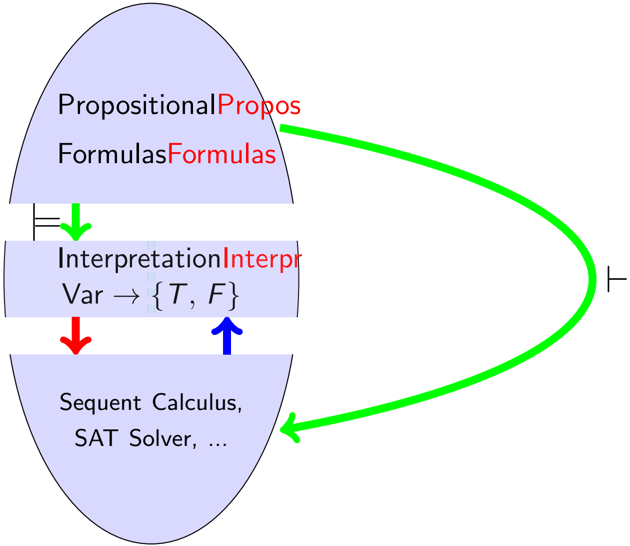
# Recapitulation: FormalisationFormalisation: Syntax, SemanticsFormalisation: Syntax, Semantics, ProvingFormal Verification: Model Checking



# The Big Picture: Syntax, Semantics, Calculus



# Simplest Case: Propositional Logic—Syntax



# Syntax of Propositional Logic

## Signature

A set of Propositional Variables $\mathcal{P}$ (with typical elements $p, q, r, \ldots$)

## Propositional Connectives

$\mathrm{true},\ \mathrm{false},\ \wedge,\ \vee,\ \neg,\ \rightarrow,\ \leftrightarrow$

## Set of Propositional Formulas $For_0$

- Truth constants $\mathrm{true}$, $\mathrm{false}$ and variables $\mathcal{P}$ are formulas
- If $\phi$ and $\psi$ are formulas then

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi, \quad \phi \leftrightarrow \psi$$

  are also formulas
- There are no other formulas (inductive definition)

# Remark on Concrete Syntax

<table>
<tr><th></th><th>Text book</th><th>Spin</th></tr>
<tr><td>Negation</td><td>$\neg$</td><td>!</td></tr>
<tr><td>Conjunction</td><td>$\land$</td><td>&&</td></tr>
<tr><td>Disjunction</td><td>$\lor$</td><td>||</td></tr>
<tr><td>Implication</td><td>$\rightarrow$, $\supset$</td><td>-></td></tr>
<tr><td>Equivalence</td><td>$\leftrightarrow$</td><td><-></td></tr>
</table>

We use mostly the textbook notation,
except for tool-specific slides, input files.

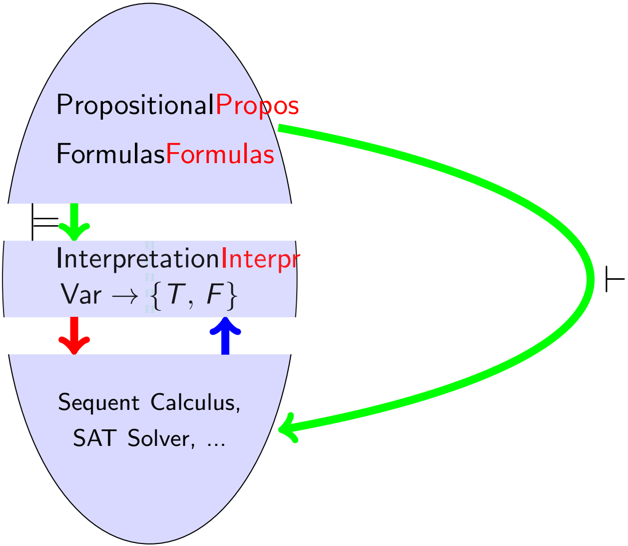
# Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \land r)) \lor p$ ✘
- $p \rightarrow (q\land)$ ✘
- $\mathit{false} \land (p \rightarrow (q \land r))$ ✔

# Simplest Case: Propositional Logic—Syntax



# Semantics of Propositional Logic

**Interpretation $\mathcal{I}$**

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

**Example**

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

| | $p$ | $q$ |
|---|---|---|
| $\mathcal{I}_1$ | $F$ | $F$ |
| $\mathcal{I}_2$ | $T$ | $F$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

# Semantics of Propositional Logic

**Interpretation $\mathcal{I}$**

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

**Valuation Function**

$val_{\mathcal{I}}$: Continuation of $\mathcal{I}$ on $For_0$

$$val_{\mathcal{I}} : For_0 \ \rightarrow \ \{T, F\}$$

$val_{\mathcal{I}}(\mathrm{true}) = T$
$val_{\mathcal{I}}(\mathrm{false}) = F$
$val_{\mathcal{I}}(p_i) = \mathcal{I}(p_i)$

(cont'd next page)

# Semantics of Propositional Logic (Cont'd)

**Valuation function (Cont'd)**

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \rightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \leftrightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = val_{\mathcal{I}}(\psi) \\ F & otherwise \end{cases}$$

# Valuation Examples

**Example**

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

| | $p$ | $q$ |
|---|---|---|
| $\mathcal{I}_1$ | $F$ | $F$ |
| $\mathcal{I}_2$ | $T$ | $F$ |

$$\ldots$$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$
\begin{aligned}
val_{\mathcal{I}_2}(\, p \rightarrow (q \rightarrow p)\,) &= T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T \\
val_{\mathcal{I}_2}(p) &= \mathcal{I}_2(p) = T \\
val_{\mathcal{I}_2}(\, q \rightarrow p\,) &= T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T \\
val_{\mathcal{I}_2}(q) &= \mathcal{I}_2(q) = F
\end{aligned}
$$

# Semantic Notions of Propositional Logic

Let $\phi \in For_0$, $\Gamma \subseteq For_0$

**Definition (Satisfying Interpretation, Consequence Relation)**

$\mathcal{I}$ satisfies $\phi$ (write: $\mathcal{I} \models \phi$) iff $val_{\mathcal{I}}(\phi) = T$

$\phi$ follows from $\Gamma$ (write: $\Gamma \models \phi$) iff for all interpretations $\mathcal{I}$:

$$\text{If } \mathcal{I} \models \psi \text{ for all } \psi \in \Gamma, \text{ then also } \mathcal{I} \models \phi$$

**Definition (Satisfiability, Validity)**

A formula is satisfiable if it is satisfied by some interpretation.
If every interpretation satisfies $\phi$ (write: $\models \phi$) then $\phi$ is called valid.

