Some Writing Examples

Assume that $\forall A: \ (|A| = n \Longrightarrow \forall x, y \in A: \ f(x) = f(y))$.
Let $|B| = n+1$, $x, y \in B$, $x \neq y$, $C := B \setminus \{y\}$, $D := B \setminus \{x\}$.
$|C| = n$, $|D| = n$, $x \in C$, $y \in D$.
Let $u \in B$, $u \neq x$, $u \neq y$.
$u \in C$, $f(x) = f(u)$, $u \in D$, $f(y) = f(u)$, $f(x) = f(y)$.
$\forall x, y \in B: \ f(x) = f(y)$.
$\forall B: \ (|B| = n+1 \Longrightarrow \forall x, y \in B: \ f(x) = f(y))$.
$\forall B: \ (|B| = 1 \Longrightarrow \forall x, y \in B: \ f(x) = f(y))$.
$\forall f \forall B \forall x, y \in B: \ f(x) = f(y)$.

Do you see what the author is trying to say here? After some back-and-forth reading one may recognize that this is supposed to be an inductive proof of sone set-theoretic assertion. Translated into more natural languange, this assertion says that for every function and in every finite set, all values of this function are equal. Of course, this assertion is absurd. Hence there must be a mistake in the alleged proof. But do you see where the mistake is hidden? Had the author connected these formulas by explaining text, and described the intentions in every step, then the author would have (probably) recognized the gap in this reasoning already in the process of writing. Actually this example is based on a classical fallacy showing a wrong use of induction. On top of being wrong it is also poorly written, which makes it harder to spot the error.

The following is a short write-up of Euclid's famous proof of the infinitude of prime numbers (a fact that is also relevant in Computer Science, e.g., for some cryptographic protocols and hashing schemes):

The proof builds the product of all primes and argues that $p+1$ contains a disjoint prime.

Already this very short piece of text comprises a whole number of cardinal sins in mathematical writing:

- Inserting an unnecessary meta-level: Instead of simply *being* the proof, the sentence is talking *about* the proof, which does not add information.
- Strange or inappropriate expressions: Can one say that a proof "builds" anything? A single number cannot be "disjoint" (from what?), but *two sets* can be disjoint.

- Undefined notation: This p that suddenly pops up seems to be the mentioned product, but this is not said.

- Unspecific description of relationships: In which sense is a prime "contained" in $p + 1$? There are many possible ways of "containment" of objects. Apparently a prime *factor* is meant here. (Another candidate for unspecific wording in mathematical texts is "combined" - combined by which operation?)

- Unclear structure: It is not announceed that this is a proof by contradiction, and what the contradiction is. The word does not even appear.

- As a result of all this, a reader is forced to infer things that could have easily been said explicitly.

Here is another write-up of Euclid's proof, which is still short (remarkably it uses less than 140 characters) but does not suffer from these problems:

Let L be the finite set of all primes. Let p be their product. The prime factors of $p + 1$ are not in L, a contradiction.

One can argue that this minimalistic version may still assume too much from the reader, in particular, abilities to follow steps with multiple thoughts and to fill in the details. We present some versions with increasing length and level of detail. There is a trade-off between length and expectations from the readers. It will depend on the audience which version is preferable. One cannot claim that "one size fits all".

Let L be the set of all primes. Assume that L is finite. Let p be the product of all primes in L. The prime factors of $p + 1$ are not in L, a contradiction.

Let L be the set of all primes. Assume that L is finite. Let p be the product of all primes in L. Integers p and $p + 1$ do not share any prime factors. Hence the prime factors of $p + 1$ are not in L, a contradiction.

Let L be the set of all primes. Assume that L is finite. Let p be the product of all primes in L. Integers p and $p+1$ do not share any prime factors. Hence the prime factors of $p + 1$ are not in L. This contradicts the assumption that L is the set of all primes.

Let L be the set of all primes. Assume for contradiction that L is finite. Then we can form the product p of all primes in L. Any two integers q and $q+1$ do not share any prime factors. Hence, in particular, the prime factors of $p+1$ are not in L. This contradicts the assumption that L is the set of all primes.

