Priority queues
(chapters 6.9, 20.1 – 20.2)
Trees
(chapter 17)

Priority queues

A *priority queue* has three operations:

- *insert*: add a new element
- *find minimum*: return the smallest element
- *delete minimum*: remove the smallest element

Similar idea to a stack or a queue, but:

- you get the *smallest* element out instead of the first-added (queue) or last-added (stack)

(Java: `PriorityQueue` class)

Applications

A simulation – models *events* happening at a particular *time*

- "At 10am the person entered the shop"

When an event happens, it can cause more events to happen in the future

- "When a person enters the shop, 1 minute later they pick up some milk"

Applications – simulation

Keep a priority queue of *future events*

- "At 10am a person will enter the shop"

Simulator's job: remove *earliest* event and run it, then repeat

- In the priority queue, earlier events will be counted as "smaller" than later events

When we run that event, it can in turn add more events to the priority queue

- When a person enters the shop, add an event "the person picked up some milk" to the priority queue at a time of 1 minute later

Applications

Sorting:

- Start with an empty priority queue
- Add each element of the input list in turn
- As long as the priority queue is not empty, find and remove the smallest element
- You get all elements out in ascending order!

Heapsort is an in-place version of this (next lecture)

This lecture

An efficient implementation of priority queues, the *binary heap*, with the help of:

- Invariants
- Trees

We'll wander off into those two topics and then come back to priority queues

Also, hopefully, a bit of: *how to think like a data structure designer*

- I will try to explain *why heaps are the way they are* rather than just how they work

An inefficient priority queue

Idea 1: implement a priority queue as a *dynamic array*

- Insert: add new element to end of array
  O(1)
- Find minimum: linear search through array
  O(n)
- Delete minimum: remove minimum element
  O(n)

Finding the minimum is quite expensive though. Another idea?

An inefficient priority queue

Idea 2: use a *sorted array*

- Insert: insert new element in right place
  O(n)
- Find minimum: minimum is first element
  O(1)
- Delete minimum: remove first element
  O(n)

Finding the minimum is cheap! Yay!
But… insertion got expensive :(
Deletion is also expensive…

An inefficient priority queue

Idea 3: implement a priority queue as a *reverse-sorted array*

- Insert: insert new element in right place
  O(n)
- Find minimum: minimum is last element
  O(1)
- Delete minimum: remove last element
  O(1)

A bit better, but O(n) insertion is not so good…

A detour: invariants

"The array is reverse-sorted" is an example of an *invariant* of a data structure

- An invariant is a property that always holds in our implementation of the data structure
- Something the data structure designer picks that helps implementing the data structure

Insert, *find minimum* and *delete minimum* can assume that the array is already reverse-sorted…

- …but they must make sure that the array remains reverse-sorted afterwards (they must *preserve the invariant*)

Checking the invariant

What happens if you *break the invariant*?

- e.g., insert simply adds the new element to the end

Answer: nothing goes wrong straight away, but later operations might fail

- A later *find minimum* might return the wrong answer!

These kind of bugs are a nightmare to track down!

Solution: *check the invariant*

Checking the invariant

Define a method

```
bool invariant()
```

that returns *true* if the invariant holds

- in this case, if the array is reverse-sorted

Then, in the implementation of every operation, do

```
assert invariant();
```

This will *throw an exception* if the invariant doesn't hold!

(Note: in Java, must run program with `-ea`)

Invariants in Haskell

Define a function

```
invariant :: Whatever → Bool
```

Then add an extra case to all operations:

```
whatever x
  | not (invariant x) = error “oops”
whatever x = …
```

[Perhaps remove this case when you've finished testing your code]

Checking invariants

Writing down and checking invariants will help you *find bugs much more easily*

- I'd say *most* data structure bugs involve breaking an invariant
- Even if you don't think about an invariant, if your data structure is at all fancy there is probably one hiding there!
- Almost all programming languages support assertions – **use them** to check invariants and make your life easier

Nick's brief guide to designing data structures

How not to do it

Here is how *not* to design a data structure:

1. Take the operations you have to implement
2. Think very hard about how to implement them
3. Bash something together that seems to work

Because:

- You will probably have lots of bugs
- You will probably miss the best solution

Looking back on older designs

We implemented bounded queues by an array and a pair of indices *front* and *back*

- The *contents of the queue* is the elements between index *front* and index *back*

Once we decide on this representation, there is only one way to implement the queue!

- Here, "representation" means – what datatype we use, plus what an instance of that datatype *means* as a queue (in this case, what the queue contains)

Looking back on older designs

We represented a priority queue by an array with the invariant that the array is reverse-sorted

Once we choose this invariant, there is only one way to implement it!

Data structure design

How to design a data structure:

- Pick a representation
  Here: we represent the priority queue by an array
- Pick an invariant
  Here: the array is reverse-sorted

Once you have the right representation and invariant, *the operations often almost "design themselves"!*

- There is often only one way to implement them

You could say...
data structure = representation + invariant

Picking a representation and invariant

How do you know which representation and invariant to go for?