# Semantics of Propositional Logic: Examples

**Formula (same as before)**

$$p \rightarrow (q \rightarrow p)$$

Is this formula valid?

$$\models p \rightarrow (q \rightarrow p)\ ?$$

# Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔
Satisfying Interpretation? $\mathcal{I}(p) = T$, $\mathcal{I}(q) = T$
Other Satisfying Interpretations? ✘
Therefore, not valid!

$$p \land ((\neg p) \lor q) \models q \lor r$$

Does it hold? Yes. Why?

# An Exercise in Formalisation

```
byte n;
active proctype [2] P() {
  n = 0;
  n = n + 1
}
```

Can we characterise the states of P propositionally?

Find a propositional formula $\phi_{\mathrm{P}}$ which is true if and only if it describes a possible state of P.

$$\phi_{\mathrm{P}} := \left( \begin{array}{l} ((PC0_3 \wedge \neg PC0_4 \wedge \neg PC0_5) \vee \cdots)\wedge \\ ((\neg PC0_5 \wedge \neg PC1_5) \longrightarrow (\neg N_0 \wedge \cdots \wedge \neg N_7)) \wedge \cdots \end{array} \right)$$

# An Exercise in Formalisation

```
1 byte n;
2 active proctype [2] P() {
3   n = 0;
4   n = n + 1
5 }
```

$\mathcal{P}$ : $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

- The variable `n` is represented by eight bits, all values possible
- A process cannot be at two positions at the same time
- If neither process 0 nor process 1 are at position 5, then `n` is zero
- ...

$$\phi_{\mathrm{P}} := \left( \begin{array}{l} ((PC0_3 \land \neg PC0_4 \land \neg PC0_5) \lor \cdots) \land \\ ((\neg PC0_5 \land \neg PC1_5) \implies (\neg N_0 \land \cdots \land \neg N_7)) \land \cdots \end{array} \right)$$

# Is Propositional Logic Enough?

Can design for a program $P$ a formula $\Phi_P$ describing all reachable states

For a given property $\Psi$ the consequence relation

$$\Phi_p \models \Psi$$

holds when $\Psi$ is true in any possible state reachable in any run of $P$

**But How to Express Properties Involving State Changes?**

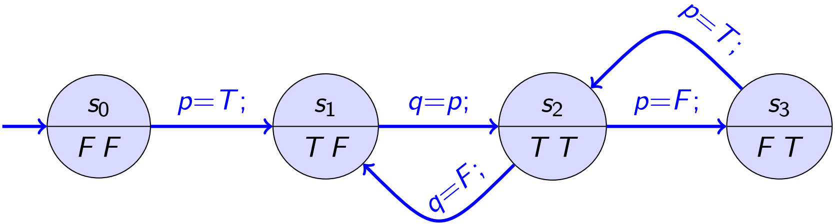
In any run of a program $P$

- $n$ will become greater than 0 eventually?
- $n$ changes its value infinitely often

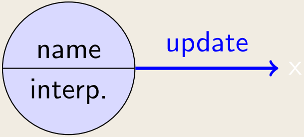
etc.

$\Rightarrow$ Need a more expressive logic: (Linear) Temporal Logic

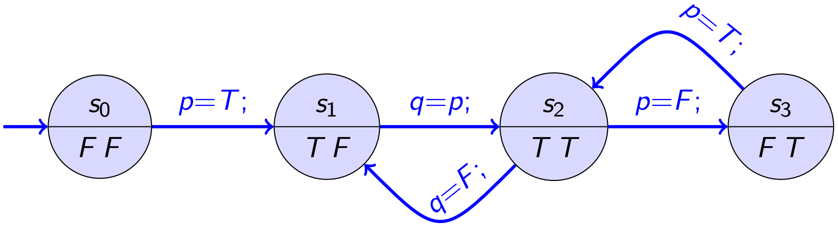
# Transition systems (aka Kripke Structures)




**Notation**



# Transition systems (aka Kripke Structures)




- Each state $s_i$ has its own propositional interpretation $\mathcal{I}_i$
  - Convention: list interpretation of variables in lexicographic order
- Computations, or runs, are *infinite* paths through states
  - Intuitively 'finite' runs modelled by looping on last state
- How to express (for example) that $p$ changes its value infinitely often in each run?

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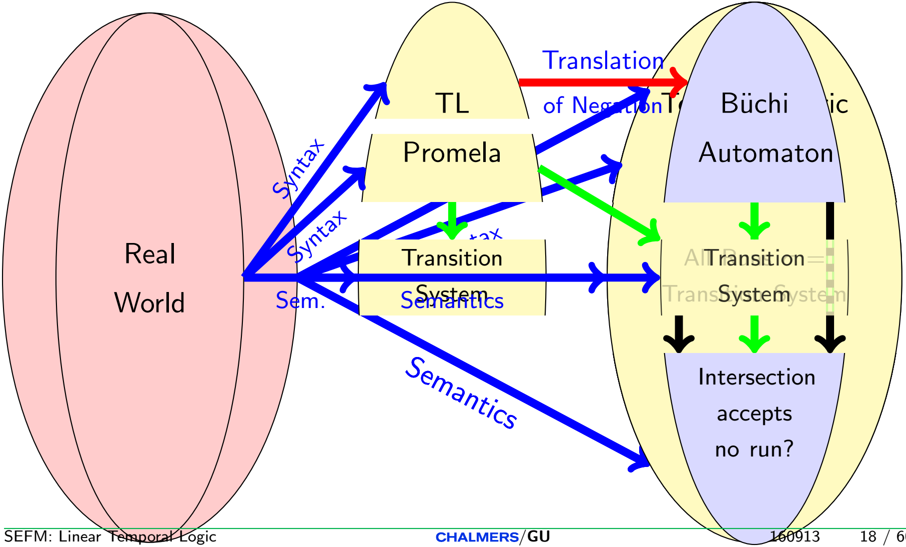


