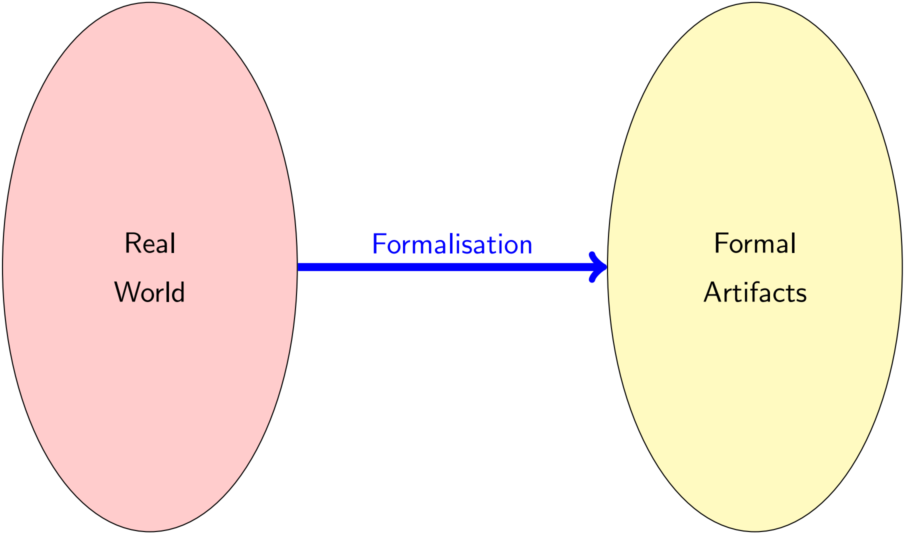
Software Engineering using Formal Methods

Propositional and (Linear) Temporal Logic

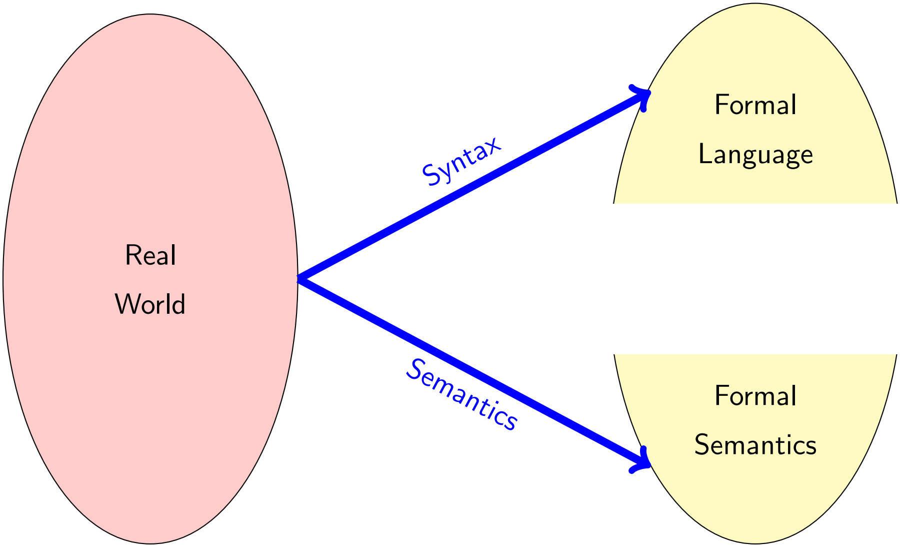
Wolfgang Ahrendt

13th September 2016

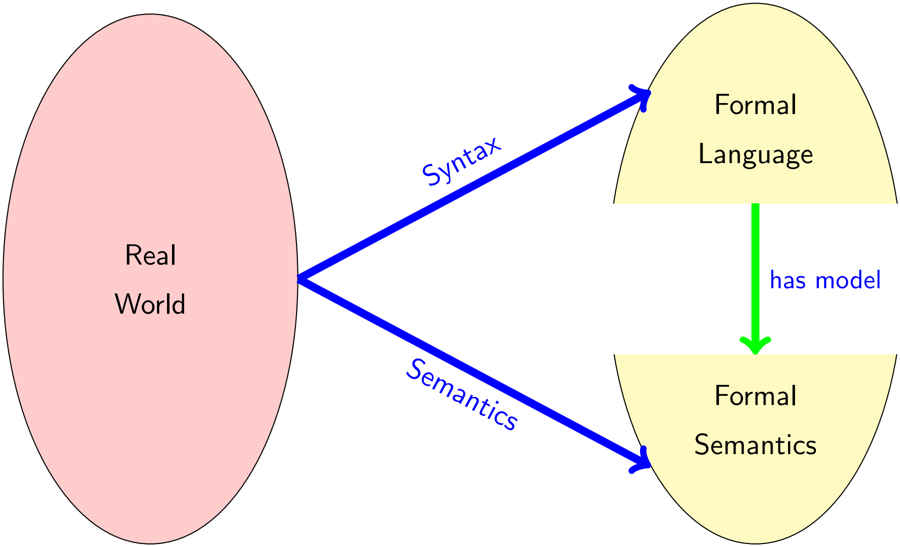
Recapitulation: Formalisation



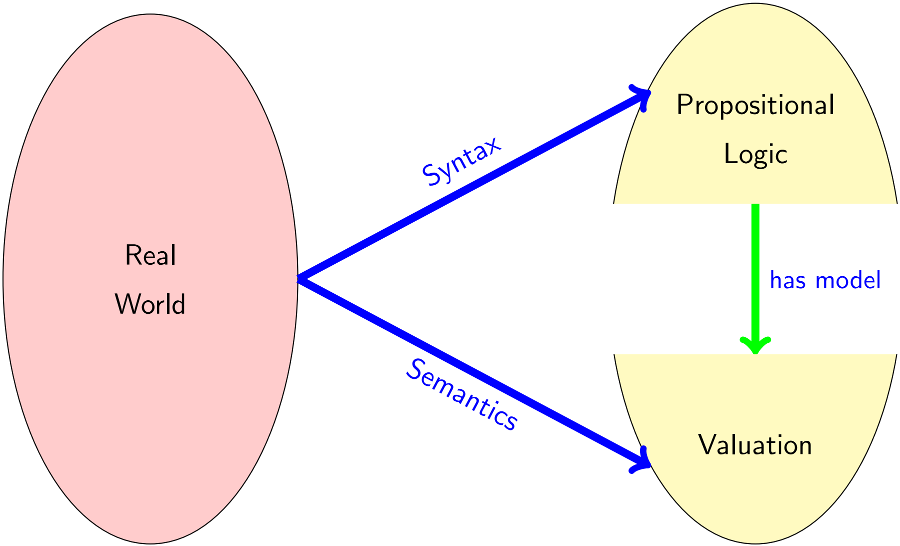
Formalisation: Syntax, Semantics



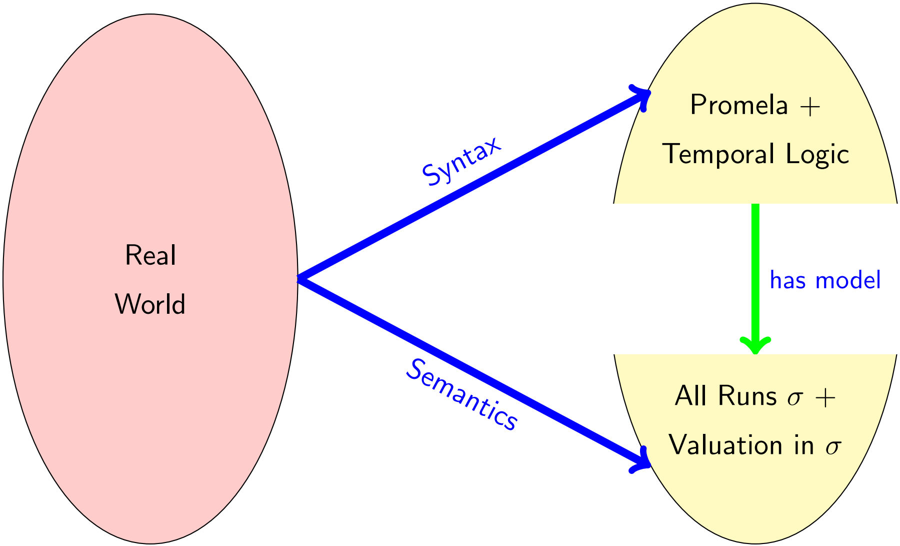
Formalisation: Syntax, Semantics



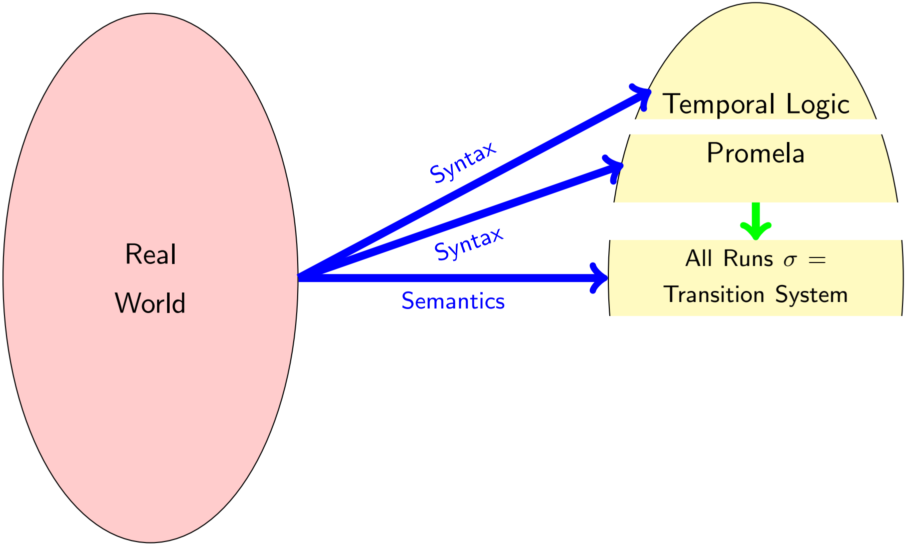
Formalisation: Syntax, Semantics



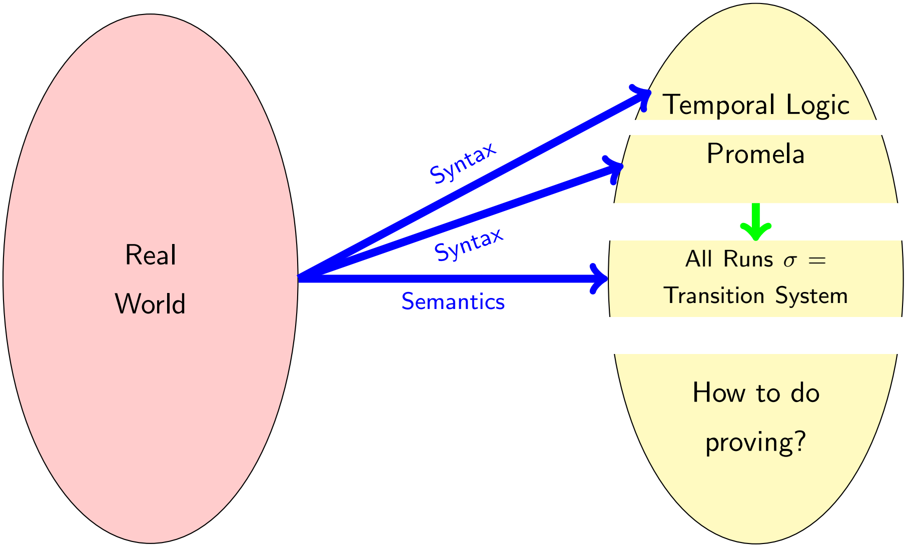
Formalisation: Syntax, Semantics



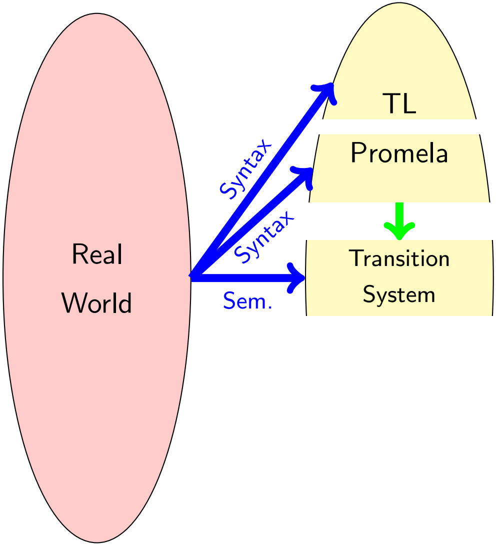
Formalisation: Syntax, Semantics



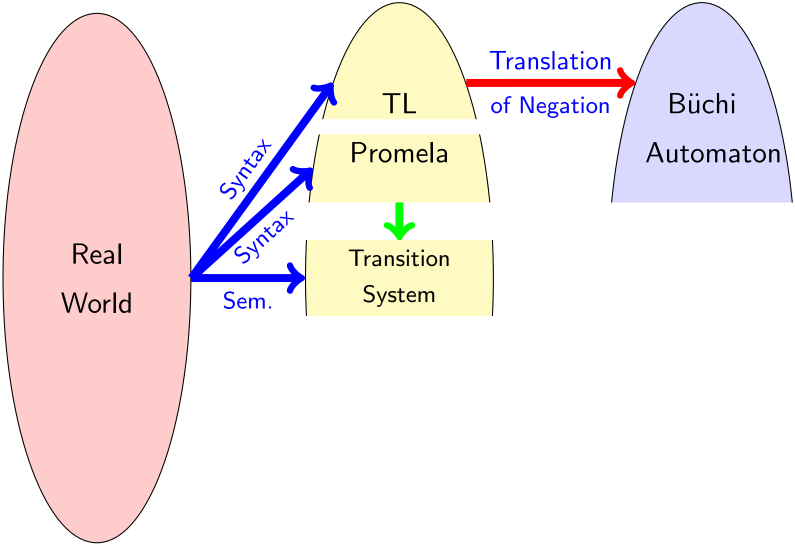
Formalisation: Syntax, Semantics, Proving