Good plan: have a first guess, see if the operations work out, then tweak it

- Queues: at first we tried a dynamic array, but there was no way to efficiently remove items, so we switched to a circular array
- Priority queues: at first we tried a sorted array, but then *remove minimum* needed to delete the first element (inefficient), so we switched to a reverse-sorted array

Takes practice!

More on invariants

A strong invariant like “the array is reverse-sorted”:

- Can make it easier to *get* information from the data structure (the data is more structured)
- Can make it harder to *update* the data structure (you have to preserve the invariant)

In our case:

- *find minimum* becomes easier (array is sorted)
- *insert* becomes harder (must make sure array is sorted afterwards)

A good invariant will provide some *extra structure* that makes the operations you want easier

- sorting the array makes it easier to find the minimum

Trees

Trees

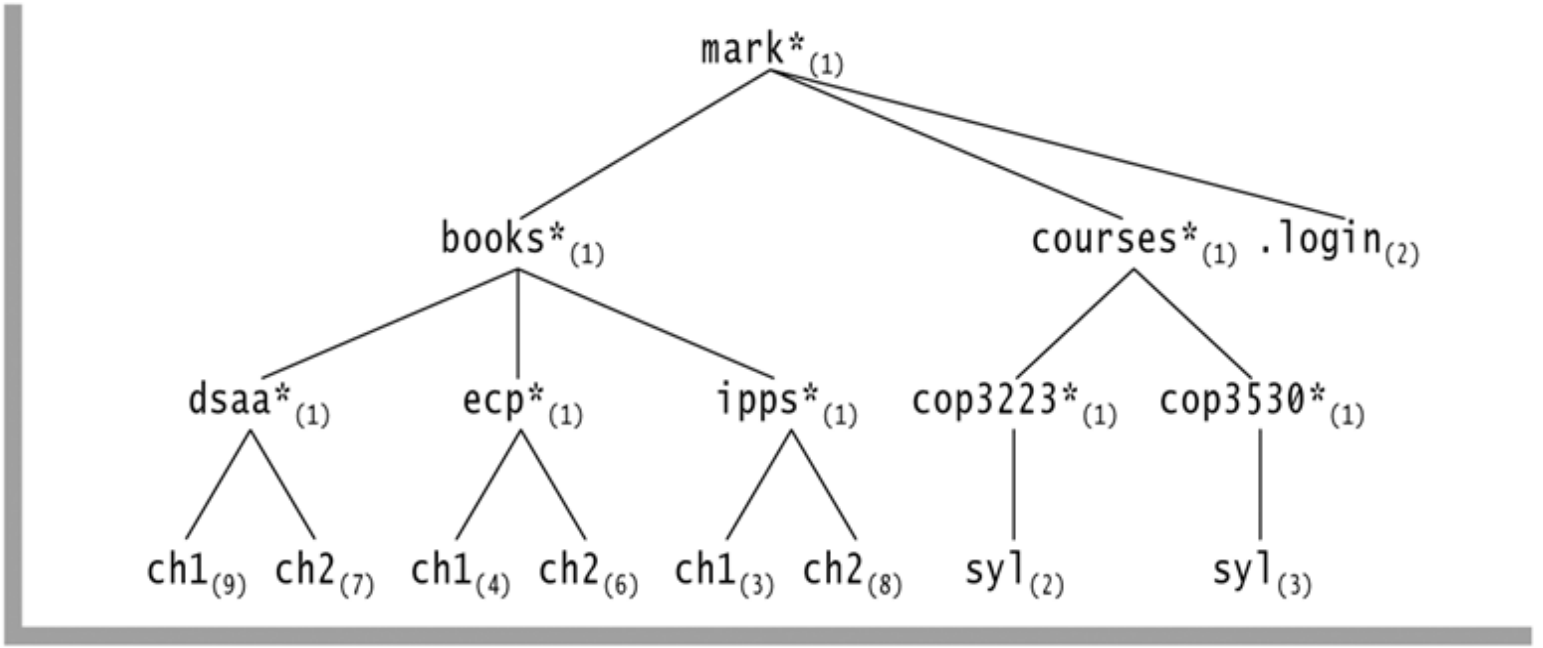
A *tree* is a hierarchical data structure

- Each node can have several *children* but only has one *parent*
- The *root* has no parents; there is only one root