# Linear Temporal Logic—Syntax

An extension of propositional logic that
allows to specify properties of all runs

## Syntax

Based on propositional signature and syntax

Extension with three connectives:

**Always** If $\phi$ is a formula, then so is $\Box\phi$

**Eventually** If $\phi$ is a formula, then so is $\Diamond\phi$

**Until** If $\phi$ and $\psi$ are formulas, then so is $\phi\mathcal{U}\psi$

## Concrete Syntax

| | text book | SPIN |
|---|---|---|
| Always | $\Box$ | [] |
| Eventually | $\Diamond$ | <> |
| Until | $\mathcal{U}$ | U |

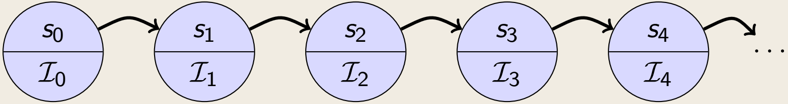
# Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- $p$
- $\mathrm{false}$
- $p \rightarrow q$
- $\Diamond p$
- $\Box q$
- $\Box\Diamond\Box(p \rightarrow q)$
- $(\Box p) \rightarrow ((\Diamond p) \vee \neg q)$
- $p\,\mathcal{U}(\Box q)$

# Temporal Logic—Semantics

**A run $\sigma$ is an infinite chain of states**




$\mathcal{I}_j$ propositional interpretation of variables in state $s_j$
Write more compactly $s_0\, s_1\, s_2\, s_3 \ldots$

If $\sigma = s_0\, s_1 \cdots$, then $\sigma|_i$ denotes the suffix $s_i\, s_{i+1} \cdots$ of $\sigma$.

# Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

**Definition (Validity Relation)**

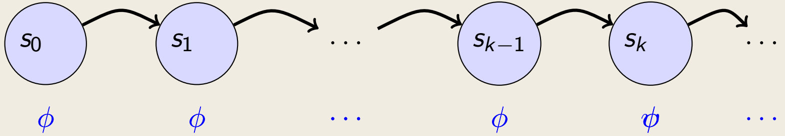
Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

| | | |
|---|---|---|
| $\sigma \models p$ | iff | $\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$. |
| $\sigma \models \neg\phi$ | iff | not $\sigma \models \phi$ (write $\sigma \not\models \phi$) |
| $\sigma \models \phi \wedge \psi$ | iff | $\sigma \models \phi$ and $\sigma \models \psi$ |
| $\sigma \models \phi \vee \psi$ | iff | $\sigma \models \phi$ or $\sigma \models \psi$ |
| $\sigma \models \phi \rightarrow \psi$ | iff | $\sigma \not\models \phi$ or $\sigma \models \psi$ |

Temporal connectives?

# Temporal Logic—Semantics (Cont'd)

**Run $\sigma$**




**Definition (Validity Relation for Temporal Connectives)**

Given a run $\sigma = s_0\, s_1 \cdots$

| | | |
|---|---|---|
| $\sigma \models \Box\phi$ | iff | $\sigma\|_k \models \phi$ for all $k \geq 0$ |
| $\sigma \models \Diamond\phi$ | iff | $\sigma\|_k \models \phi$ for some $k \geq 0$ |
| $\sigma \models \phi\,\mathcal{U}\,\psi$ | iff | $\sigma\|_k \models \psi$ for some $k \geq 0$, and $\sigma\|_j \models \phi$ for all $0 \leq j < k$ |
| | | (if $k = 0$ then $\phi$ needs never hold) |

# Safety and Liveness Properties

## Safety Properties

- Always-formulas called safety properties:
  "something bad never happens"
- Let `mutex` ("mutual exclusion") be a variable that is true when two processes do not access a critical resource at the same time
- $\Box$ `mutex` expresses that simultaneous access never happens

## Liveness Properties

- Eventually-formulas called liveness properties:
  "something good happens eventually"
- Let `s` be variable that is true when a process delivers a service
- $\Diamond$ `s` expresses that service is eventually provided

# Complex Properties

**What does this mean?Infinitely Often**

$$\sigma \models \Box\Diamond\phi$$

"During run $\sigma$ the formula $\phi$ becomes true infinitely often"

# Validity of Temporal Logic

**Definition (Validity)**

$\phi$ is valid, write $\models \phi$, iff $\sigma \models \phi$ for all runs $\sigma = s_0\, s_1 \cdots$.

Recall that each run $s_0\, s_1 \cdots$ essentially is an infinite sequence of interpretations $\mathcal{I}_0\, \mathcal{I}_1 \cdots$

**Representation of Runs**

Can represent a set of runs as a sequence of propositional formulas:

- $\phi_0\, \phi_1, \cdots$ represents all runs $s_0\, s_1 \cdots$ such that $s_i \models \phi_i$ for $i \geq 0$

# Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

$\Box\phi \rightarrow \phi$ $\qquad (\neg\Box\phi) \leftrightarrow (\Diamond\neg\phi)$ $\qquad \Diamond\phi \leftrightarrow (\mathrm{true}\,\mathcal{U}\phi)$

All are valid! (proof is exercise)

- $\Box$ is reflexive
- $\Box$ and $\Diamond$ are dual connectives
- $\Box$ and $\Diamond$ can be expressed with only using $\mathcal{U}$

# Transition Systems: Formal Definition

**Definition (Transition System)**

A transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$ is composed of a set of states $S$, a set $\emptyset \neq Ini \subseteq S$ of initial states, a transition relation $\delta \subseteq S \times S$, and a labeling $\mathcal{I}$ of each state $s \in S$ with a propositional interpretation $\mathcal{I}_s$.