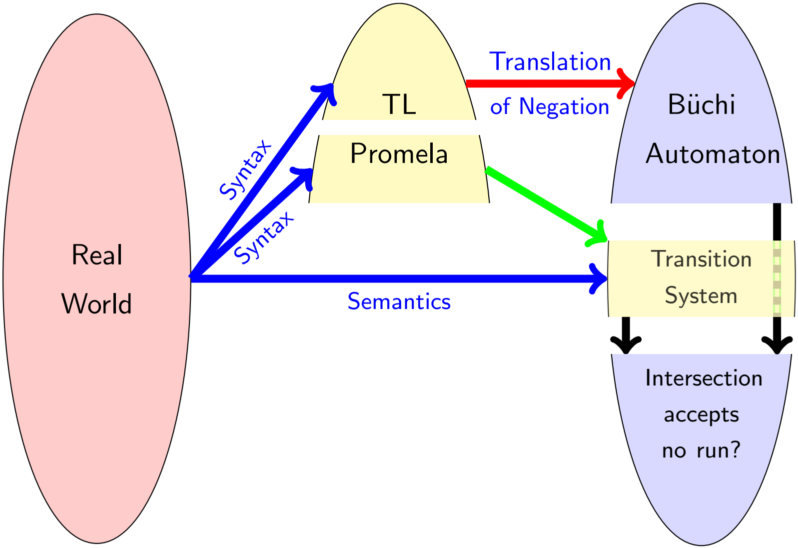
Formal Verification: Model Checking



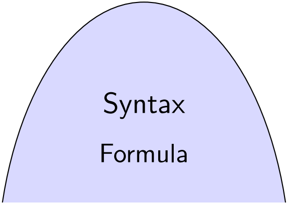
Formal Verification: Model Checking



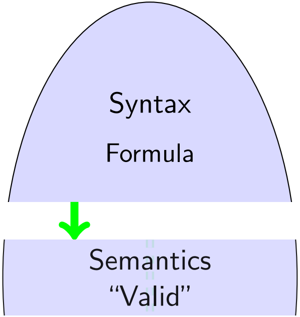
Formal Verification: Model Checking



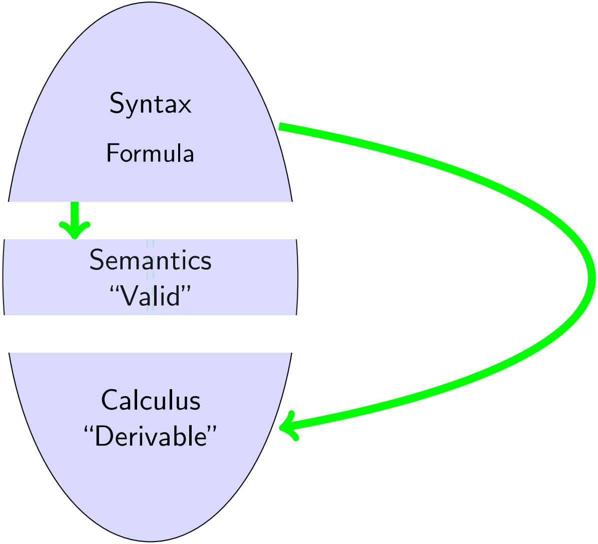
The Big Picture: Syntax, Semantics, Calculus



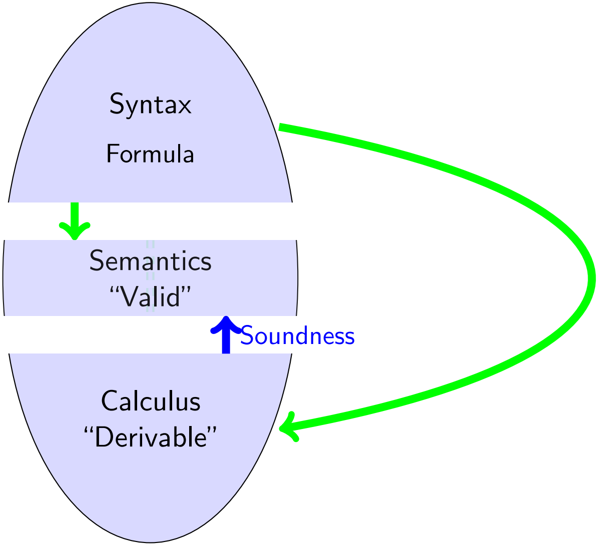
The Big Picture: Syntax, Semantics, Calculus



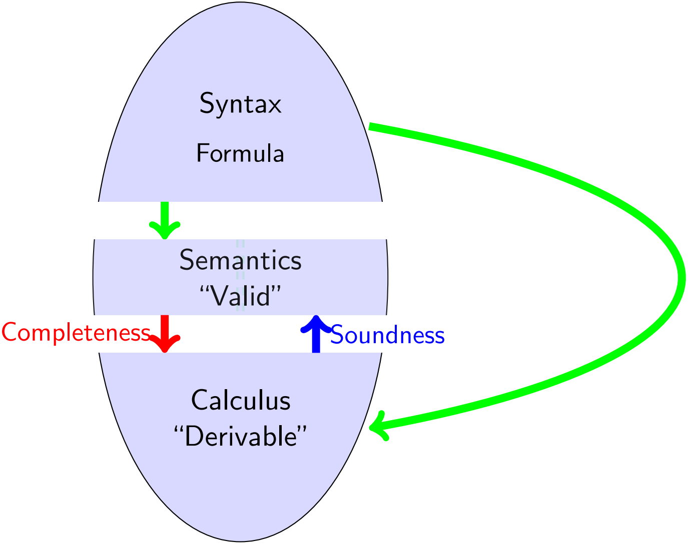
The Big Picture: Syntax, Semantics, Calculus



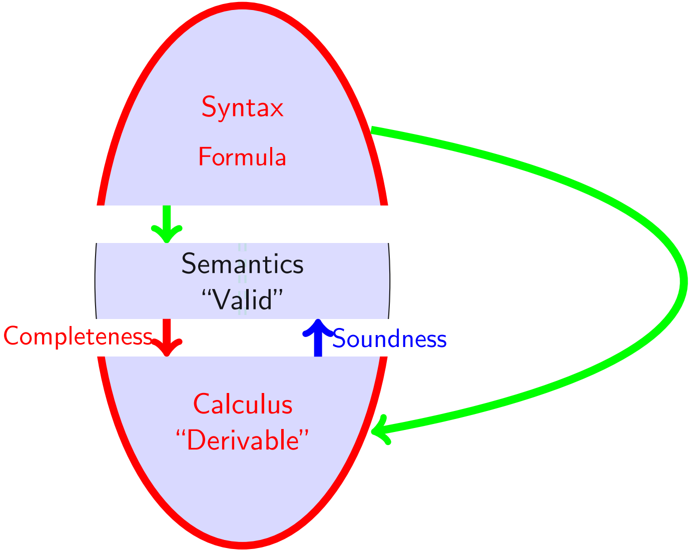
The Big Picture: Syntax, Semantics, Calculus



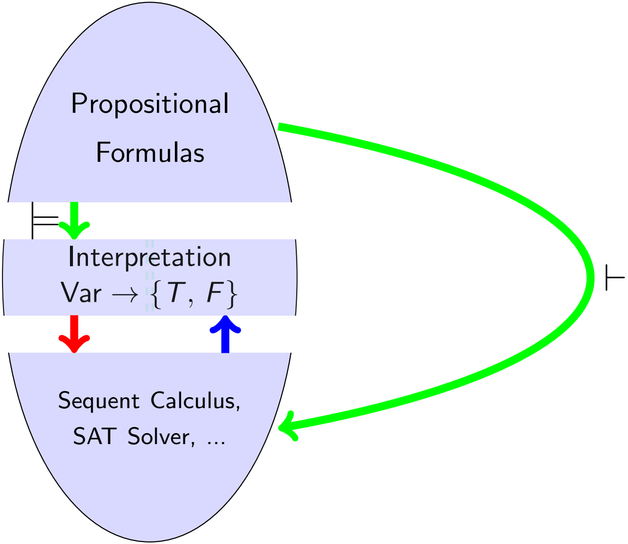
The Big Picture: Syntax, Semantics, Calculus



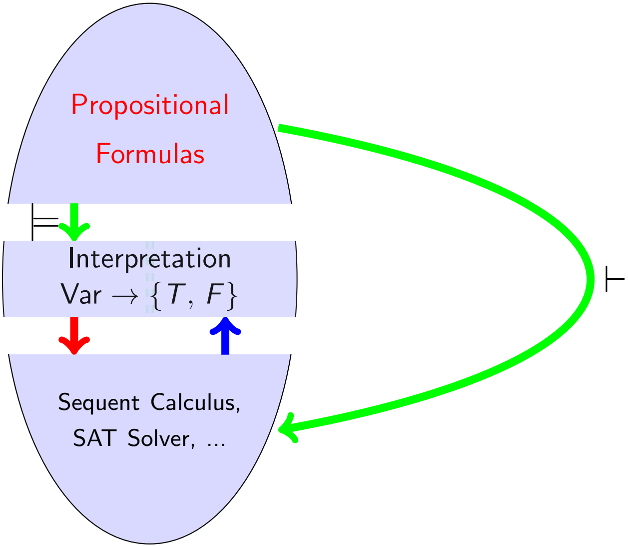
The Big Picture: Syntax, Semantics, Calculus



Simplest Case: Propositional Logic



Simplest Case: Propositional Logic—Syntax



Syntax of Propositional Logic

Signature

A set of Propositional Variables $\mathcal{P}$ (with typical elements $p, q, r, \ldots$)

Syntax of Propositional Logic

Signature

A set of Propositional Variables $\mathcal{P}$ (with typical elements $p, q, r, \ldots$)

Propositional Connectives

true, false, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$

Syntax of Propositional Logic

Signature

A set of Propositional Variables $\mathcal{P}$ (with typical elements $p, q, r, \ldots$)

Propositional Connectives

$\mathrm{true},\ \mathrm{false},\ \wedge,\ \vee,\ \neg,\ \rightarrow,\ \leftrightarrow$

Set of Propositional Formulas For_0

- Truth constants true, false and variables $\mathcal{P}$ are formulas
- If ϕ and ψ are formulas then

  $$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi, \quad \phi \leftrightarrow \psi$$

  are also formulas
- There are no other formulas (inductive definition)

Remark on Concrete Syntax

	Text book	SPIN
Negation	$\neg$	!
Conjunction	$\wedge$	&&
Disjunction	$\vee$	||
Implication	$\rightarrow$, $\supset$	->
Equivalence	$\leftrightarrow$	<->

Remark on Concrete Syntax

<table>
<tr><th></th><th>Text book</th><th>SPIN</th></tr>
<tr><td>Negation</td><td>$\neg$</td><td>!</td></tr>
<tr><td>Conjunction</td><td>$\wedge$</td><td>&&</td></tr>
<tr><td>Disjunction</td><td>$\vee$</td><td>||</td></tr>
<tr><td>Implication</td><td>$\rightarrow$, $\supset$</td><td>-></td></tr>
<tr><td>Equivalence</td><td>$\leftrightarrow$</td><td><-></td></tr>
</table>

We use mostly the textbook notation,
except for tool-specific slides, input files.

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \land r)) \lor p$

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \wedge r)) \vee p$ ✘

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \wedge r)) \vee p$ ✘
- $p \rightarrow (q \wedge)$

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \wedge r)) \vee p$ ✘
- $p \rightarrow (q \wedge)$ ✘

Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \land r)) \lor p$ ✘
- $p \rightarrow (q\land)$ ✘
- $\mathit{false} \land (p \rightarrow (q \land r))$

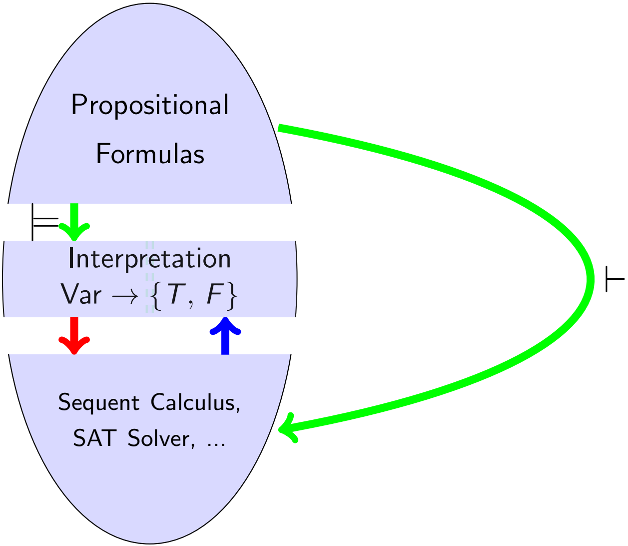
Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

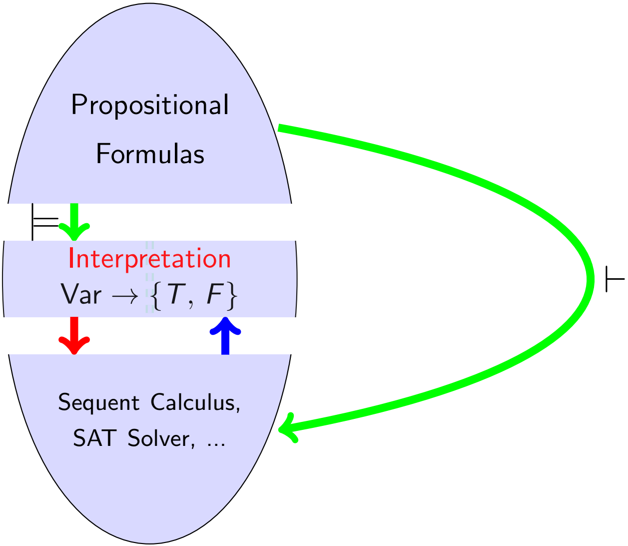
Are the following character sequences also propositional formulas?

- $\mathrm{true} \rightarrow p$ ✔
- $(p(q \land r)) \lor p$ ✘
- $p \rightarrow (q\land)$ ✘
- $\mathit{false} \land (p \rightarrow (q \land r))$ ✔

Simplest Case: Propositional Logic



Simplest Case: Propositional Logic



Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \to \{T, F\}$$

Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
$\vdots$	$\vdots$	$\vdots$

Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
$\vdots$	$\vdots$	$\vdots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in each interpretation $\mathcal{I}_i$?