Example: directory hierarchy

figure 18.7

The Unix directory with file sizes



Binary trees

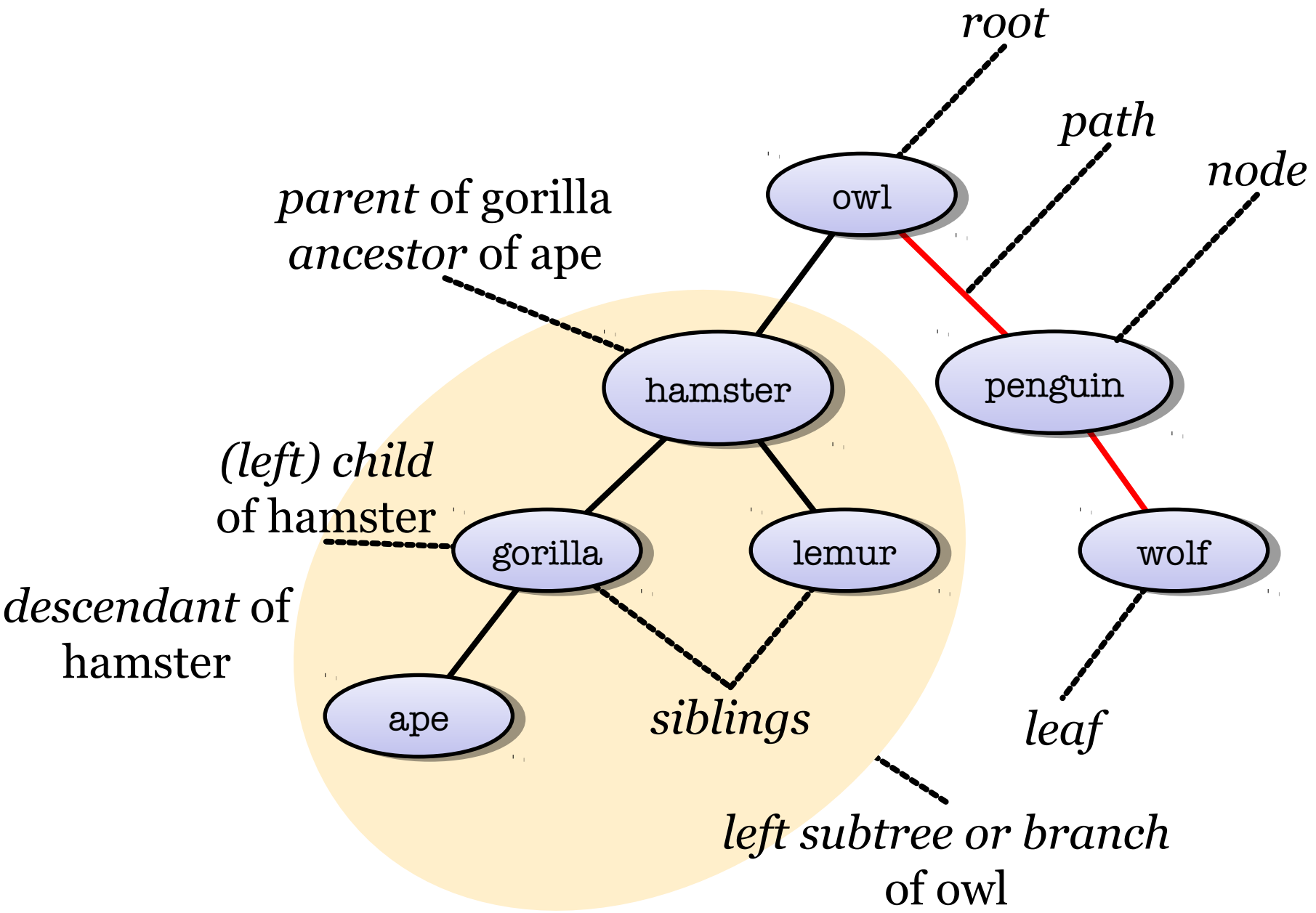
Most often we use *binary trees*, where each node has at most two children

```
class Node<E> {
  E value;
  Node<E> left, right;
}
```