**Definition (Run of Transition System)**

A run of $\mathcal{T}$ is a sequence of states $\sigma = s_0\, s_1 \cdots$ such that $s_0 \in Ini$ and for all $i$ is $s_i \in S$ as well as $(s_i, s_{i+1}) \in \delta$.

# Temporal Logic—Semantics (Cont'd)

Extension of validity of temporal formulas to transition systems:

**Definition (Validity Relation)**

Given a transition system $\mathcal{T} = (S, \mathit{Ini}, \delta, \mathcal{I})$, a temporal formula $\phi$ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs $\sigma$ of $\mathcal{T}$.

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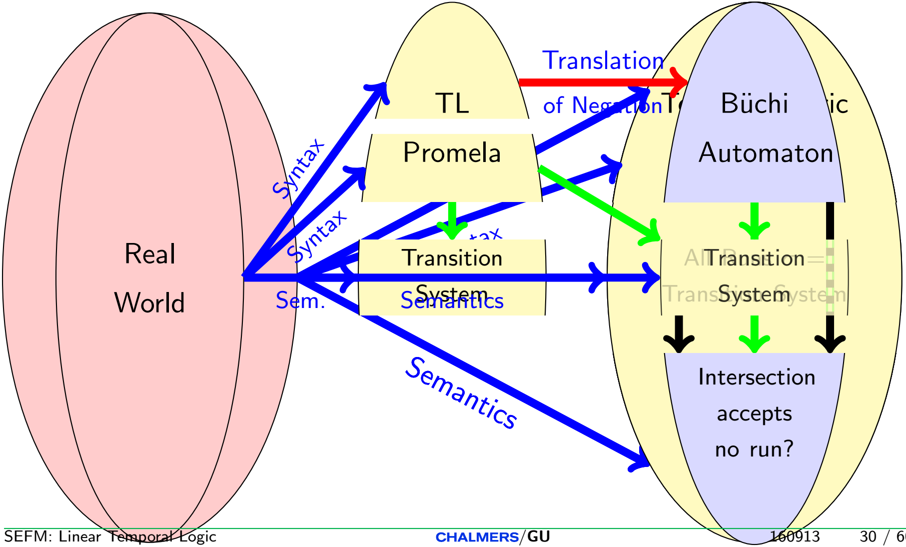


# $\omega$-Languages

Given a finite alphabet (vocabulary) $\Sigma$

An $\omega$-word $w \in \Sigma^{*\omega}$ is a n infinite sequence

$$w = a_o \cdots a_{nk} \cdots$$

with $a_i \in \Sigma, i \in \{0, \ldots, n\}\mathbb{N}$

$\mathcal{L}^\omega \subseteq \Sigma^{*\omega}$ is called a n $\omega$-language

# Büchi Automaton

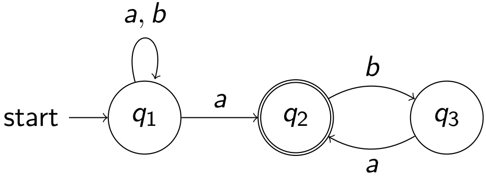
**Definition (Büchi Automaton)**

A (non-deterministic) Büchi automaton over an alphabet $\Sigma$ consists of a

- finite, non-empty set of locations $Q$
- a non-empty set of initial/start locations $I \subseteq Q$
- a set of accepting locations $F = \{F_1, \ldots, F_n\} \subseteq Q$
- a transition relation $\delta \subseteq Q \times \Sigma \times Q$

**Example**

$\Sigma = \{a, b\}, Q = \{q_1, q_2, q_3\}, I = \{q_1\}, F = \{q_2\}$



# Büchi Automaton—Executions and Accepted Words

**Definition (Execution)**

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton over alphabet $\Sigma$.
An execution of $\mathcal{B}$ is a pair $(w, v)$, with

- $w = a_o \cdots a_k \cdots \in \Sigma^\omega$
- $v = q_o \cdots q_k \cdots \in Q^\omega$

where $q_0 \in I$, and $(q_i, a_i, q_{i+1}) \in \delta$, for all $i \in \mathbb{N}$

**Definition (Accepted Word)**

A Büchi automaton $\mathcal{B}$ accepts a word $w \in \Sigma^\omega$, if there exists an execution $(w, v)$ of $\mathcal{B}$ where some accepting location $f \in F$ appears infinitely often in $v$.

# Büchi Automaton—Language

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton, then

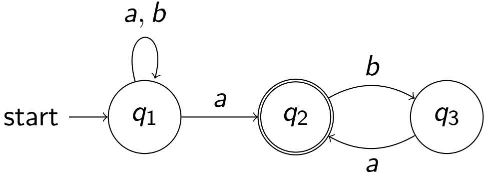
$$\mathcal{L}^\omega(\mathcal{B}) = \{w \in \Sigma^\omega | w \in \Sigma^\omega \text{ is an accepted word of } \mathcal{B}\}$$

denotes the $\omega$-language recognised by $\mathcal{B}$.

An $\omega$-language for which an accepting Büchi automaton exists is called $\omega$-regular language.

# Example, $\omega$-Regular Expression

Which language is accepted by the following Büchi automaton?