Semantics of Propositional Logic

Interpretation $\mathcal{I}$

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

Valuation Function

$val_{\mathcal{I}}$: Continuation of $\mathcal{I}$ on For_0

$$val_{\mathcal{I}} : For_0 \; \rightarrow \; \{T, F\}$$

$val_{\mathcal{I}}(\text{true}) = T$
$val_{\mathcal{I}}(\text{false}) = F$
$val_{\mathcal{I}}(p_i) = \mathcal{I}(p_i)$

(cont'd next page)

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$val_{\mathcal{I}}(\neg\phi) =$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) =$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & \text{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \land \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & \text{otherwise} \end{cases}$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) =$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \rightarrow \psi) =$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \to \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \rightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & \textit{otherwise} \end{cases}$$

$$val_{\mathcal{I}}(\phi \leftrightarrow \psi) =$$

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$val_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ and } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = T \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \rightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = F \textbf{ or } val_{\mathcal{I}}(\psi) = T \\ F & otherwise \end{cases}$$

$$val_{\mathcal{I}}(\phi \leftrightarrow \psi) = \begin{cases} T & \text{if } val_{\mathcal{I}}(\phi) = val_{\mathcal{I}}(\psi) \\ F & otherwise \end{cases}$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$$\ldots$$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$\ldots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\, p \rightarrow (q \rightarrow p)\,) \quad =$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$\ldots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\, p \rightarrow (q \rightarrow p)\,) \quad = \quad T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$$\dots$$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) \quad = \quad T$ iff $val_{\mathcal{I}_2}(p) = F$ **or** $val_{\mathcal{I}_2}(q \rightarrow p) = T$

$val_{\mathcal{I}_2}(p) \quad =$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
$\dots$		

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\, p \rightarrow (q \rightarrow p)\,) = T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$
$$val_{\mathcal{I}_2}(p) = \mathcal{I}_2(p) =$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
$\dots$		

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) = T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$
$$val_{\mathcal{I}_2}(p) = \mathcal{I}_2(p) = T$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$\ldots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) = T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$

$$val_{\mathcal{I}_2}(p) = \mathcal{I}_2(p) = T$$

$$val_{\mathcal{I}_2}(\ q \rightarrow p\ ) =$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
	$\dots$	

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) \quad = \quad T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$
$$val_{\mathcal{I}_2}(p) \quad = \quad \mathcal{I}_2(p) \quad = \quad T$$
$$val_{\mathcal{I}_2}(\ q \rightarrow p\ ) \quad = \quad T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
	$\dots$	

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) = T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T$$

$$val_{\mathcal{I}_2}(p) = \mathcal{I}_2(p) = T$$

$$val_{\mathcal{I}_2}(\ q \rightarrow p\ ) = T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T$$

$$val_{\mathcal{I}_2}(q) =$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F
	$\ldots$	

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$\begin{array}{lcl}
val_{\mathcal{I}_2}(\, p \rightarrow (q \rightarrow p)\,) & = & T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T \\
val_{\mathcal{I}_2}(p) & = & \mathcal{I}_2(p) \quad = \quad T \\
val_{\mathcal{I}_2}(\, q \rightarrow p\,) & = & T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T \\
val_{\mathcal{I}_2}(q) & = & \mathcal{I}_2(q) \quad =
\end{array}$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
$\mathcal{I}_1$	F	F
$\mathcal{I}_2$	T	F

$\ldots$

How to evaluate $p \rightarrow (q \rightarrow p)$ in $\mathcal{I}_2$?

$$\begin{aligned}
val_{\mathcal{I}_2}(\ p \rightarrow (q \rightarrow p)\ ) &= T \text{ iff } val_{\mathcal{I}_2}(p) = F \textbf{ or } val_{\mathcal{I}_2}(q \rightarrow p) = T \\
val_{\mathcal{I}_2}(p) &= \mathcal{I}_2(p) = T \\
val_{\mathcal{I}_2}(\ q \rightarrow p\ ) &= T \text{ iff } val_{\mathcal{I}_2}(q) = F \textbf{ or } val_{\mathcal{I}_2}(p) = T \\
val_{\mathcal{I}_2}(q) &= \mathcal{I}_2(q) = F
\end{aligned}$$

Semantic Notions of Propositional Logic

Let $\phi \in \mathit{For}_0$, $\Gamma \subseteq \mathit{For}_0$

Definition (Satisfying Interpretation, Consequence Relation)

$\mathcal{I}$ satisfies ϕ (write: $\mathcal{I} \models \phi$) iff $\mathit{val}_{\mathcal{I}}(\phi) = T$

ϕ follows from Γ (write: $\Gamma \models \phi$) iff for all interpretations $\mathcal{I}$:

If $\mathcal{I} \models \psi$ for all $\psi \in \Gamma$, then also $\mathcal{I} \models \phi$

Semantic Notions of Propositional Logic

Let $\phi \in For_0$, $\Gamma \subseteq For_0$

Definition (Satisfying Interpretation, Consequence Relation)

$\mathcal{I}$ satisfies ϕ (write: $\mathcal{I} \models \phi$) iff $val_{\mathcal{I}}(\phi) = T$

ϕ follows from Γ (write: $\Gamma \models \phi$) iff for all interpretations $\mathcal{I}$:

If $\mathcal{I} \models \psi$ for all $\psi \in \Gamma$, then also $\mathcal{I} \models \phi$

Definition (Satisfiability, Validity)

A formula is satisfiable if it is satisfied by some interpretation.
If every interpretation satisfies ϕ (write: $\models \phi$) then ϕ is called valid.

Semantics of Propositional Logic: Examples

Formula (same as before)

$$p \rightarrow (q \rightarrow p)$$

Semantics of Propositional Logic: Examples

Formula (same as before)

$$p \rightarrow (q \rightarrow p)$$

Is this formula valid?

$$\models p \rightarrow (q \rightarrow p) \; ?$$

Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable?

Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔

Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔

Satisfying Interpretation?

Semantics of Propositional Logic: Examples

$$p \wedge ((\neg p) \vee q)$$

Satisfiable? ✔

Satisfying Interpretation? $\mathcal{I}(p) = T,\ \mathcal{I}(q) = T$

Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔

Satisfying Interpretation? $\mathcal{I}(p) = T,\ \mathcal{I}(q) = T$

Other Satisfying Interpretations?

Semantics of Propositional Logic: Examples

$$p \wedge ((\neg p) \vee q)$$

Satisfiable?	✔
Satisfying Interpretation?	$\mathcal{I}(p) = T,\ \mathcal{I}(q) = T$
Other Satisfying Interpretations?	✘

Semantics of Propositional Logic: Examples

$$p \ \wedge \ ((\neg p) \ \vee \ q)$$

Satisfiable?	✔
Satisfying Interpretation?	$\mathcal{I}(p) = T, \ \mathcal{I}(q) = T$
Other Satisfying Interpretations?	✘

Therefore, not valid!

Semantics of Propositional Logic: Examples

$$p \land ((\neg p) \lor q)$$

Satisfiable? ✔
Satisfying Interpretation? $\mathcal{I}(p) = T,\ \mathcal{I}(q) = T$
Other Satisfying Interpretations? ✘
Therefore, not valid!

$$p \land ((\neg p) \lor q) \models q \lor r$$

Does it hold?

Semantics of Propositional Logic: Examples

$$p \wedge ((\neg p) \vee q)$$

Satisfiable? ✔
Satisfying Interpretation? $\mathcal{I}(p) = T,\ \mathcal{I}(q) = T$
Other Satisfying Interpretations? ✘
Therefore, not valid!

$$p \wedge ((\neg p) \vee q) \models q \vee r$$

Does it hold? Yes. Why?

An Exercise in Formalisation

```
byte n;
active proctype [2] P() {
  n = 0;
  n = n + 1
}
```

Can we characterise the states of P propositionally?

An Exercise in Formalisation

```
byte n;
active proctype [2] P() {
  n = 0;
  n = n + 1
}
```

Can we characterise the states of P propositionally?

Find a propositional formula ϕ_{P} which is true if and only if it describes a possible state of P.

An Exercise in Formalisation

```
1 byte n;
2 active proctype [2] P() {
3   n = 0;
4   n = n + 1
5 }
```

$\mathcal{P}$: $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

$$\phi_{\mathrm{P}} := \left(\qquad\qquad\qquad\qquad \right)$$

An Exercise in Formalisation

```
1 byte n;
2 active proctype [2] P() {
3   n = 0;
4   n = n + 1
5 }
```

$\mathcal{P}$: $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

- The variable `n` is represented by eight bits, all values possible

$$\phi_{\mathrm{P}} := \left(\qquad\qquad\qquad\qquad \right)$$

An Exercise in Formalisation

```
byte n;
active proctype [2] P() {
  n = 0;
  n = n + 1
}
```

$\mathcal{P}$: $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

- The variable `n` is represented by eight bits, all values possible
- A process cannot be at two positions at the same time

$$\phi_{\mathrm{P}} := \left(((PC0_3 \land \neg PC0_4 \land \neg PC0_5) \lor \cdots) \land \right)$$

An Exercise in Formalisation

```
1 byte n;
2 active proctype [2] P() {
3   n = 0;
4   n = n + 1
5 }
```

$\mathcal{P}$: $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

- The variable `n` is represented by eight bits, all values possible
- A process cannot be at two positions at the same time
- If neither process 0 nor process 1 are at position 5, then `n` is zero

$$\phi_{\mathrm{P}} := \left(\begin{array}{l} ((PC0_3 \land \neg PC0_4 \land \neg PC0_5) \lor \cdots) \land \\ ((\neg PC0_5 \land \neg PC1_5) \implies (\neg N_0 \land \cdots \land \neg N_7)) \end{array} \right)$$

An Exercise in Formalisation

```
1 byte n;
2 active proctype [2] P() {
3   n = 0;
4   n = n + 1
5 }
```

$\mathcal{P}$: $N_0, N_1, N_2, \ldots, N_7$ 8-bit representation of **byte**
$PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to "exclude"?

- The variable `n` is represented by eight bits, all values possible
- A process cannot be at two positions at the same time
- If neither process 0 nor process 1 are at position 5, then `n` is zero
- ...

$$\phi_{\mathrm{P}} := \left(\begin{array}{l} ((PC0_3 \land \neg PC0_4 \land \neg PC0_5) \lor \cdots) \land \\ ((\neg PC0_5 \land \neg PC1_5) \implies (\neg N_0 \land \cdots \land \neg N_7)) \land \cdots \end{array} \right)$$

Is Propositional Logic Enough?

Can design for a program P a formula Φ_P describing all reachable states

For a given property Ψ the consequence relation

$$\Phi_p \models \Psi$$

holds when Ψ is true in any possible state reachable in any run of P

Is Propositional Logic Enough?

Can design for a program P a formula Φ_P describing all reachable states

For a given property Ψ the consequence relation

$$\Phi_p \models \Psi$$

holds when Ψ is true in any possible state reachable in any run of P

But How to Express Properties Involving State Changes?

In any run of a program P

- n will become greater than 0 eventually?
- n changes its value infinitely often

etc.

Is Propositional Logic Enough?

Can design for a program P a formula Φ_P describing all reachable states

For a given property Ψ the consequence relation

$$\Phi_p \models \Psi$$

holds when Ψ is true in any possible state reachable in any run of P

But How to Express Properties Involving State Changes?

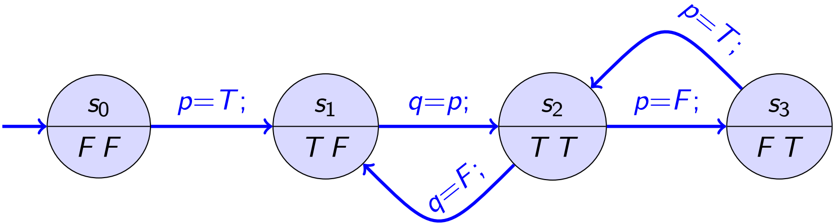
In any run of a program P

- n will become greater than 0 eventually?
- n changes its value infinitely often

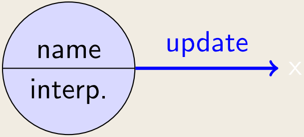
etc.

$\Rightarrow$ Need a more expressive logic: (Linear) Temporal Logic

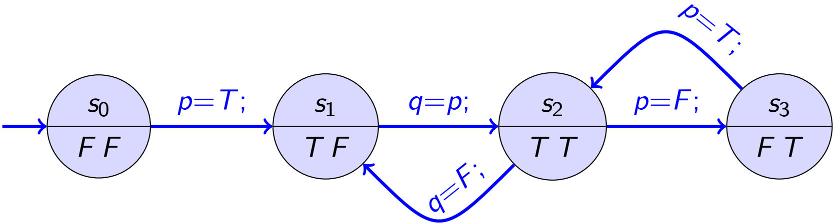
Transition systems (aka Kripke Structures)



Notation

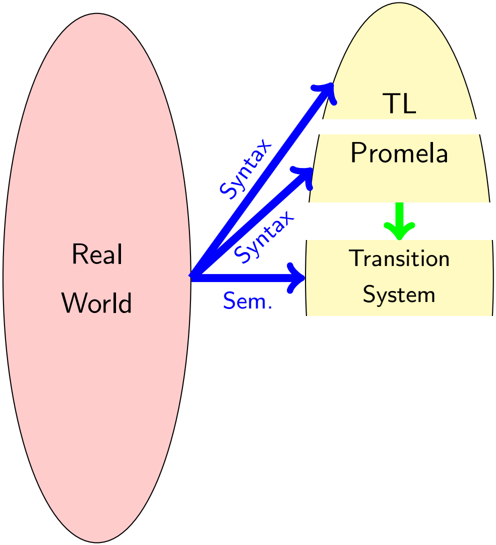


Transition systems (aka Kripke Structures)



- Each state s_i has its own propositional interpretation $\mathcal{I}_i$
  - Convention: list interpretation of variables in lexicographic order
- Computations, or runs, are *infinite* paths through states
  - Intuitively 'finite' runs modelled by looping on last state
- How to express (for example) that p changes its value infinitely often in each run?