Can be `null`

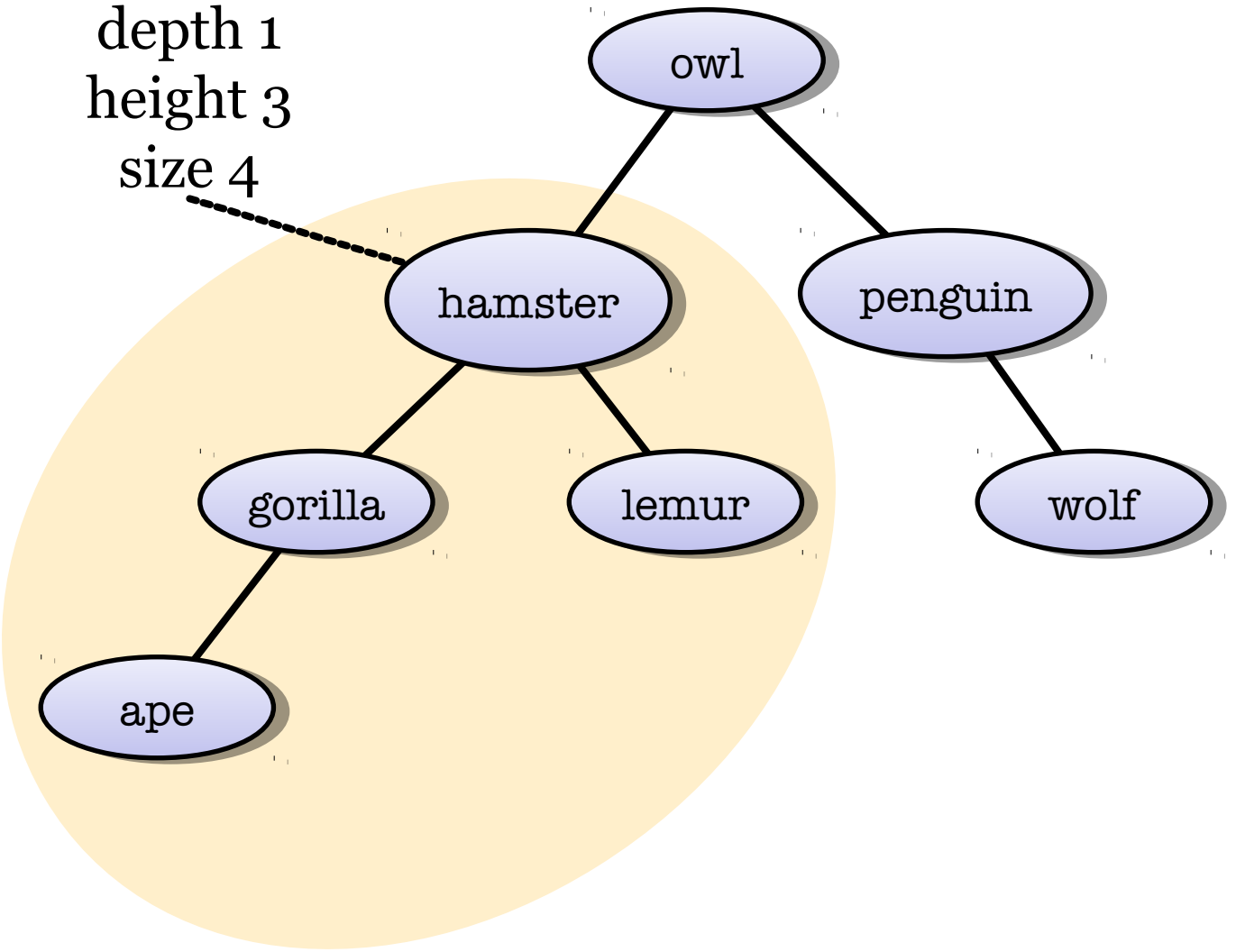
```
data Tree a
  = Node a (Tree a) (Tree a)
  | Nil
```

Terminology



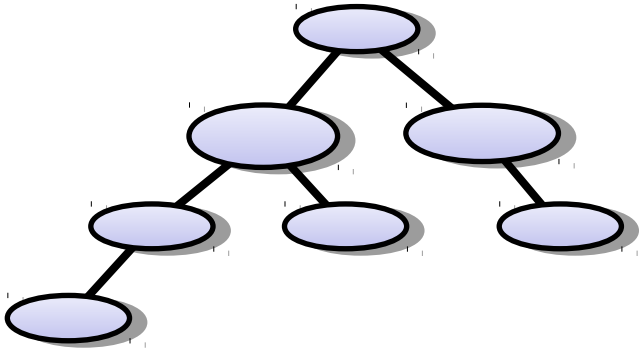
Terminology

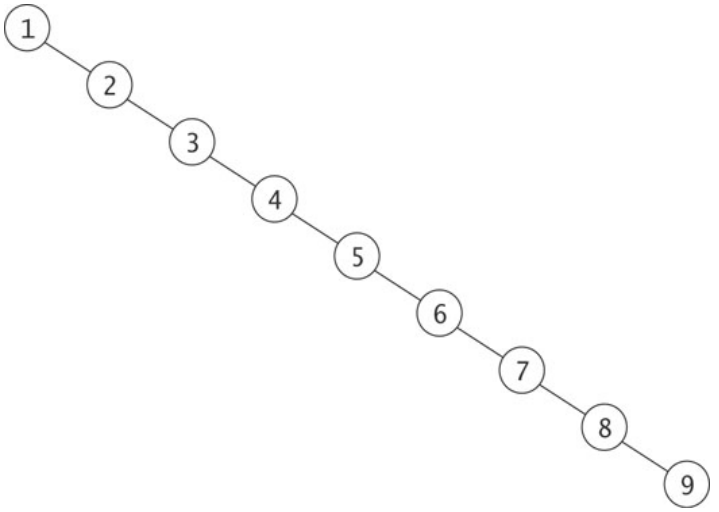
The *depth* of a node is the distance from the root
The *height* of a tree is the number of levels in the tree
The *size* of a tree is the number of nodes in it



Balanced trees

A tree can be *balanced* or *unbalanced*





If a tree of size *n* is

- balanced, its height is O(log n)
- unbalanced, its height could be O(n)