Solution: $(a+b)^*(ab)^\omega$ [NB: $(ab)^\omega = a(ba)^\omega$]

$\omega$-regular expressions similar to standard regular expression

- $ab$ $a$ **followed by** $b$
- $a+b$ $a$ **or** $b$
- $a^*$ arbitrarily, but finitely often $a$
- **new:** $a^\omega$ infinitely often $a$

# Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

**Theorem (Decidability)**

*It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton $\mathcal{B}$ is empty.*

**Theorem (Closure properties)**

*The set of $\omega$-regular languages is closed with respect to intersection, union and complement:*

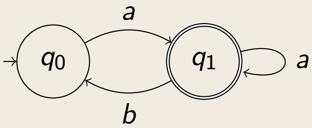
- *if $\mathcal{L}_1, \mathcal{L}_2$ are $\omega$-regular then $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are $\omega$-regular*
- *$\mathcal{L}$ is $\omega$-regular then $\Sigma^\omega \setminus \mathcal{L}$ is $\omega$-regular*

**But in contrast to regular finite automata:**

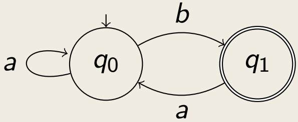
Non-deterministic Büchi automata are strictly more expressive than deterministic ones.

# Büchi Automata—More Examples

**Language:** $a(a + ba)^\omega$




**Language:** $(a^* ba)^\omega$



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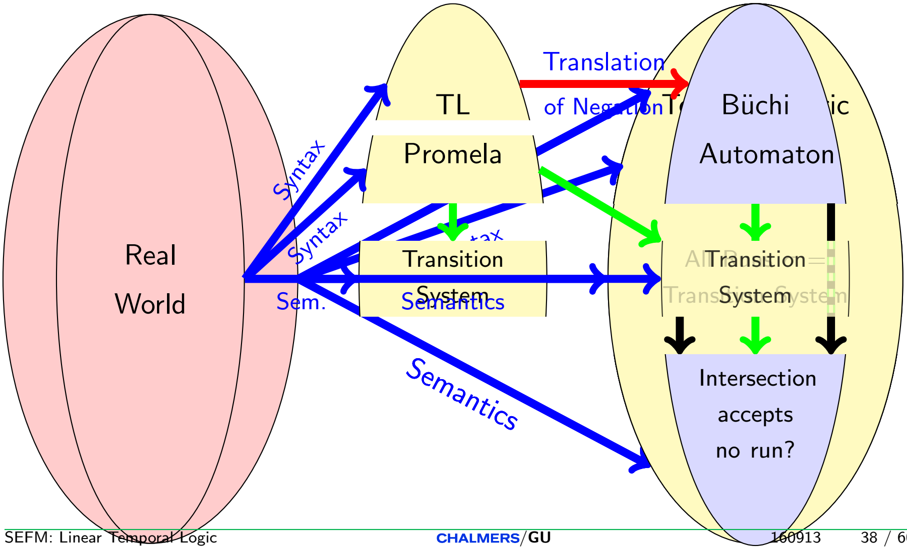


# Linear Temporal Logic and Büchi Automata

LTL and Büchi Automata are connected

Recall

**Definition (Validity Relation)**

Given a transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$, a temporal formula $\phi$ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs $\sigma$ of $\mathcal{T}$.

A run of the transition system is an infinite sequence of interpretations $\mathcal{I}$.

**Intended Connection**

Given an LTL formula $\phi$:

Construct a Büchi automaton accepting exactly those runs (infinite sequences of interpretations) that satisfy $\phi$.

# Encoding an LTL Formula as a Büchi Automaton

$\mathcal{P}$ set of propositional variables, e.g., $\mathcal{P} = \{r, s\}$

Suitable alphabet $\Sigma$ for Büchi automaton?

A state transition of Büchi automaton must represent an interpretation

Choose $\Sigma$ to be the set of all interpretations over $\mathcal{P}$, encoded as $2^{\mathcal{P}}$

**Example**

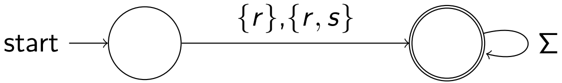
$\Sigma = \big\{\emptyset, \{r\}, \{s\}, \{r, s\}\big\}$

$$I_\emptyset(r) = F, I_\emptyset(s) = F, I_{\{r\}}(r) = T, I_{\{r\}}(s) = F, \ldots$$

# Büchi Automaton for LTL Formula By Example

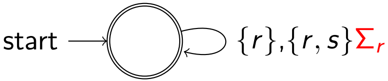
**Example (Büchi automaton for formula $r$ over $\mathcal{P} = \{r, s\}$)**

A Büchi automaton $\mathcal{B}$ accepting exactly those runs $\sigma$ satisfying $r$




In the first state $s_0$ (of $\sigma$) at least $r$ must hold, the rest is arbitrary

**Example (Büchi automaton for formula $\Box r$ over $\mathcal{P} = \{r, s\}$)**

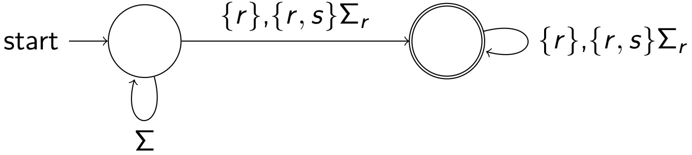



$$\Sigma_r := \{I | I \in \Sigma, r \in I\}$$

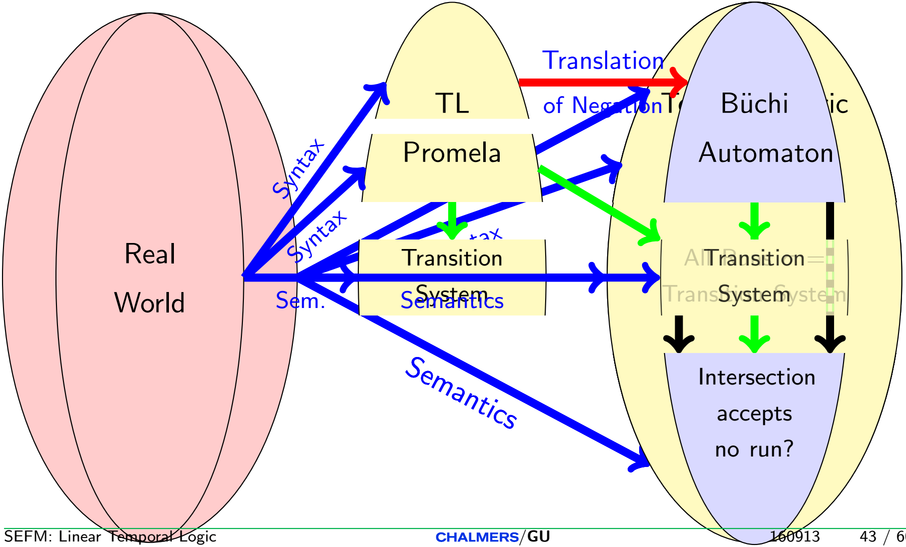
In *all* states $s$ (of $\sigma$) at least $r$ must hold

