Advanced Algorithms 2016. Exercises 6-8

Remark: Technically the following exercises should be rather simple. But what we also evaluate is the clean and correct use of probability-theoretic concepts. Show and explain your calculations, not only the final answers. Prove all claims, e.g., about expected numbers.

Exercise 6

Consider the following gambling strategy: on the first play, stake \$1; on the second play \$2; on the third play \$4; on the fourth play \$8; and so on. That is, the player is choosing the amount to stake on the i^{th} play to be $\$2^{i-1}$.

When the player wins the first time, he/she stops playing and leaves the casino!

(a) Show that, when leaving the casino, the player is \$1 richer with probability 1.
(b) What is the maximum loss before winning?

Exercise 7

A bus in a city starts from its designated station S_0 with m passengers on board and travels through intermediate stations S_1, S_2, S_3, …, until it stops at the final destination S_n of the bus. Each passengers gets off at one of the S_1, S_2, S_3, … S_n stations uniformly at random (independently of everybody else).
What is the expected number of stations the bus stops at?
Note: Obviously, we don't count S_0.

Exercise 8

Let a simple online auction system works as follows. There are n bidding agents; agent i has a bid b_i. All bids are distinct positive natural numbers. The bidding agents appear in an order chosen uniformly at random. Each proposes its bid b_i in turn, and at all times the system maintains a variable b^* equal to the highest bid seen so far. Assume an initial value of 0 for b^*.
Give an upper bound on the expected number of times the b^* is updated when this process is executed, as a function of the parameters in the problem.

Example: Suppose $b_1 = 20$, $b_2 = 25$, and $b_3 = 10$, and the bidders arrive in the order 1, 3, 2.
Then b^* is updated for 1 and 2, but not for 3.