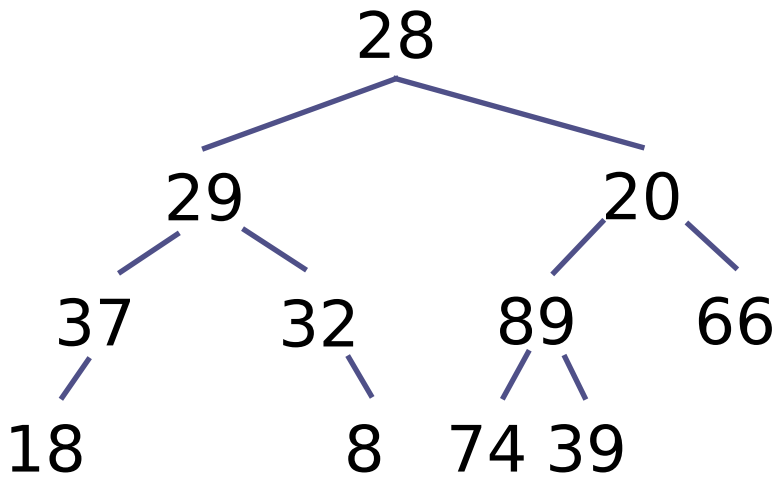
Many tree algorithms have complexity O(height of tree), so are efficient on balanced trees and less so on unbalanced trees

Normally: balanced trees good, unbalanced bad!

Heaps

Heaps – representation

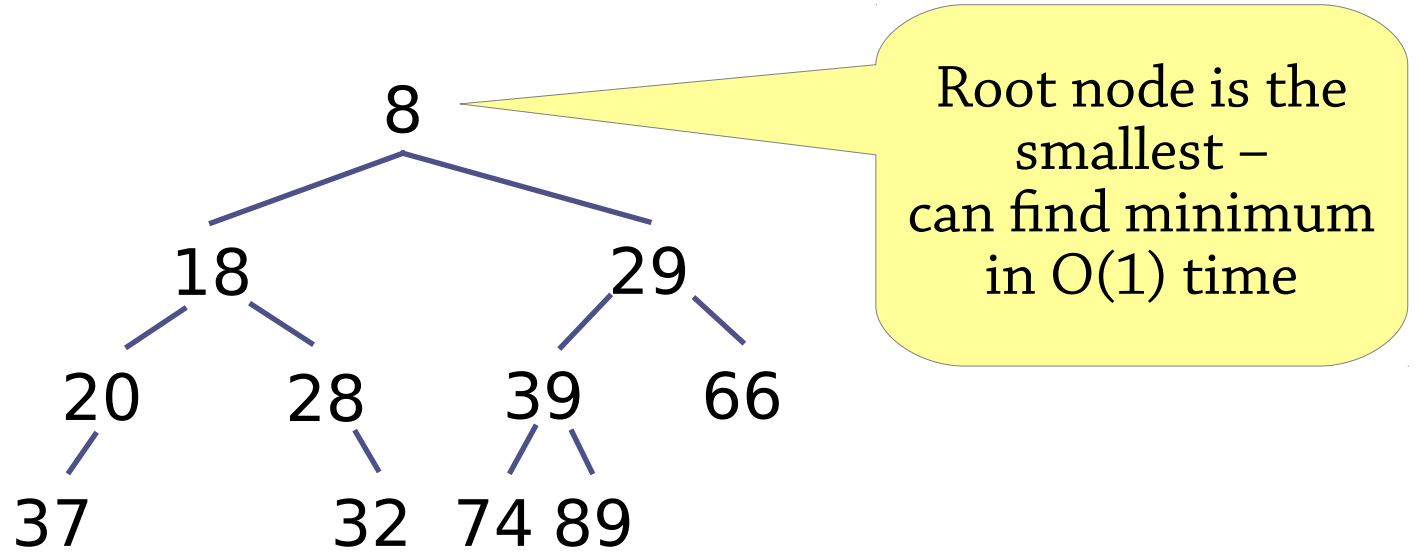
A heap implements a priority queue as a tree. Here is a tree:



This is not yet a heap. We need to add an invariant that makes it easy to find the minimum element.

The heap property

A tree satisfies the *heap property* if the value of each node is less than (or equal to) the value of its children:



Where can we find the smallest element?

Why the heap property

Why did we pick this invariant? One reason:

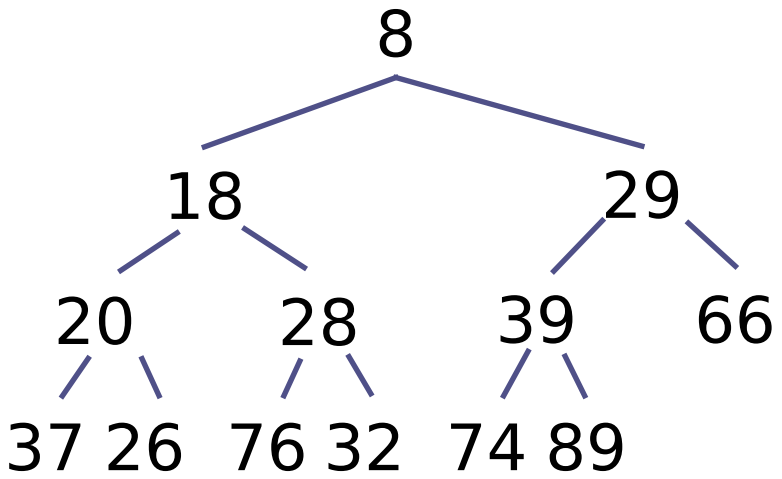
- It puts the smallest element at the root of the tree, so we can find it in O(1) time