# Büchi Automaton for LTL Formula By Example

**Example (Büchi automaton for formula $\Diamond\Box r$ over $\mathcal{P} = \{r, s\}$)**



# Recapitulation: FormalisationFormalisation: Syntax, SemanticsFormalisation: Syntax, Semantics, ProvingFormal Verification: Model Checking



# Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system $\mathcal{T}$ (e.g., derived from a PROMELA program).

Verification task: is the LTL formula $\phi$ satisfied in all runs of $\mathcal{T}$, i.e.,

$$\mathcal{T} \models \phi \quad ?$$

Temporal model checking with SPIN: Topic of next lecture

Today: Basic principle behind SPIN model checking

# SPIN Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for negation of formula $\phi$
3. If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) = \emptyset$$

   then $\mathcal{T} \models \phi$ holds.

   If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) \neq \emptyset$$

   then each element of the set is a counterexample for $\phi$.

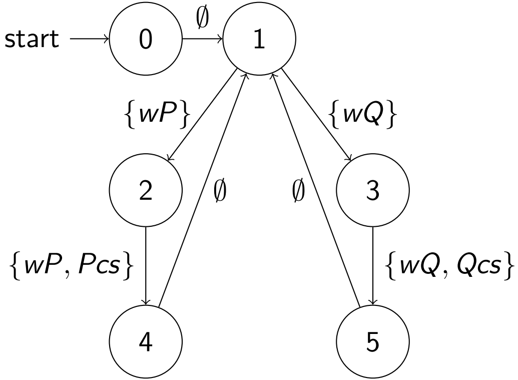
   To check $\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi})$ construct intersection automaton and search for cycle through accepting state.

# Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of $\mathcal{T}$

**Example**

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```




Similar code for process Q.

Second **atomic** block just to keep automaton small.

# Büchi Automaton $B_{\neg\phi}$ for $\neg\phi$

Second Step:

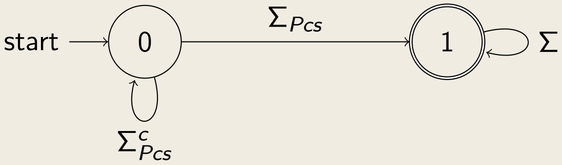
Construct Büchi automaton corresponding to negated LTL formula

$\mathcal{T} \models \phi$ holds iff there is no accepting run $\sigma$ of $\mathcal{T}$ s.t. $\sigma \models \neg\phi$

Simplify $\neg\phi = \neg\Box\neg Pcs = \Diamond Pcs$

**Büchi Automaton $\mathcal{B}_{\neg\phi}$**

$$\mathcal{P} = \{wP, wQ, Pcs, Qcs\},\ \Sigma = 2^{\mathcal{P}}$$

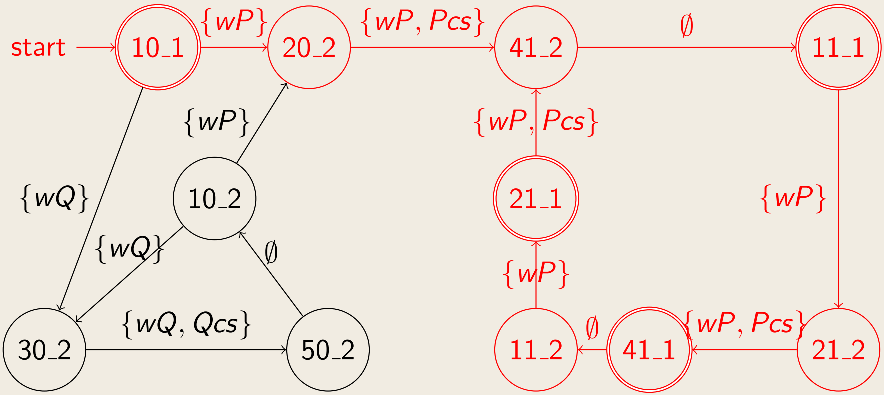



$$\Sigma_{Pcs} = \{I | I \in \Sigma, Pcs \in I\}, \quad \Sigma^c_{Pcs} = \Sigma - \Sigma_{Pcs}$$

Third Step: $\mathcal{L}^\omega(\mathcal{B}_\mathcal{T}) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) = \neq \emptyset$ ?

Counterexample Construction of intersection automaton: Appendix

**Intersection Automaton** (skipping first step of $\mathcal{T}$ for simplicity)



# Literature for this Lecture

**Ben-Ari** Section 5.2.1
(only syntax of LTL)

**Baier and Katoen** Principles of Model Checking,
May 2008, The MIT Press,
ISBN: 0-262-02649-X

# Appendix I:

# Intersection Automaton

# –

# Construction

# Construction of Intersection Automaton

Given: two Büchi automata $\mathcal{B}_i = (Q_i, \delta_i, I_i, F_i)$, $i = 1, 2$

Wanted: a Büchi automaton

$$\mathcal{B}_{1\cap 2} = (Q_{1\cap 2}, \delta_{1\cap 2}, I_{1\cap 2}, F_{1\cap 2})$$

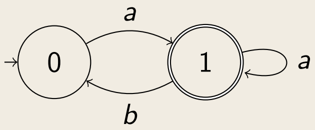
accepting a word $w$ iff $w$ is accepted by $\mathcal{B}_1$ and $\mathcal{B}_2$

Maybe just the product automaton as for regular automata?