Formal Verification: Model Checking



Linear Temporal Logic

An extension of propositional logic that
allows to specify properties of all runs

Linear Temporal Logic—Syntax

An extension of propositional logic that
allows to specify properties of all runs

Syntax

Based on propositional signature and syntax

Extension with three connectives:

Always If ϕ is a formula, then so is $\Box\phi$

Eventually If ϕ is a formula, then so is $\Diamond\phi$

Until If ϕ and ψ are formulas, then so is $\phi\mathcal{U}\psi$

Concrete Syntax

	text book	SPIN
Always	$\Box$	`[]`
Eventually	$\Diamond$	`<>`
Until	$\mathcal{U}$	`U`

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \to q$

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \rightarrow q$
- $\Diamond p$

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \rightarrow q$
- $\Diamond p$
- $\Box q$

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \rightarrow q$
- $\Diamond p$
- $\Box q$
- $\Box\Diamond\Box(p \rightarrow q)$

Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \rightarrow q$
- $\Diamond p$
- $\Box q$
- $\Box\Diamond\Box(p \rightarrow q)$
- $(\Box p) \rightarrow ((\Diamond p) \vee \neg q)$

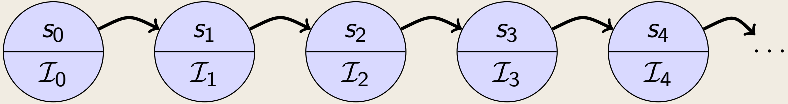
Linear Temporal Logic Syntax: Examples

Let $\mathcal{P} = \{p, q\}$ be the set of propositional variables.

- p
- false
- $p \to q$
- $\Diamond p$
- $\Box q$
- $\Box\Diamond\Box(p \to q)$
- $(\Box p) \to ((\Diamond p) \vee \neg q)$
- $p\,\mathcal{U}(\Box q)$

Temporal Logic—Semantics

A run σ is an infinite chain of states

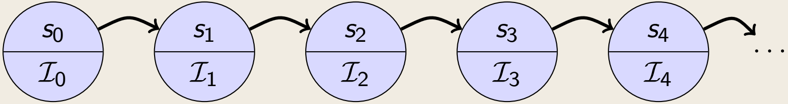


$\mathcal{I}_j$ propositional interpretation of variables in state s_j

Write more compactly $s_0\, s_1\, s_2\, s_3 \ldots$

Temporal Logic—Semantics

A run σ is an infinite chain of states



$\mathcal{I}_j$ propositional interpretation of variables in state s_j
Write more compactly $s_0\, s_1\, s_2\, s_3 \ldots$

If $\sigma = s_0\, s_1 \cdots$, then $\sigma|_i$ denotes the suffix $s_i\, s_{i+1} \cdots$ of σ.

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Definition (Validity Relation)

Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

$\sigma \models p$ iff $\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Definition (Validity Relation)

Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

$\sigma \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.
$\sigma \models \neg\phi$	iff	not $\sigma \models \phi$ (write $\sigma \not\models \phi$)

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Definition (Validity Relation)

Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

$\sigma \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.
$\sigma \models \neg\phi$	iff	not $\sigma \models \phi$ (write $\sigma \not\models \phi$)
$\sigma \models \phi \wedge \psi$	iff	$\sigma \models \phi$ and $\sigma \models \psi$

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Definition (Validity Relation)

Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

$\sigma \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.
$\sigma \models \neg\phi$	iff	not $\sigma \models \phi$ (write $\sigma \not\models \phi$)
$\sigma \models \phi \wedge \psi$	iff	$\sigma \models \phi$ and $\sigma \models \psi$
$\sigma \models \phi \vee \psi$	iff	$\sigma \models \phi$ or $\sigma \models \psi$
$\sigma \models \phi \rightarrow \psi$	iff	$\sigma \not\models \phi$ or $\sigma \models \psi$

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to run (infinite sequence of states)

Definition (Validity Relation)

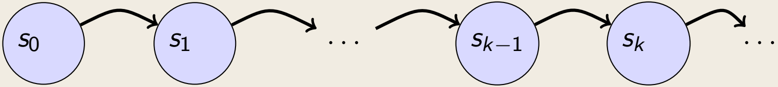
Validity of temporal formula depends on runs $\sigma = s_0\, s_1 \ldots$

$\sigma \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.
$\sigma \models \neg\phi$	iff	not $\sigma \models \phi$ (write $\sigma \not\models \phi$)
$\sigma \models \phi \land \psi$	iff	$\sigma \models \phi$ and $\sigma \models \psi$
$\sigma \models \phi \lor \psi$	iff	$\sigma \models \phi$ or $\sigma \models \psi$
$\sigma \models \phi \rightarrow \psi$	iff	$\sigma \not\models \phi$ or $\sigma \models \psi$

Temporal connectives?

Temporal Logic—Semantics (Cont'd)

Run σ

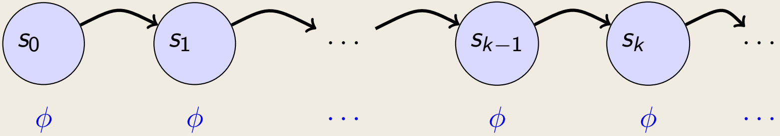


Definition (Validity Relation for Temporal Connectives)

Given a run $\sigma = s_0\, s_1 \cdots$

Temporal Logic—Semantics (Cont'd)

Run σ



Definition (Validity Relation for Temporal Connectives)

Given a run $\sigma = s_0\, s_1 \cdots$

$\sigma \models \Box\phi$ iff $\sigma|_k \models \phi$ for all $k \geq 0$

Temporal Logic—Semantics (Cont'd)

Run σ

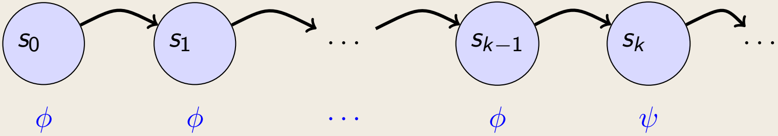
Definition (Validity Relation for Temporal Connectives)

Given a run $\sigma = s_0\, s_1 \cdots$

$\sigma \models \Box\phi$	iff	$\sigma\|_k \models \phi$ for all $k \geq 0$
$\sigma \models \Diamond\phi$	iff	$\sigma\|_k \models \phi$ for some $k \geq 0$

Temporal Logic—Semantics (Cont'd)

Run σ



Definition (Validity Relation for Temporal Connectives)

Given a run $\sigma = s_0\, s_1 \cdots$

$\sigma \models \Box\phi$ iff $\sigma|_k \models \phi$ for all $k \geq 0$

$\sigma \models \Diamond\phi$ iff $\sigma|_k \models \phi$ for some $k \geq 0$

$\sigma \models \phi\,\mathcal{U}\,\psi$ iff $\sigma|_k \models \psi$ for some $k \geq 0$, and $\sigma|_j \models \phi$ for all $0 \leq j < k$
(if $k = 0$ then ϕ needs never hold)

Safety and Liveness Properties

Safety Properties

- Always-formulas called safety properties:
  "something bad never happens"
- Let `mutex` ("mutual exclusion") be a variable that is true when two processes do not access a critical resource at the same time
- $\Box$ `mutex` expresses that simultaneous access never happens

Safety and Liveness Properties

Safety Properties

- Always-formulas called safety properties:
  "something bad never happens"
- Let `mutex` ("mutual exclusion") be a variable that is true when two processes do not access a critical resource at the same time
- $\Box$ `mutex` expresses that simultaneous access never happens

Liveness Properties

- Eventually-formulas called liveness properties:
  "something good happens eventually"
- Let `s` be variable that is true when a process delivers a service
- $\Diamond$ `s` expresses that service is eventually provided

Complex Properties

What does this mean?

$$\sigma \models \Box\Diamond\phi$$

Complex Properties

Infinitely Often

$$\sigma \models \Box\Diamond\phi$$

“During run σ the formula ϕ becomes true infinitely often”

Validity of Temporal Logic

Definition (Validity)

ϕ is valid, write $\models \phi$, iff $\sigma \models \phi$ for all runs $\sigma = s_0\, s_1 \cdots$.

Validity of Temporal Logic

Definition (Validity)

ϕ is valid, write $\models \phi$, iff $\sigma \models \phi$ for all runs $\sigma = s_0\, s_1 \cdots$.

Recall that each run $s_0\, s_1 \cdots$ essentially is an infinite sequence of interpretations $\mathcal{I}_0\, \mathcal{I}_1 \cdots$

Representation of Runs

Can represent a set of runs as a sequence of propositional formulas:

- $\phi_0\, \phi_1, \cdots$ represents all runs $s_0\, s_1 \cdots$ such that $s_i \models \phi_i$ for $i \geq 0$

Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

Semantics of Temporal Logic: Examples

$\Diamond\Box\phi$

Valid?

No, there is a run where it is not valid:

Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

No, there is a run where it is not valid:

$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Semantics of Temporal Logic: Examples

$\Diamond\Box\phi$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Semantics of Temporal Logic: Examples

$\Diamond\Box\phi$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

Semantics of Temporal Logic: Examples

$\Diamond\Box\phi$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

$\Box\phi \rightarrow \phi$ $\quad(\neg\Box\phi) \leftrightarrow (\Diamond\neg\phi)$ $\quad\Diamond\phi \leftrightarrow (\mathrm{true}\ \mathcal{U}\phi)$

Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

$$\Box\phi \rightarrow \phi \qquad (\neg\Box\phi) \leftrightarrow (\Diamond\neg\phi) \qquad \Diamond\phi \leftrightarrow (\mathrm{true}\;\mathcal{U}\phi)$$

All are valid! (proof is exercise)

Semantics of Temporal Logic: Examples

$$\Diamond\Box\phi$$

Valid?

No, there is a run where it is not valid:
$(\neg\phi\,\neg\phi\,\neg\phi\,\ldots)$

Valid in some run?

Yes, for example: $(\neg\phi\,\phi\,\phi\,\ldots)$

$$\Box\phi \rightarrow \phi \qquad (\neg\Box\phi) \leftrightarrow (\Diamond\neg\phi) \qquad \Diamond\phi \leftrightarrow (\mathrm{true}\ \mathcal{U}\phi)$$

All are valid! (proof is exercise)

- $\Box$ is reflexive
- $\Box$ and $\Diamond$ are dual connectives
- $\Box$ and $\Diamond$ can be expressed with only using $\mathcal{U}$

Transition Systems: Formal Definition

Definition (Transition System)

A transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$ is composed of a set of states S, a set $\emptyset \neq Ini \subseteq S$ of initial states, a transition relation $\delta \subseteq S \times S$, and a labeling $\mathcal{I}$ of each state $s \in S$ with a propositional interpretation $\mathcal{I}_s$.

Definition (Run of Transition System)

A run of $\mathcal{T}$ is a sequence of states $\sigma = s_0\, s_1 \cdots$ such that
$s_0 \in Ini$ and for all i is $s_i \in S$ as well as $(s_i, s_{i+1}) \in \delta$.

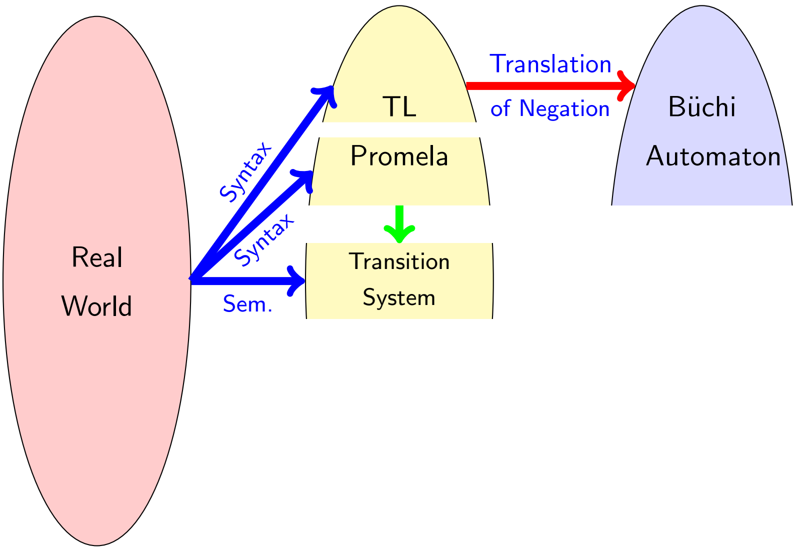
Temporal Logic—Semantics (Cont'd)

Extension of validity of temporal formulas to transition systems:

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, \mathit{Ini}, \delta, \mathcal{I})$, a temporal formula ϕ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs σ of $\mathcal{T}$.

Formal Verification: Model Checking



ω-Languages

Given a finite alphabet (vocabulary) Σ

A word $w \in \Sigma^*$ is a finite sequence

$$w = a_o \cdots a_n$$

with $a_i \in \Sigma, i \in \{0, \dots, n\}$

$\mathcal{L} \subseteq \Sigma^*$ is called a language

ω-Languages

Given a finite alphabet (vocabulary) Σ

An ω-word $w \in \Sigma^\omega$ is an infinite sequence

$$w = a_o \cdots a_k \cdots$$

with $a_i \in \Sigma, i \in \mathbb{N}$

$\mathcal{L}^\omega \subseteq \Sigma^\omega$ is called an ω-language

Büchi Automaton

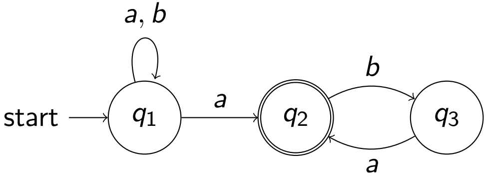
Definition (Büchi Automaton)

A (non-deterministic) Büchi automaton over an alphabet Σ consists of a

- finite, non-empty set of locations Q
- a non-empty set of initial/start locations $I \subseteq Q$
- a set of accepting locations $F = \{F_1, \ldots, F_n\} \subseteq Q$
- a transition relation $\delta \subseteq Q \times \Sigma \times Q$

Example

$\Sigma = \{a, b\}, Q = \{q_1, q_2, q_3\}, I = \{q_1\}, F = \{q_2\}$



Büchi Automaton—Executions and Accepted Words

Definition (Execution)

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton over alphabet Σ.
An execution of $\mathcal{B}$ is a pair (w, v), with

- $w = a_o \cdots a_k \cdots \in \Sigma^\omega$
- $v = q_o \cdots q_k \cdots \in Q^\omega$

where $q_0 \in I$, and $(q_i, a_i, q_{i+1}) \in \delta$, for all $i \in \mathbb{N}$

Büchi Automaton—Executions and Accepted Words

Definition (Execution)

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton over alphabet Σ.
An execution of $\mathcal{B}$ is a pair (w, v), with

- $w = a_o \cdots a_k \cdots \in \Sigma^\omega$
- $v = q_o \cdots q_k \cdots \in Q^\omega$

where $q_0 \in I$, and $(q_i, a_i, q_{i+1}) \in \delta$, for all $i \in \mathbb{N}$

Definition (Accepted Word)

A Büchi automaton $\mathcal{B}$ accepts a word $w \in \Sigma^\omega$, if there exists an execution (w, v) of $\mathcal{B}$ where some accepting location $f \in F$ appears infinitely often in v.