Why not just have the invariant "the root node is the smallest"? Because:

- Trees are a *recursive* structure – the children of a node are also trees
- It's then a good rule of thumb to have a recursive invariant – each node of the tree should satisfy the same sort of property
- In this case, instead of "the root node is smaller than its descendants", we pick "each node is smaller than its descendants"

Binary heap

A *binary heap* is a *complete* binary tree that satisfies the heap property:



Complete means that all levels except the bottom one are full, and the bottom level is filled from left to right (see above)

Why completeness?

There are a couple of reasons why we choose to have a complete tree:

- It makes sure the tree is balanced
- When we insert a new element, it means there is only one place the element can go – this is one less design decision we have to make

There is a third reason which trumps the first two, but that will have to wait for next time!

Binary heap invariant

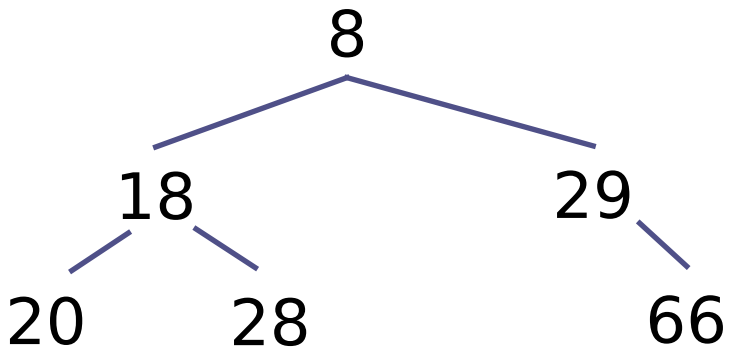
The binary heap invariant:

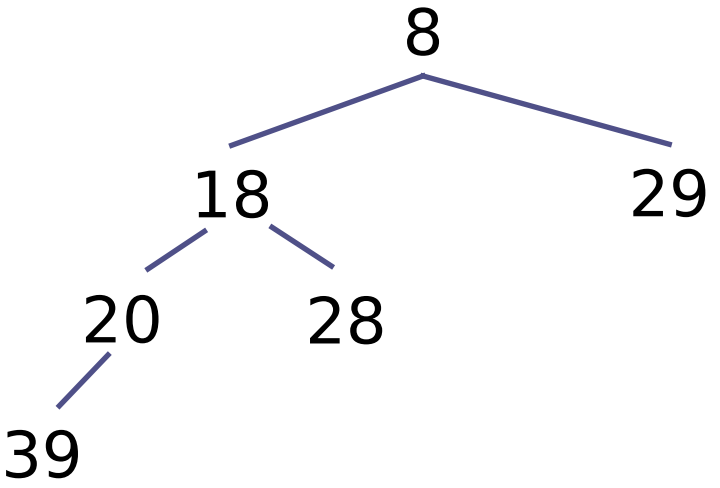
- The tree must be *complete*
- It must have the *heap property* (each node is less than or equal to its children)

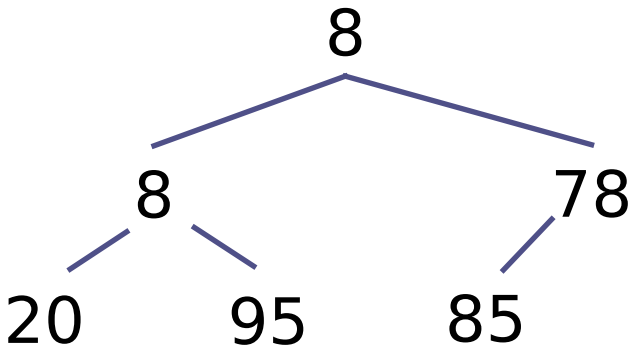
Remember, all our operations must preserve this invariant

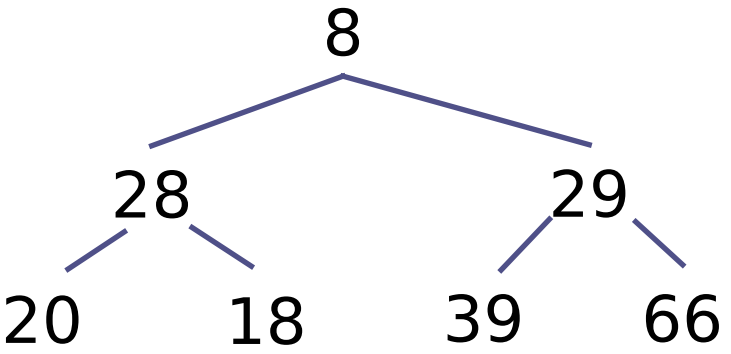
Once we have picked this invariant, there is only one sensible way to implement the operations!