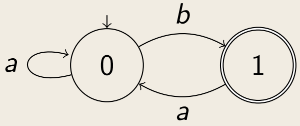
# First Attempt: Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^* ba)^\omega = \emptyset$? No, e.g., $a(ba)^\omega$

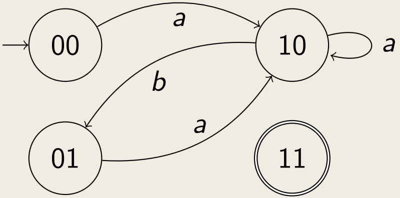
$a(a + ba)^\omega$ :




$(a^* ba)^\omega$ :

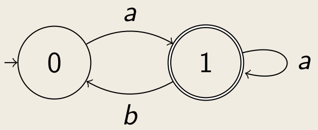



**Product Automaton: accepting location 11 never reached**

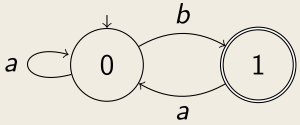


# Explicit Construction of Intersection Automaton

$a(a + ba)^\omega$ :




$(a^* ba)^\omega$ :

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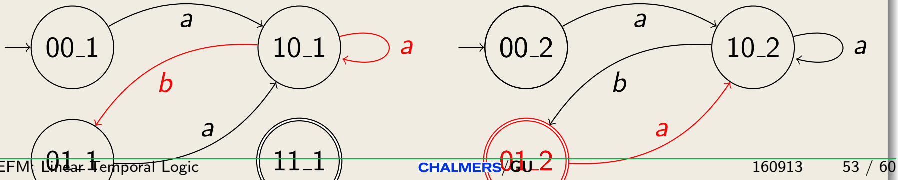
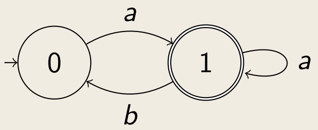
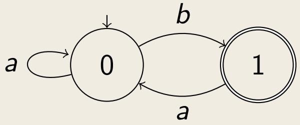


# Explicit Construction of Intersection Automaton

$a(a+ba)^\omega$ :




$(a^*ba)^\omega$ :

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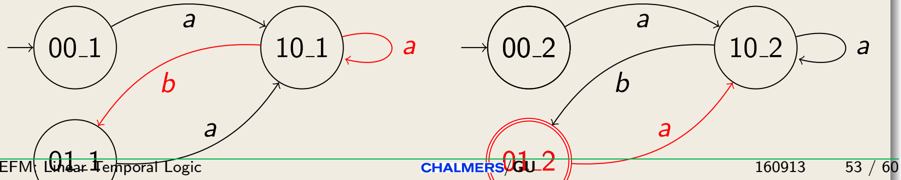
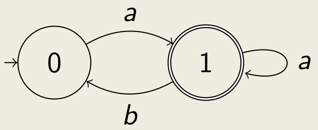
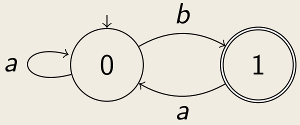


# Explicit Construction of Intersection Automaton

$a(a + ba)^\omega$ :




$(a^* ba)^\omega$ :

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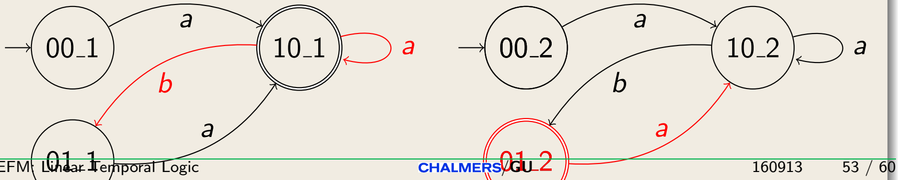
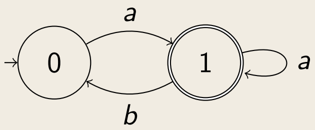
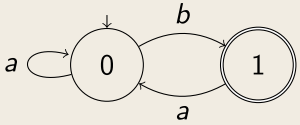


# Explicit Construction of Intersection Automaton

$a(a+ba)^\omega$ :




$(a^*ba)^\omega$ :

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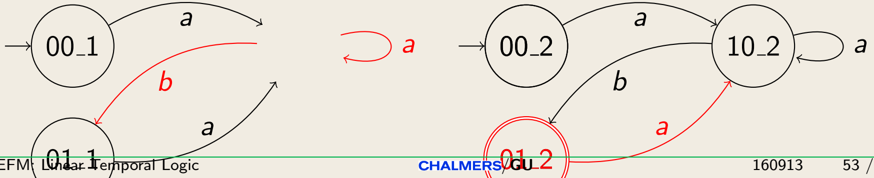
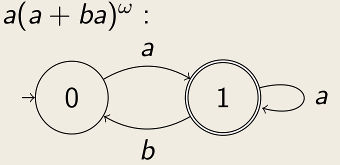
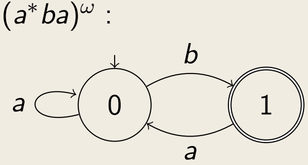


# Explicit Construction of Intersection Automaton

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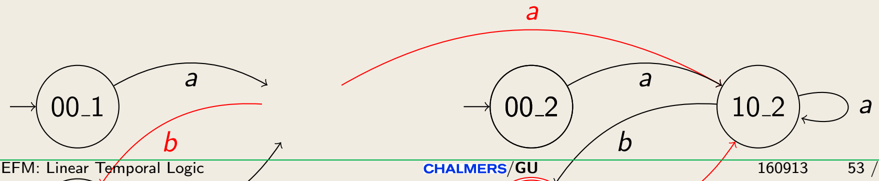
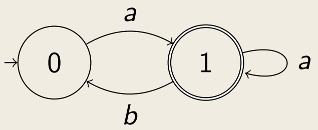
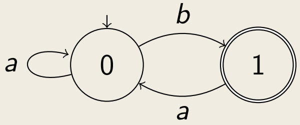