Büchi Automaton—Language

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton, then

$$\mathcal{L}^\omega(\mathcal{B}) = \{w \in \Sigma^\omega | w \in \Sigma^\omega \text{ is an accepted word of } \mathcal{B}\}$$

denotes the ω-language recognised by $\mathcal{B}$.

Büchi Automaton—Language

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton, then

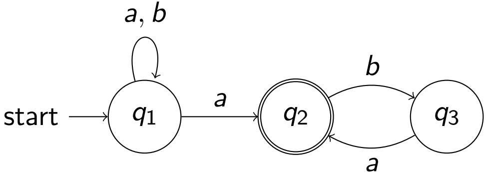
$$\mathcal{L}^\omega(\mathcal{B}) = \{w \in \Sigma^\omega | w \in \Sigma^\omega \text{ is an accepted word of } \mathcal{B}\}$$

denotes the ω-language recognised by $\mathcal{B}$.

An ω-language for which an accepting Büchi automaton exists is called ω-regular language.

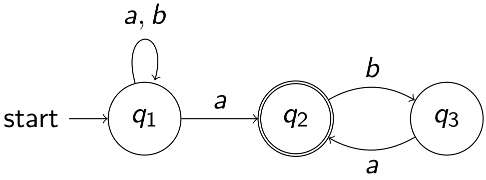
Example, ω-Regular Expression

Which language is accepted by the following Büchi automaton?



Example, ω-Regular Expression

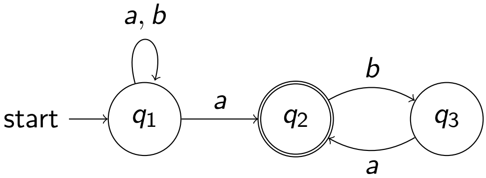
Which language is accepted by the following Büchi automaton?



Solution: $(a+b)^*(ab)^\omega$ [NB: $(ab)^\omega = a(ba)^\omega$]

Example, ω-Regular Expression

Which language is accepted by the following Büchi automaton?



Solution: $(a+b)^*(ab)^\omega$ [NB: $(ab)^\omega = a(ba)^\omega$]

ω-regular expressions similar to standard regular expression

- ab a **followed by** b
- $a+b$ a **or** b
- a^* arbitrarily, but finitely often a
- **new:** a^ω infinitely often a

Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

Theorem (Decidability)

It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton $\mathcal{B}$ is empty.

Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

Theorem (Decidability)

It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton $\mathcal{B}$ is empty.

Theorem (Closure properties)

The set of ω-regular languages is closed with respect to intersection, union and complement:

- *if $\mathcal{L}_1, \mathcal{L}_2$ are ω-regular then $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are ω-regular*
- *$\mathcal{L}$ is ω-regular then $\Sigma^\omega \backslash \mathcal{L}$ is ω-regular*

Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

Theorem (Decidability)

It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton $\mathcal{B}$ is empty.

Theorem (Closure properties)

The set of ω-regular languages is closed with respect to intersection, union and complement:

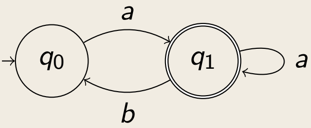
- *if $\mathcal{L}_1, \mathcal{L}_2$ are ω-regular then $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are ω-regular*
- *$\mathcal{L}$ is ω-regular then $\Sigma^\omega \backslash \mathcal{L}$ is ω-regular*

But in contrast to regular finite automata:

Non-deterministic Büchi automata are strictly more expressive than deterministic ones.

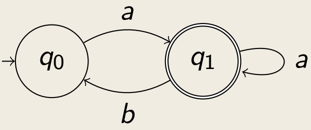
Büchi Automata—More Examples

Language:



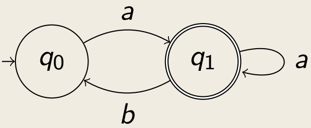
Büchi Automata—More Examples

Language: $a(a + ba)^\omega$

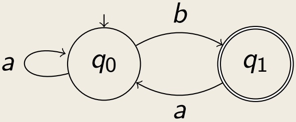


Büchi Automata—More Examples

Language: $a(a + ba)^\omega$

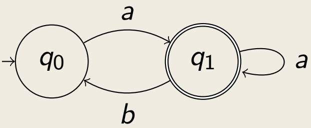


Language:

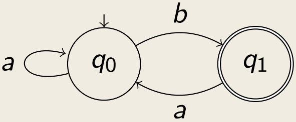


Büchi Automata—More Examples

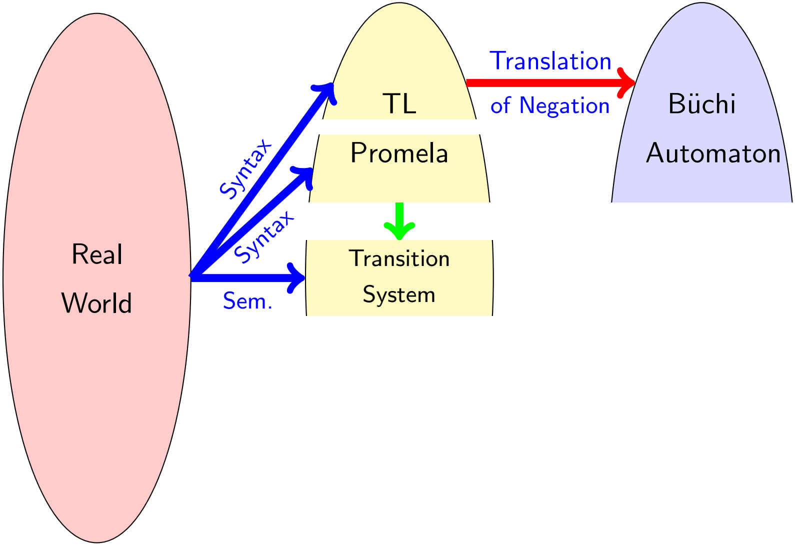
Language: $a(a + ba)^\omega$



Language: $(a^* ba)^\omega$



Formal Verification: Model Checking



Linear Temporal Logic and Büchi Automata

LTL and Büchi Automata are connected

Recall

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, \mathit{Ini}, \delta, \mathcal{I})$, a temporal formula ϕ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs σ of $\mathcal{T}$.

A run of the transition system is an infinite sequence of interpretations $\mathcal{I}$.

Linear Temporal Logic and Büchi Automata

LTL and Büchi Automata are connected

Recall

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$, a temporal formula ϕ is valid in $\mathcal{T}$ (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs σ of $\mathcal{T}$.

A run of the transition system is an infinite sequence of interpretations $\mathcal{I}$.

Intended Connection

Given an LTL formula ϕ:

Construct a Büchi automaton accepting exactly those runs (infinite sequences of interpretations) that satisfy ϕ.

Encoding an LTL Formula as a Büchi Automaton

$\mathcal{P}$ set of propositional variables, e.g., $\mathcal{P} = \{r, s\}$

Suitable alphabet Σ for Büchi automaton?

Encoding an LTL Formula as a Büchi Automaton

$\mathcal{P}$ set of propositional variables, e.g., $\mathcal{P} = \{r, s\}$

Suitable alphabet Σ for Büchi automaton?

A state transition of Büchi automaton must represent an interpretation

Encoding an LTL Formula as a Büchi Automaton

$\mathcal{P}$ set of propositional variables, e.g., $\mathcal{P} = \{r, s\}$

Suitable alphabet Σ for Büchi automaton?

A state transition of Büchi automaton must represent an interpretation

Choose Σ to be the set of all interpretations over $\mathcal{P}$, encoded as $2^{\mathcal{P}}$

Example

$\Sigma = \big\{\emptyset, \{r\}, \{s\}, \{r, s\}\big\}$

$$I_\emptyset(r) = F, I_\emptyset(s) = F, I_{\{r\}}(r) = T, I_{\{r\}}(s) = F, \ldots$$

Büchi Automaton for LTL Formula By Example

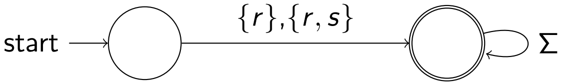
Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those runs σ satisfying r

Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those runs σ satisfying r

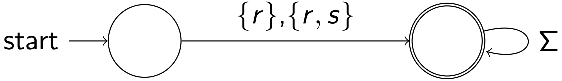


In the first state s_0 (of σ) at least r must hold, the rest is arbitrary

Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those runs σ satisfying r



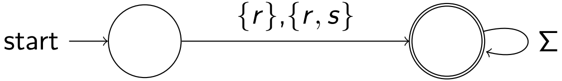
In the first state s_0 (of σ) at least r must hold, the rest is arbitrary

Example (Büchi automaton for formula $\Box r$ over $\mathcal{P} = \{r, s\}$)

Büchi Automaton for LTL Formula By Example

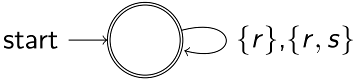
Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those runs σ satisfying r



In the first state s_0 (of σ) at least r must hold, the rest is arbitrary

Example (Büchi automaton for formula $\Box r$ over $\mathcal{P} = \{r, s\}$)

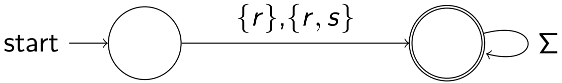


In *all* states s (of σ) at least r must hold

Büchi Automaton for LTL Formula By Example

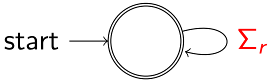
Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton $\mathcal{B}$ accepting exactly those runs σ satisfying r



In the first state s_0 (of σ) at least r must hold, the rest is arbitrary

Example (Büchi automaton for formula $\Box r$ over $\mathcal{P} = \{r, s\}$)



$\Sigma_r := \{I | I \in \Sigma, r \in I\}$

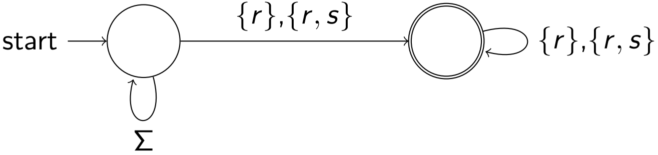
In *all* states s (of σ) at least r must hold

Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula $\Diamond\Box r$ over $\mathcal{P} = \{r, s\}$)

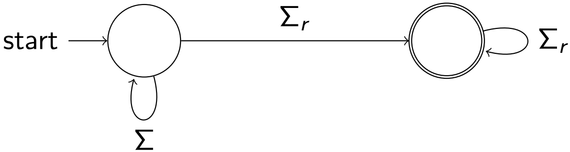
Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula $\Diamond\Box r$ over $\mathcal{P} = \{r, s\}$)

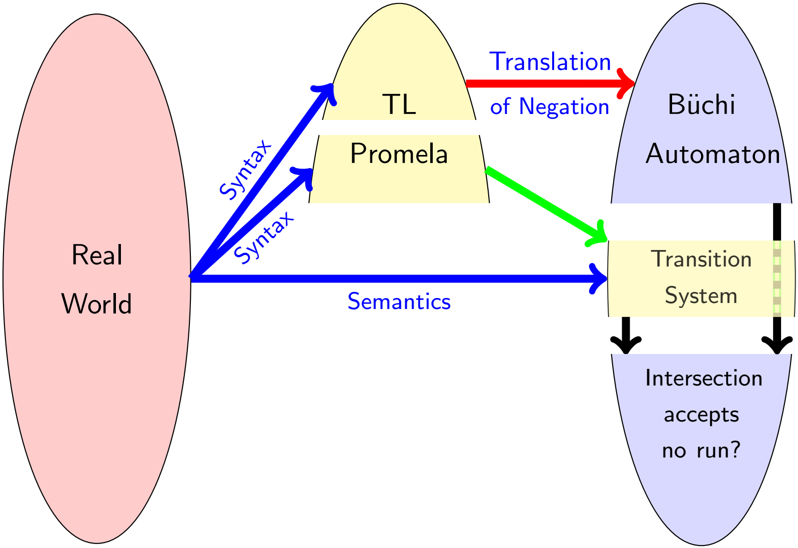


Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula $\Diamond\Box r$ over $\mathcal{P} = \{r, s\}$)



Formal Verification: Model Checking



Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system $\mathcal{T}$ (e.g., derived from a PROMELA program).

Verification task: is the LTL formula ϕ satisfied in all runs of $\mathcal{T}$, i.e.,

$$\mathcal{T} \models \phi \quad ?$$

Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system $\mathcal{T}$ (e.g., derived from a PROMELA program).

Verification task: is the LTL formula ϕ satisfied in all runs of $\mathcal{T}$, i.e.,

$$\mathcal{T} \models \phi \quad ?$$

Temporal model checking with SPIN: Topic of next lecture

Model Checking

Check whether a formula is valid in all runs of a transition system.

Given a transition system $\mathcal{T}$ (e.g., derived from a PROMELA program).

Verification task: is the LTL formula ϕ satisfied in all runs of $\mathcal{T}$, i.e.,

$$\mathcal{T} \models \phi \quad ?$$

Temporal model checking with SPIN: Topic of next lecture

Today: Basic principle behind SPIN model checking

SPIN Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$

Spin Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for negation of formula ϕ

Spin Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for negation of formula ϕ
3. If

$$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) = \emptyset$$

then $\mathcal{T} \models \phi$ holds.

SPIN Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for negation of formula ϕ
3. If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) = \emptyset$$

   then $\mathcal{T} \models \phi$ holds.

   If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) \neq \emptyset$$

   then each element of the set is a counterexample for ϕ.

Spin Model Checking—Overview

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through $\mathcal{T}$
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for negation of formula ϕ
3. If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) = \emptyset$$

   then $\mathcal{T} \models \phi$ holds.

   If

   $$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) \neq \emptyset$$

   then each element of the set is a counterexample for ϕ.

   To check $\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi})$ construct intersection automaton and search for cycle through accepting state.

Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of $\mathcal{T}$

Example

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```

Similar code for process `Q`.

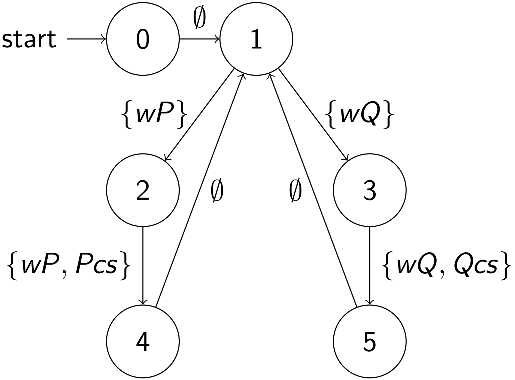
Second **atomic** block just to keep automaton small.

Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of $\mathcal{T}$

Example

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```



Similar code for process Q.

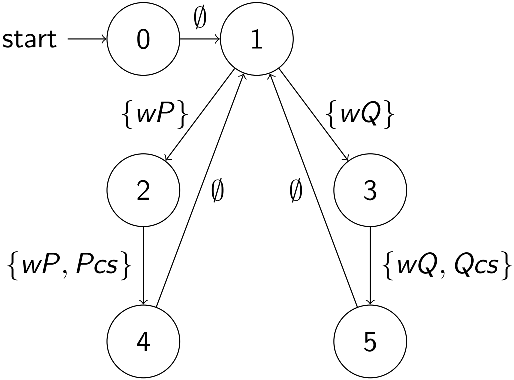
Second **atomic** block just to keep automaton small.

Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of $\mathcal{T}$

Example

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```



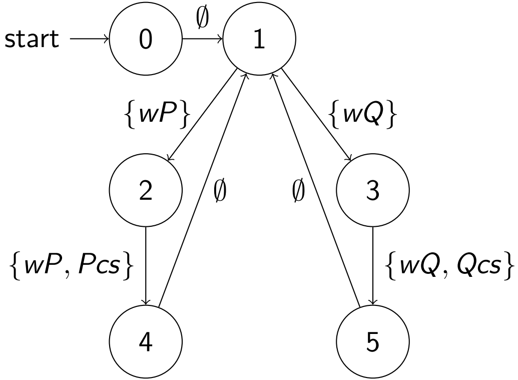
Which are the accepting locations?

Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_\mathcal{T}$ accepting exactly those words representing a run of $\mathcal{T}$

Example

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```



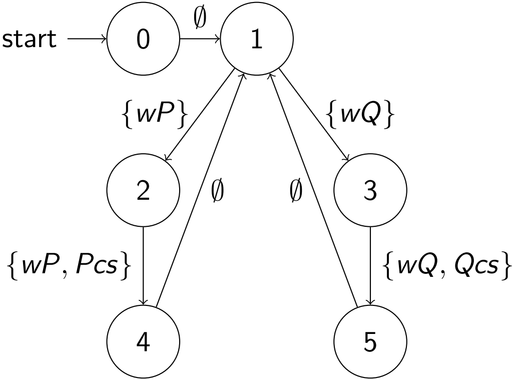
Which are the accepting locations? All!

Representing a Model as a Büchi Automaton

First Step: Represent transition system $\mathcal{T}$ as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of $\mathcal{T}$

Example

```
active proctype P () {
do
  :: atomic {
      !wQ; wP = true
     };
     Pcs = true;
     atomic {
      Pcs = false;
      wP = false
     }
od }
```



The property we want to check is $\phi = \Box\neg Pcs$ (which does not hold).

Büchi Automaton $B_{\neg\phi}$ for $\neg\phi$

Second Step:

Construct Büchi automaton corresponding to negated LTL formula

$\mathcal{T} \models \phi$ holds iff there is no accepting run σ of $\mathcal{T}$ s.t. $\sigma \models \neg\phi$

Simplify $\neg\phi = \neg\Box\neg \mathit{Pcs} = \Diamond \mathit{Pcs}$

Büchi Automaton $B_{\neg\phi}$ for $\neg\phi$

Second Step:

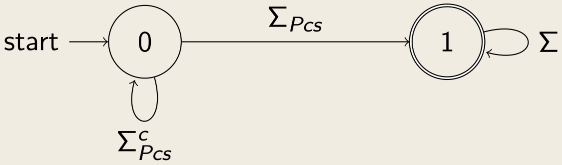
Construct Büchi automaton corresponding to negated LTL formula

$\mathcal{T} \models \phi$ holds iff there is no accepting run σ of $\mathcal{T}$ s.t. $\sigma \models \neg\phi$

Simplify $\neg\phi = \neg\Box\neg Pcs = \Diamond Pcs$

Büchi Automaton $\mathcal{B}_{\neg\phi}$

$$\mathcal{P} = \{wP, wQ, Pcs, Qcs\},\ \Sigma = 2^{\mathcal{P}}$$



$$\Sigma_{Pcs} = \{I | I \in \Sigma, Pcs \in I\}, \quad \Sigma^c_{Pcs} = \Sigma - \Sigma_{Pcs}$$

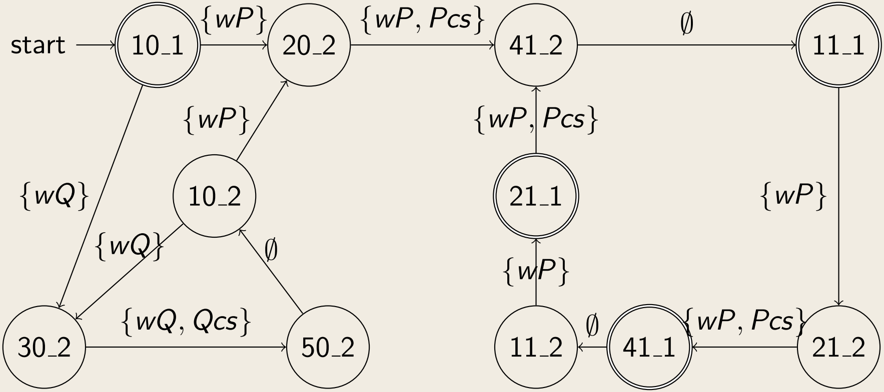
Checking for Emptiness of Intersection Automaton

Third Step: $\mathcal{L}^\omega(\mathcal{B}_\mathcal{T}) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) = \emptyset$?

Checking for Emptiness of Intersection Automaton

Third Step: $\mathcal{L}^\omega(\mathcal{B}_\mathcal{T}) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) = \emptyset$?

Intersection Automaton (skipping first step of $\mathcal{T}$ for simplicity)

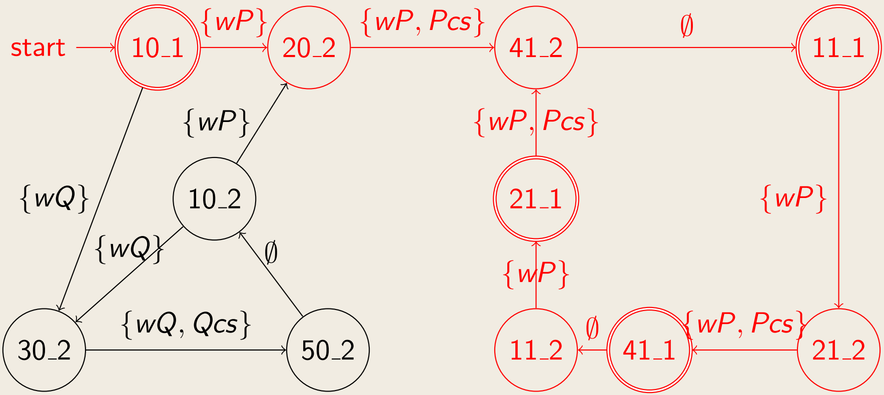


Checking for Emptiness of Intersection Automaton

Third Step: $\mathcal{L}^\omega(\mathcal{B}_\mathcal{T}) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) \neq \emptyset$

Counterexample

Intersection Automaton (skipping first step of $\mathcal{T}$ for simplicity)

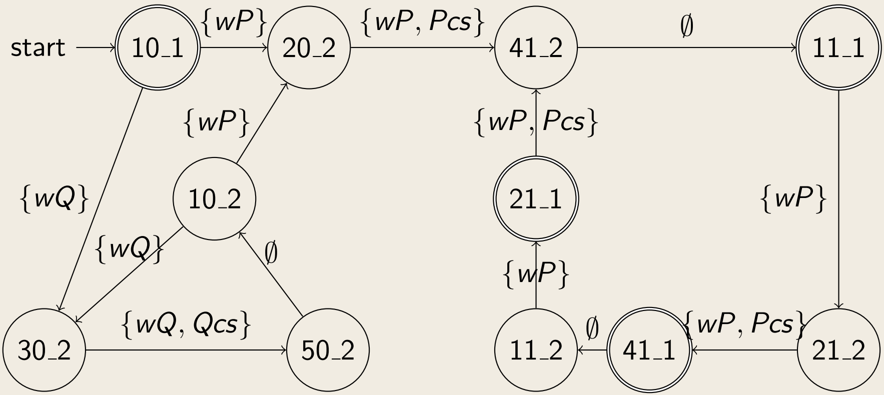


Checking for Emptiness of Intersection Automaton

Third Step: $\mathcal{L}^\omega(\mathcal{B}_\mathcal{T}) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) \neq \emptyset$

Counterexample   Construction of intersection automaton: Appendix

Intersection Automaton (skipping first step of $\mathcal{T}$ for simplicity)



Literature for this Lecture

Ben-Ari Section 5.2.1
(only syntax of LTL)

Baier and Katoen Principles of Model Checking,
May 2008, The MIT Press,
ISBN: 0-262-02649-X

Appendix I:

Intersection Automaton

–

Construction

Construction of Intersection Automaton

Given: two Büchi automata $\mathcal{B}_i = (Q_i, \delta_i, I_i, F_i)$, $i = 1, 2$

Wanted: a Büchi automaton

$$\mathcal{B}_{1\cap 2} = (Q_{1\cap 2}, \delta_{1\cap 2}, I_{1\cap 2}, F_{1\cap 2})$$

accepting a word w iff w is accepted by $\mathcal{B}_1$ and $\mathcal{B}_2$

Construction of Intersection Automaton

Given: two Büchi automata $\mathcal{B}_i = (Q_i, \delta_i, I_i, F_i)$, $i = 1, 2$

Wanted: a Büchi automaton

$$\mathcal{B}_{1\cap2} = (Q_{1\cap2}, \delta_{1\cap2}, I_{1\cap2}, F_{1\cap2})$$

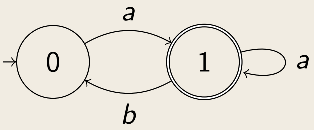
accepting a word w iff w is accepted by $\mathcal{B}_1$ and $\mathcal{B}_2$

Maybe just the product automaton as for regular automata?

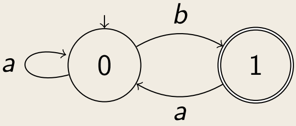
Product Automata for Intersection

$\Sigma = \{a, b\}$

$a(a + ba)^\omega$:



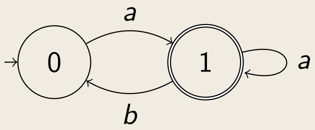
$(a^* ba)^\omega$:



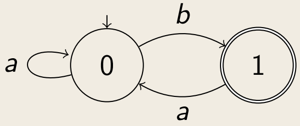
Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^* ba)^\omega = \emptyset$?

$a(a + ba)^\omega$:



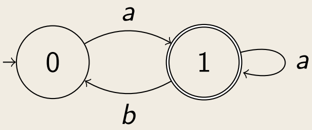
$(a^* ba)^\omega$:



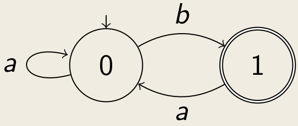
Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^* ba)^\omega = \emptyset$? No, e.g., $a(ba)^\omega$

$a(a + ba)^\omega$:



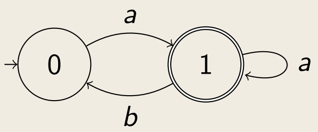
$(a^* ba)^\omega$:



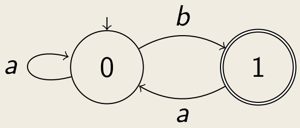
Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^* ba)^\omega = \emptyset$? No, e.g., $a(ba)^\omega$

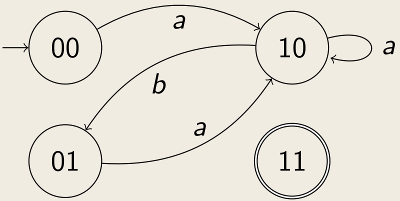
$a(a + ba)^\omega$:



$(a^* ba)^\omega$:

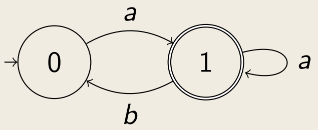


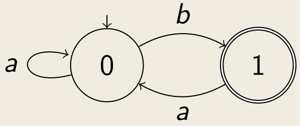
Product Automaton:



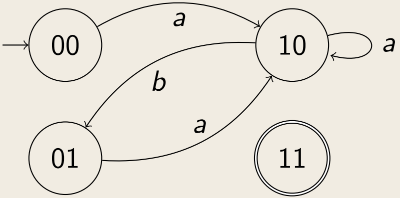
First Attempt: Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^* ba)^\omega = \emptyset$? No, e.g., $a(ba)^\omega$



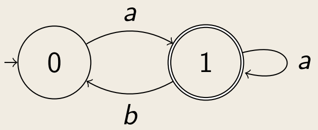


Product Automaton: accepting location 11 never reached

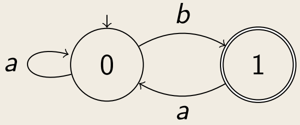


Explicit Construction of Intersection Automaton

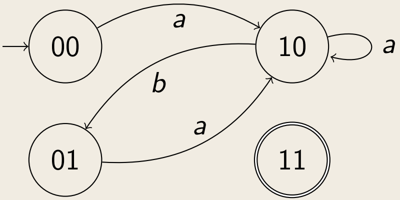
$a(a + ba)^\omega$:



$(a^* ba)^\omega$:



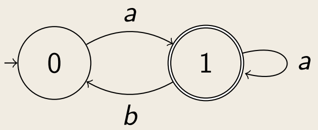
(i) Product Automaton



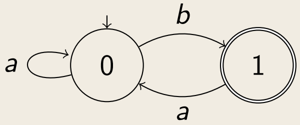
$Q_\cap = Q_1 \times Q_2$

Explicit Construction of Intersection Automaton

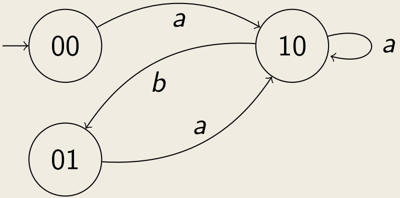
$a(a + ba)^\omega$:



$(a^* ba)^\omega$:



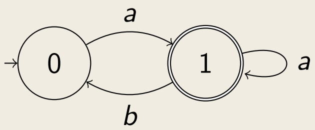
(ii) Reachable Locations



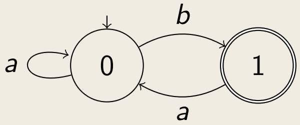
$Q_\cap = Q_1 \times Q_2$

Explicit Construction of Intersection Automaton

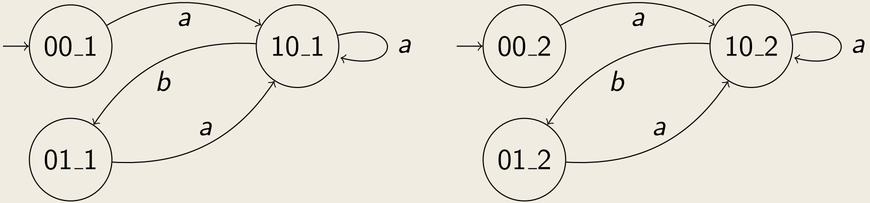
$a(a+ba)^\omega$:



$(a^* ba)^\omega$:



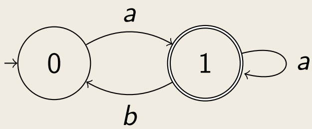
(iii) Clone



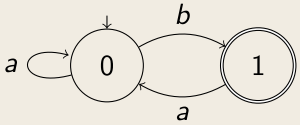
$Q_\cap = Q_1 \times Q_2 \times \{1, 2\}$

Explicit Construction of Intersection Automaton

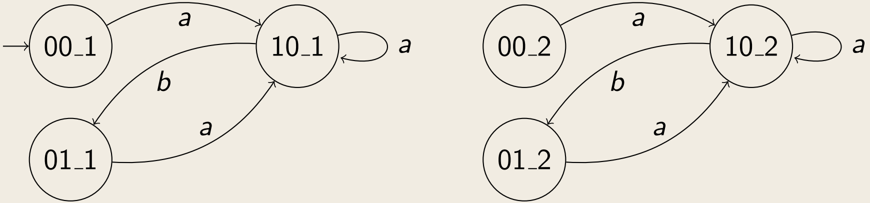
$a(a + ba)^\omega$:



$(a^* ba)^\omega$:



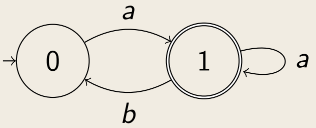
(iv) Initial Locations Restricted to First Copy



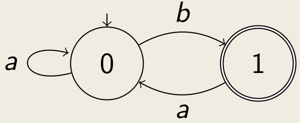
$Q_\cap = Q_1 \times Q_2 \times \{1, 2\}, I_\cap = I_1 \times I_2 \times \{1\}$

Explicit Construction of Intersection Automaton

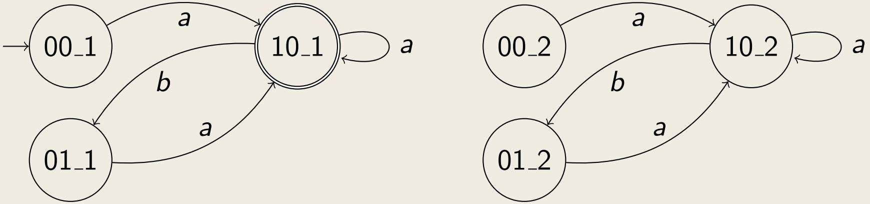
$a(a+ba)^\omega$:



$(a^*ba)^\omega$:



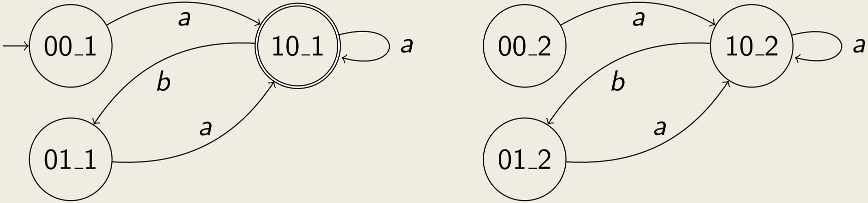
(v) Final Locations Restricted to First Atomaton of First Copy



$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

Explicit Construction of Intersection Automaton

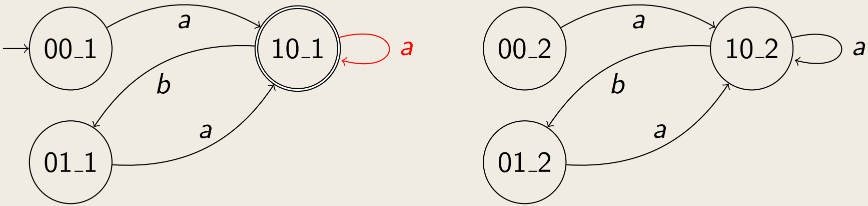
(v) Final Locations Restricted to First Atomaton of First Copy



$Q_\cap = Q_1 \times Q_2 \times \{1, 2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

Explicit Construction of Intersection Automaton

(vi) Ensure Acceptance in Both Copies $1 \to 2$



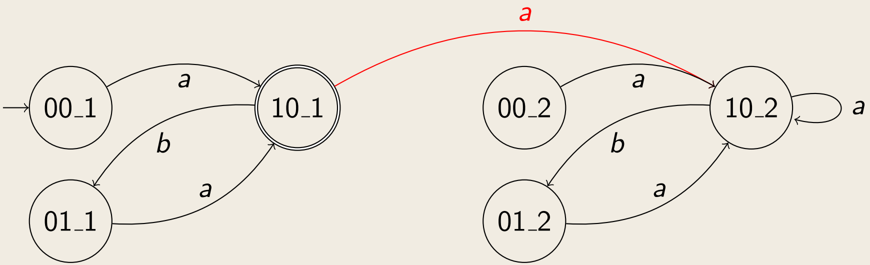
$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma$:

if $s_1 \in F_1$: $\quad \delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(vi) Ensure Acceptance in Both Copies $1 \to 2$



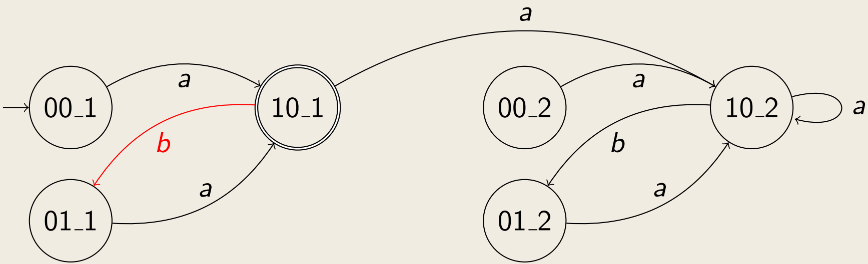
$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma$:

if $s_1 \in F_1$: $\quad \delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(vi) Ensure Acceptance in Both Copies $1 \to 2$



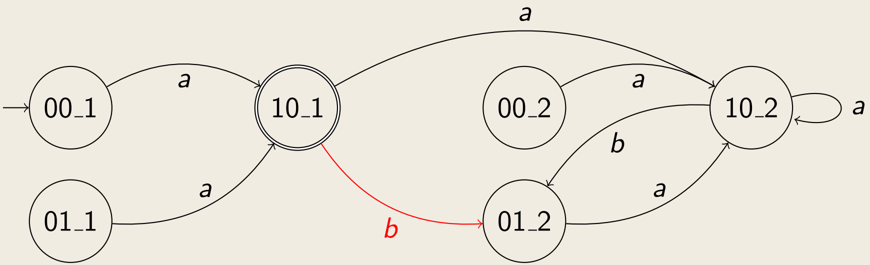
$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma:$

if $s_1 \in F_1$: $\quad \delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(vi) Ensure Acceptance in Both Copies $1 \to 2$



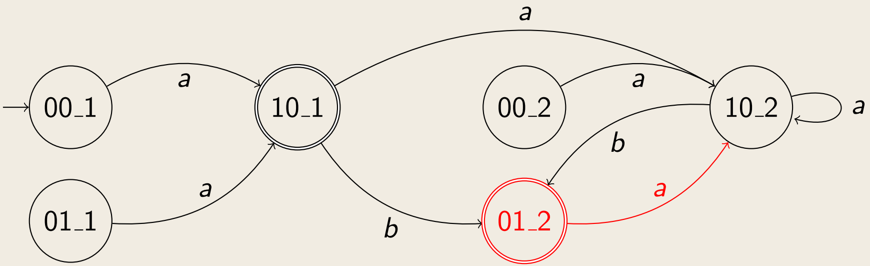
$Q_\cap = Q_1 \times Q_2 \times \{1, 2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma :$

if $s_1 \in F_1$: $\delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(vii) Ensure Acceptance in Both Copies $2 \rightarrow 1$



$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

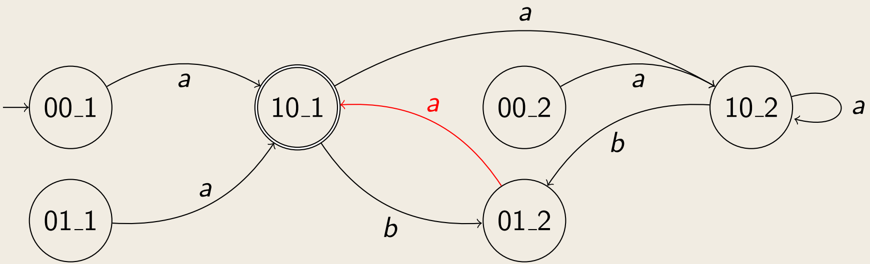
$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma$:

if $s_1 \in F_1$: $\delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

if $s_2 \in F_2$: $\delta_\cap((s_1, s_2, 2), \alpha) = \{(s_1', s_2', 1) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(vii) Ensure Acceptance in Both Copies $2 \to 1$



$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

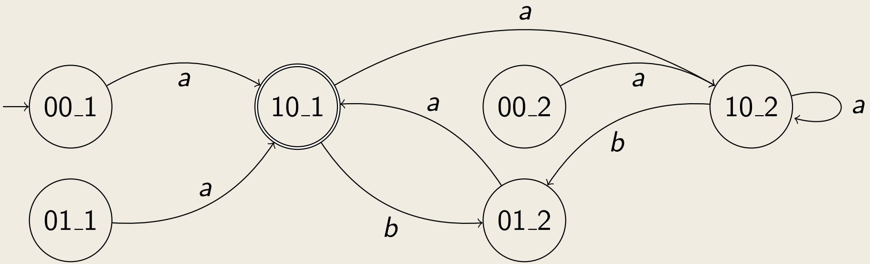
$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma :$

if $s_1 \in F_1$: $\delta_\cap((s_1, s_2, 1), \alpha) = \{(s_1', s_2', 2) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

if $s_2 \in F_2$: $\delta_\cap((s_1, s_2, 2), \alpha) = \{(s_1', s_2', 1) | s_1' \in \delta_1(s_1, \alpha), s_2' \in \delta_2(s_2, \alpha)\}$

Explicit Construction of Intersection Automaton

(viii) Transitions of Product Automaton



$Q_\cap = Q_1 \times Q_2 \times \{1,2\}, I_\cap = I_1 \times I_2 \times \{1\}, F = F_1 \times Q_2 \times \{1\}$

$s_1 \in Q_1, s_2 \in Q_2, \alpha \in \Sigma :$

if $s_1 \in F_1 :$ $\delta_\cap((s_1, s_2, 1), \alpha) = \{(s'_1, s'_2, 2) | s'_1 \in \delta_1(s_1, \alpha), s'_2 \in \delta_2(s_2, \alpha)\}$

if $s_2 \in F_2 :$ $\delta_\cap((s_1, s_2, 2), \alpha) = \{(s'_1, s'_2, 1) | s'_1 \in \delta_1(s_1, \alpha), s'_2 \in \delta_2(s_2, \alpha)\}$

else: $\delta_\cap((s_1, s_2, i), \alpha) = \{(s'_1, s'_2, i) | s'_1 \in \delta_1(s_1, \alpha), s'_2 \in \delta_2(s_2, \alpha)\}$

Appendix II:

Construction of a Büchi Automaton $\mathcal{B}_\phi$ for an LTL-Formula ϕ

The General Case: Generalised Büchi Automata

A generalised Büchi automaton is defined as:

$$\mathcal{B}^g = (Q, \delta, I, \mathbb{F})$$

Q, δ, I as for standard Büchi automata

$\mathbb{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$, where $\mathcal{F}_i = \{q_{i1}, \ldots, q_{im_i}\} \subseteq Q$

The General Case: Generalised Büchi Automata

A generalised Büchi automaton is defined as:

$$\mathcal{B}^g = (Q, \delta, I, \mathbb{F})$$

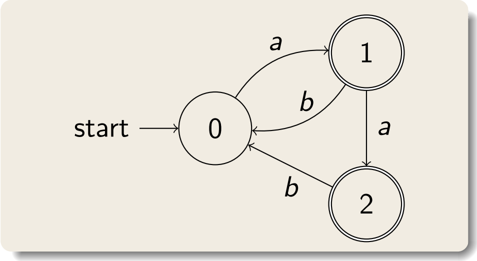
Q, δ, I as for standard Büchi automata

$\mathbb{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$, where $\mathcal{F}_i = \{q_{i1}, \ldots, q_{im_i}\} \subseteq Q$

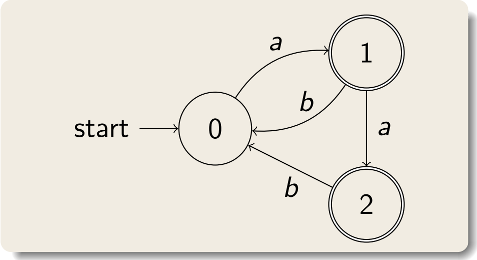
Definition (Acceptance for generalised Büchi automata)

A generalised Büchi automaton accepts an ω-word $w \in \Sigma^\omega$ iff for every $i \in \{1, \ldots, n\}$ at least one $q_{ik} \in \mathcal{F}_i$ is visited infinitely often.