Heap or not?

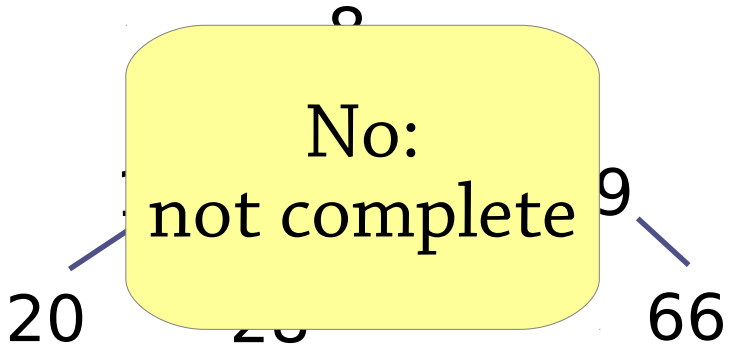


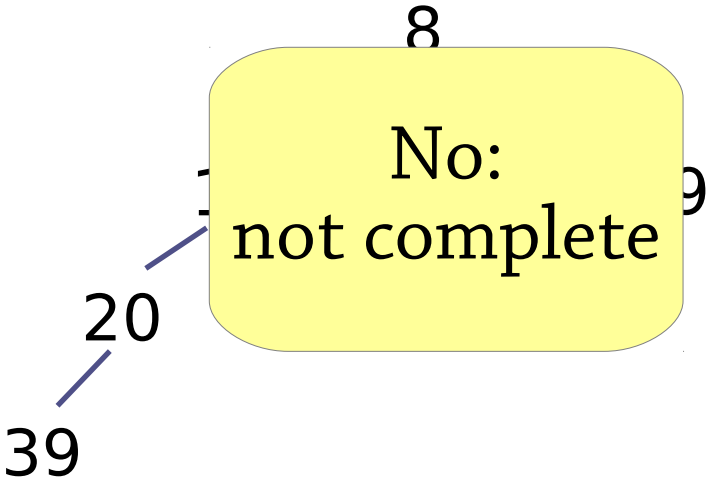


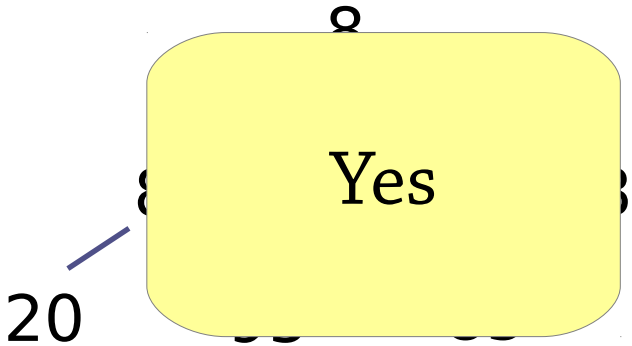


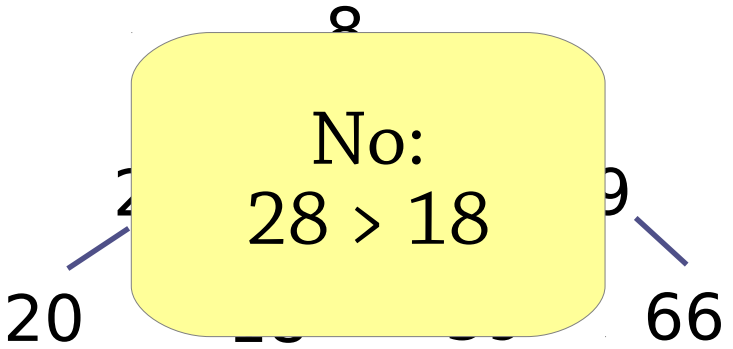


Heap or not?



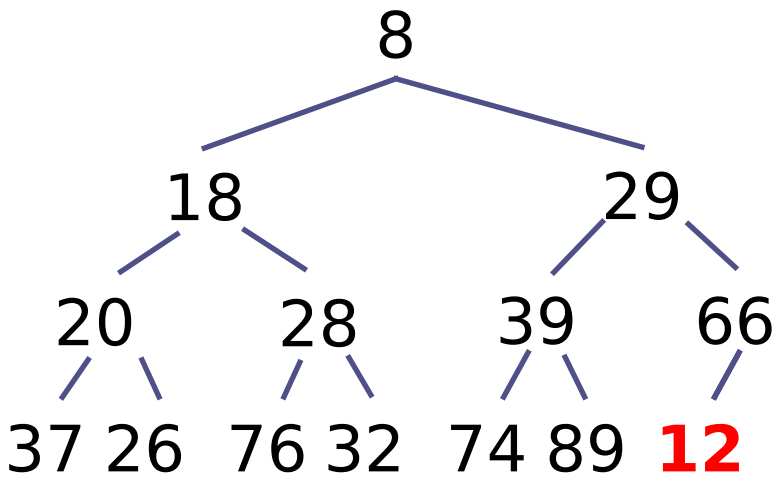






Adding an element to a binary heap

Step 1: insert the element at the next empty position in the tree



This might break the heap invariant!