# Explicit Construction of Intersection Automaton

$a(a+ba)^\omega$ :




$(a^*ba)^\omega$ :

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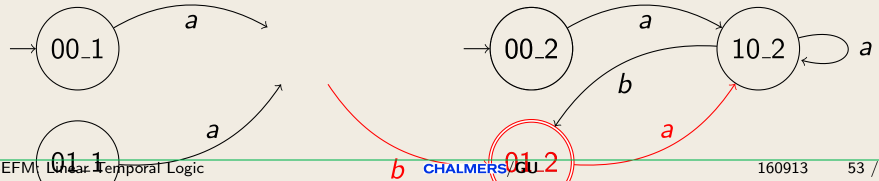
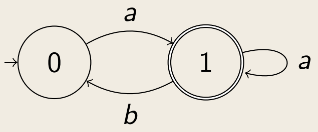
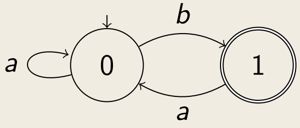


# Explicit Construction of Intersection Automaton

$a(a + ba)^\omega$ :




$(a^* ba)^\omega$ :

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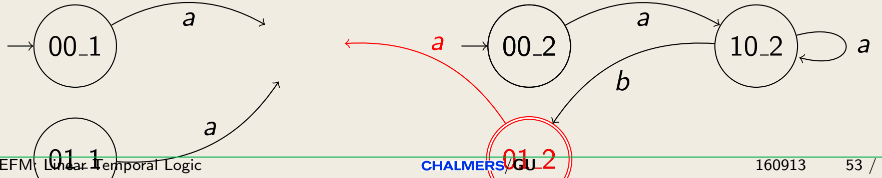


# Appendix II:

# Construction of a Büchi Automaton $\mathcal{B}_\phi$ for an LTL-Formula $\phi$

# The General Case: Generalised Büchi Automata

A generalised Büchi automaton is defined as:

$$\mathcal{B}^g = (Q, \delta, I, \mathbb{F})$$

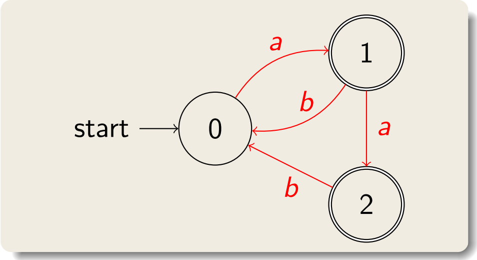
$Q, \delta, I$ as for standard Büchi automata

$\mathbb{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$, where $\mathcal{F}_i = \{q_{i1}, \dots, q_{im_i}\} \subseteq Q$

**Definition (Acceptance for generalised Büchi automata)**

A generalised Büchi automaton accepts an $\omega$-word $w \in \Sigma^\omega$ iff for every $i \in \{1, \dots, n\}$ at least one $q_{ik} \in \mathcal{F}_i$ is visited infinitely often.

# Normal vs. Generalised Büchi Automata: Example




$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\quad \mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which $\omega$-word is accepted by which automaton?

| $\omega$-word | $\mathcal{B}^{normal}$ | $\mathcal{B}^{general}$ |
|---|---|---|
| $(ab)^\omega$ | ✔ | ✘ |
| $(aab)^\omega$ | ✔ | ✔ |

# Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula $\phi$

$$FL(\phi) = \{\varphi | \varphi \text{ is subformula or negated subformula of } \phi\}$$

($\neg\neg\varphi$ is identified with $\varphi$)

**Example**

$$FL(r\,\mathcal{U}s) = \{r, \neg r, s, \neg s, r\,\mathcal{U}s, \neg(r\,\mathcal{U}s)\}$$

# $\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

**Consistent, Total**

- $\psi \in FL(\phi)$: exactly one of $\psi$ and $\neg\psi$ in $q$
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$

**Downward Closed**

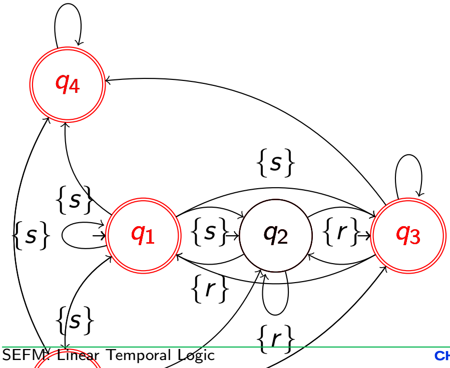
- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ... other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r\mathcal{U}s) = \{r, \neg r, s, \neg s, r\mathcal{U}s, \neg(r\mathcal{U}s)\}$$

| | $\in Q$ |
|---|---|
| $\{r\mathcal{U}s, \neg r, s\}$ | ✔ |
| $\{r\mathcal{U}s, \neg r, \neg s\}$ | ✘ |
| $\{\neg(r\mathcal{U}s), r, s\}$ | ✘ |
| $\{\neg(r\mathcal{U}s), r, \neg s\}$ | ✔ |

# $\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U} s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U} s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U} s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U} s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U} s), \neg r, \neg s\}}_{q_5}$$




Transitions $(q, \alpha, q') \in \delta_\phi$:

$\alpha = q \cap \mathcal{P}$

$\mathcal{P}$ set of propositional variables outgoing edges of $q_1$ labeled $\{s\}$, of $q_2$ labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \mathcal{U} \psi_2 \in q'$
2. If $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \backslash q)$ and $\psi_1 \in q$ then $\psi_1 \mathcal{U} \psi_2 \notin q'$

Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

Accepting locations

# Remarks on Generalized Büchi Automata

- Construction always gives exponential number of states in $|\phi|$
- Satisfiability checking of LTL is PSPACE-complete
- There exist (more complex) constructions that minimize number of required states
  - One of these is used in SPIN, which moreover computes the states lazily