Normal vs. Generalised Büchi Automata: Example

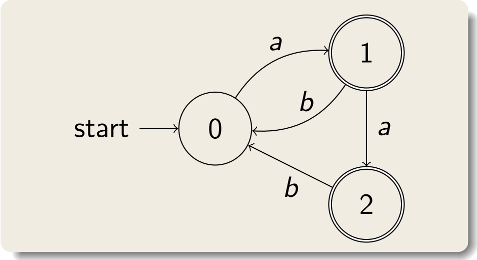


Normal vs. Generalised Büchi Automata: Example



$$\mathcal{B}^{normal} \text{ with } \mathcal{F} = \{1, 2\}, \qquad \mathcal{B}^{general} \text{ with } \mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$$

Normal vs. Generalised Büchi Automata: Example

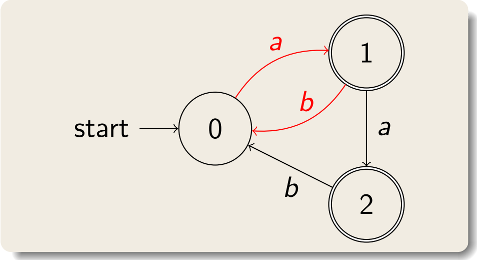


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\quad \mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$

Normal vs. Generalised Büchi Automata: Example

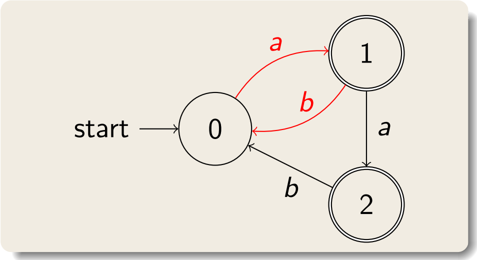


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$		

Normal vs. Generalised Büchi Automata: Example

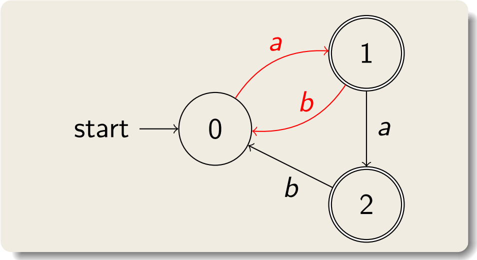


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$	✔	

Normal vs. Generalised Büchi Automata: Example

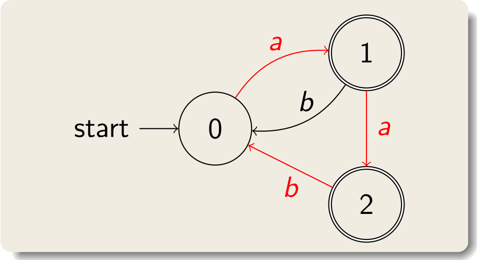


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$	✔	✘

Normal vs. Generalised Büchi Automata: Example

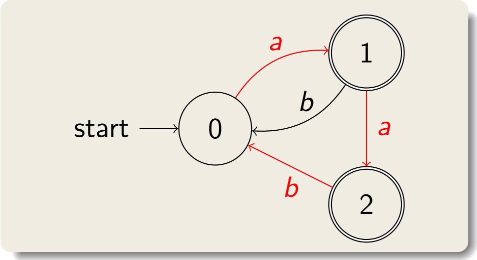


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\quad \mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$	✔	✘
$(aab)^\omega$		

Normal vs. Generalised Büchi Automata: Example

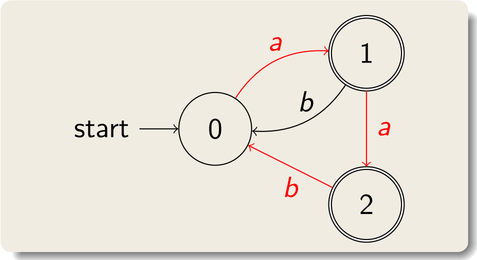


$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$	✔	✘
$(aab)^\omega$	✔	

Normal vs. Generalised Büchi Automata: Example



$\mathcal{B}^{normal}$ with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω-word is accepted by which automaton?

ω-word	$\mathcal{B}^{normal}$	$\mathcal{B}^{general}$
$(ab)^\omega$	✔	✘
$(aab)^\omega$	✔	✔

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

$$FL(\phi) = \{\varphi | \varphi \text{ is subformula or negated subformula of } \phi\}$$

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

$$FL(\phi) = \{\varphi | \varphi \text{ is subformula or negated subformula of } \phi\}$$

($\neg\neg\varphi$ is identified with φ)

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

$$FL(\phi) = \{\varphi | \varphi \text{ is subformula or negated subformula of } \phi\}$$

($\neg\neg\varphi$ is identified with φ)

Example

$$FL(r\,\mathcal{U}s) = \{r, \neg r, s, \neg s, r\,\mathcal{U}s, \neg(r\,\mathcal{U}s)\}$$

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total

- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$

Downward Closed

- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total

- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$

Downward Closed

- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r\,\mathcal{U}\,s) = \{r, \neg r, s, \neg s, r\,\mathcal{U}\,s, \neg(r\,\mathcal{U}\,s)\}$$

$$\underline{\qquad\qquad\qquad\qquad\qquad \in Q}$$

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total
- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$

Downward Closed
- $\psi_1 \land \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r\mathcal{U}s) = \{r, \neg r, s, \neg s, r\mathcal{U}s, \neg(r\mathcal{U}s)\}$$

	$\in Q$
$\{r\mathcal{U}s, \neg r, s\}$	✔

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total
- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$

Downward Closed
- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r\mathcal{U}s) = \{r, \neg r, s, \neg s, r\mathcal{U}s, \neg(r\mathcal{U}s)\}$$

	$\in Q$
$\{r\mathcal{U}s, \neg r, s\}$	✔
$\{r\mathcal{U}s, \neg r, \neg s\}$	✘

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total
- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \backslash q)$ then $\psi_2 \notin q$

Downward Closed
- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r \mathcal{U} s) = \{r, \neg r, s, \neg s, r \mathcal{U} s, \neg(r \mathcal{U} s)\}$$

	$\in Q$
$\{r \mathcal{U} s, \neg r, s\}$	✔
$\{r \mathcal{U} s, \neg r, \neg s\}$	✘
$\{\neg(r \mathcal{U} s), r, s\}$	✘

$\mathcal{B}_\phi$-Construction: Locations

Assumption:

$\mathcal{U}$ only temporal logic operator in LTL-formula (can express $\Box, \Diamond$ with $\mathcal{U}$)

Locations of $\mathcal{B}_\phi$ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

Consistent, Total

- $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
- $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \backslash q)$ then $\psi_2 \notin q$

Downward Closed

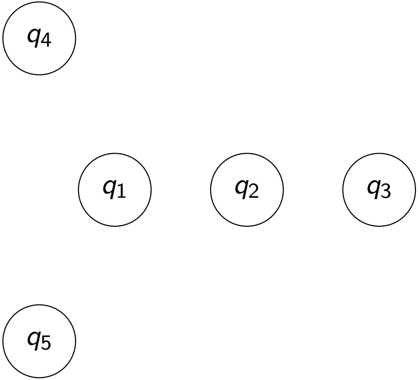
- $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
- ...other propositional connectives similar
- $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(r\mathcal{U}s) = \{r, \neg r, s, \neg s, r\mathcal{U}s, \neg(r\mathcal{U}s)\}$$

	$\in Q$
$\{r\mathcal{U}s, \neg r, s\}$	✔
$\{r\mathcal{U}s, \neg r, \neg s\}$	✘
$\{\neg(r\mathcal{U}s), r, s\}$	✘
$\{\neg(r\mathcal{U}s), r, \neg s\}$	✔

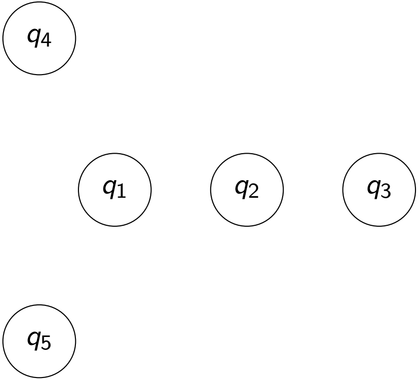
$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



Transitions $(q, \alpha, q') \in \delta_\phi$:

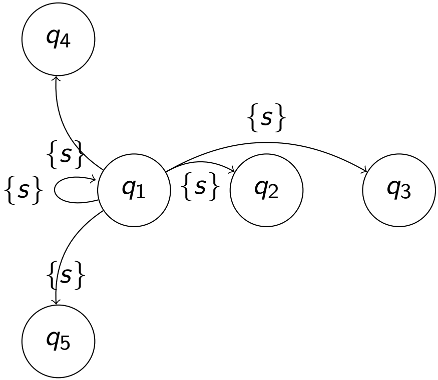
$\alpha = q \cap \mathcal{P}$

$\mathcal{P}$ set of propositional variables outgoing edges of q_1 labeled $\{s\}$, of q_2 labeled $\{r\}$, etc.

1. If $\psi_1\,\mathcal{U}\psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1\,\mathcal{U}\psi_2 \in q'$
2. If $\psi_1\,\mathcal{U}\psi_2 \in (FL(\phi)\backslash q)$ and $\psi_1 \in q$ then $\psi_1\,\mathcal{U}\psi_2 \notin q'$

$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U} s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U} s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U} s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U} s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U} s), \neg r, \neg s\}}_{q_5}$$



Transitions $(q, \alpha, q') \in \delta_\phi$:

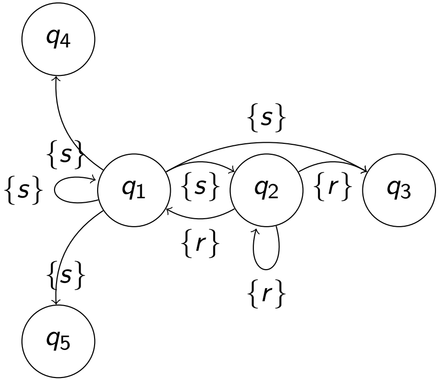
$\alpha = q \cap \mathcal{P}$

$\mathcal{P}$ set of propositional variables
outgoing edges of q_1 labeled $\{s\}$, of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \mathcal{U} \psi_2 \in q'$
2. If $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \backslash q)$ and $\psi_1 \in q$ then $\psi_1 \mathcal{U} \psi_2 \notin q'$

$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



Transitions $(q, \alpha, q') \in \delta_\phi$:

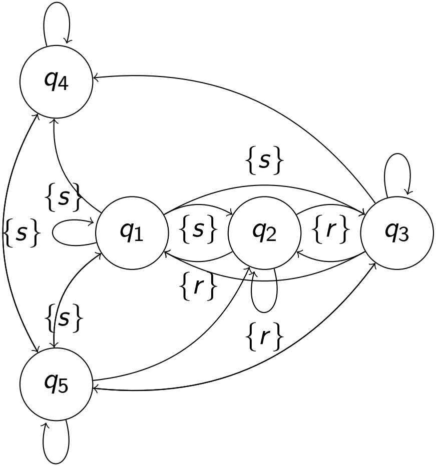
$\alpha = q \cap \mathcal{P}$

$\mathcal{P}$ set of propositional variables
outgoing edges of q_1 labeled $\{s\}$, of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \mathcal{U} \psi_2 \in q'$
2. If $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \backslash q)$ and $\psi_1 \in q$ then $\psi_1 \mathcal{U} \psi_2 \notin q'$

$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



Transitions $(q, \alpha, q') \in \delta_\phi$:

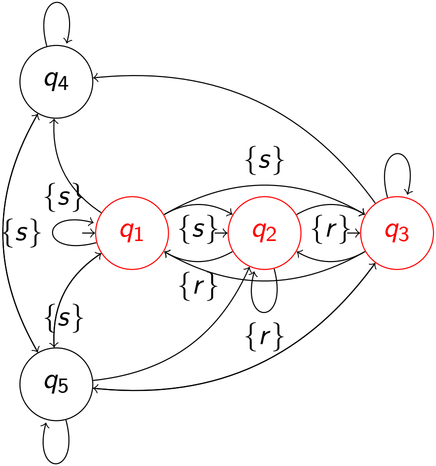
$\alpha = q \cap \mathcal{P}$

$\mathcal{P}$ set of propositional variables outgoing edges of q_1 labeled $\{s\}$, of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 \mathcal{U} \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 \mathcal{U} \psi_2 \in q'$
2. If $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ and $\psi_1 \in q$ then $\psi_1 \mathcal{U} \psi_2 \notin q'$

$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$

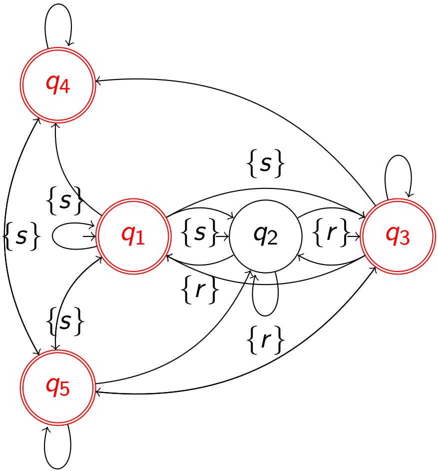


Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

$\mathcal{B}_\phi$-Construction: Transitions

$$\underbrace{\{r\,\mathcal{U}s, \neg r, s\}}_{q_1}, \underbrace{\{r\,\mathcal{U}s, r, \neg s\}}_{q_2}, \underbrace{\{r\,\mathcal{U}s, r, s\}}_{q_3}, \underbrace{\{\neg(r\,\mathcal{U}s), r, \neg s\}}_{q_4}, \underbrace{\{\neg(r\,\mathcal{U}s), \neg r, \neg s\}}_{q_5}$$



Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

Accepting locations

$$\mathbb{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$$

- One $\mathcal{F}_i$ for each $\psi_{i1}\,\mathcal{U}\psi_{i2} \in FL(\phi)$;
  Example: $\mathbb{F} = \{\mathcal{F}_1\}$
- $\mathcal{F}_i$ set of locations that do *not* contain $\psi_{i1}\,\mathcal{U}\psi_{i2}$ or that contain ψ_{i2}
  Ex.: $\mathcal{F}_1 = \{q_1, q_3, q_4, q_5\}$

Remarks on Generalized Büchi Automata

- Construction always gives exponential number of states in $|\phi|$
- Satisfiability checking of LTL is PSPACE-complete
- There exist (more complex) constructions that minimize number of required states
  - One of these is used in Spin, which moreover computes the states lazily