... and so on.

The following text describes Erathostenes' sieve method to find all primes.

Starting with $n = 1$, repeat the following steps forever: Set $n := n + 1$. If there is no pebble on n, then create a pebble with label n. Take each pebble P found on n and move it to $n + \ell(P)$, where $\ell(P)$ denotes the label of P.

Of coruse, this passage needs some context explaining that we work with labeled pebbles that are moved on an infinite tape with fields for all positive integers n, and that the generated labels are exactly all primes (a claim that should also be proved). Apart from that:

As opposed to usual pseudocode descriptions of algorithms, this text uses a metaphor that one hopefully keeps in mind. Moreover, this may be more fun to read than a "dry" description using variables, functions, and their values. Nevertheless it does not lack clarity.

Now look at a proof that the sum of even numbers is always even:

Let x and y be even. We have $x = 2z$ for some integer z. Similarly we have $y = 2z$ for some integer z. Thus $x+y = 2z+2z = 2(2z)$ showing that $x+y$ is even.

Here the mistake is rather obvious: variable mismatch! The second sentence is true in isolation, and so is the third one. But the same symbol z is used for different objects. The conjunction is built in the wrong way.

The example may appear stupid, but variable mismatch does happen very often. In the best case it only causes moments of confusion, in the worst case it leads to real mistakes, such as invalid calculations and logical mistakes as above, especially when the false "re-uses" of a symbol are far away from each other in the text. Similarly, two contradictory statements may be hard to notice if there is a lot of other text in between.

This is also an example of a "howler proof" that uses invalid arguments but incidentally (or "innocently") yields a correct conclusion. Another case:

All gemstones are minerals. Some minerals are opaque. Hence some gemstones are opaque.

This is eesy to believe if one hears this reasoning in passing, or if one reads it superficially. The conclusion is also true. But a moment of thinking reveals that it does NOT follow from the first two statements. One can easily accept a reasoning uncritically if the conclusion is expected, plausible, not surprising. This also holds for own solutions or research results: A plausible result does not automatically mean that the method was valid. Easy to forget!

Further typical cases of mismatch and ambiguities:

One can often read statements like $\sum_{i=1}^{n} k = O(k^2)$ rather than the correct $O(n^2)$. Here the summation index is confused with the upper limit, which is a real mistake, not only a formality.

If ALG denotes a recursive algorithm with one input parameter, what does "run $ALG(i)$ for any i" mean? Run it for all i, or run it for some (arbitrarily chosen) i? The use of "any" is delicate. Some authors even recommend to avoid "any" in mathematical texts altogether. Perhaps one does not have to be that strict. But check in every case whether "any" gives rise to misunderstandings.

The following phrase, dealing with graphs, contains yet another ambiguity: "any two nodes are connected" - connected by a path or by an edge? In the latter case one should prefer the established term "adjacent". The combinations allow three different interpretations of the graph class described here: non-empty graphs (having at least one edge), connected graphs, complete graphs (cliques).

Our "Euclidean" example also has an ambiguity: "The prime factors of $p+1$ are not in L." What is "the"? Neither of them is in L, or only some of them are not in L? But in this case it doesn't matter: Both interpretations work as correct arguments.

Depending on the context, in the phrase "a simple graph" it may be unclear whether "simple" is meant in the usual sense of "not complicated", or the technical term is used: A simple graph is a graph without loops or multiple edges. This usage of "simple graph" is very common. Talking about definitions: "A simple graph *is when* it has no loops or multiple edges" is bad style.

A more subtle issue is shown here:

Every multiple of 4 is also an even number. If $x = 4z$ then $x = 2(2z)$. ...

Is the second sentence meant to be the explanation of the first one? Or should one understand the first statement immediately (easy enough in this case), and the next sentence is the beginning of a new thought (introducing notations that may be used afterwards for something else)? One stumbles upon such questions. – Ideally, always tell the reader what you will do next, and how things are connected.

Instead of specific recommendations for further reading: A web search with keywords “mathematical” and “writing” (or similar) returns several useful writing guides with similar examples.