In this case, 12 is less than 66, its parent.

An aside

To modify a data structure with an invariant, we have to

- modify it,
- while preserving the invariant

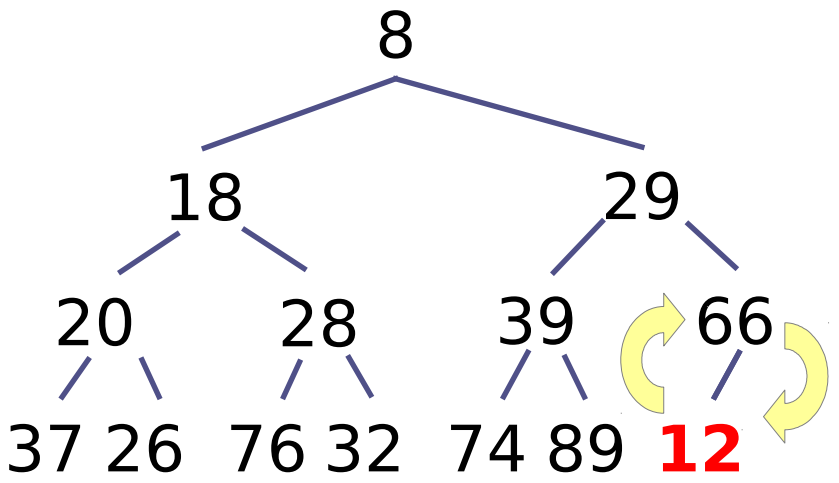
Often it's easier to separate these:

- first modify the data structure, possibly breaking the invariant in the process
- then “repair” the data structure, making the invariant true again

This is what we are going to do here

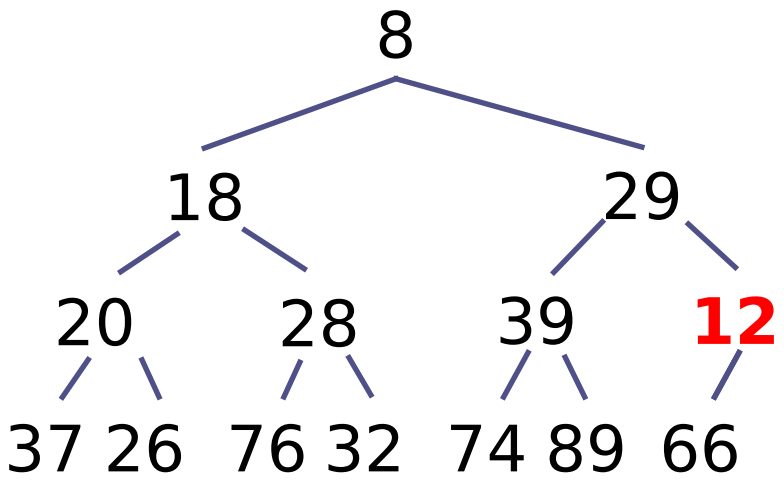
Adding an element to a binary heap

Step 2: if the new element is less than its parent, swap it with its parent



Adding an element to a binary heap

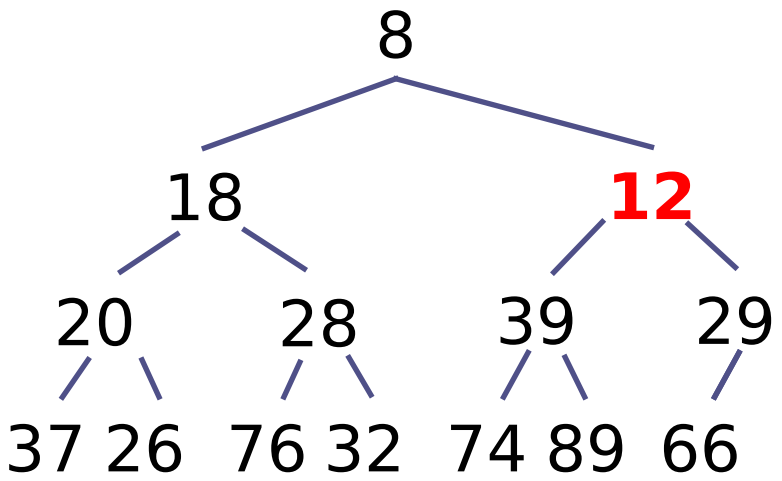
Step 2: if the new element is less than its parent, swap it with its parent



The invariant is still broken, since 12 is less than 29, its new parent

Adding an element to a binary heap

Repeat step 2 until the new element is greater than or equal to its parent.

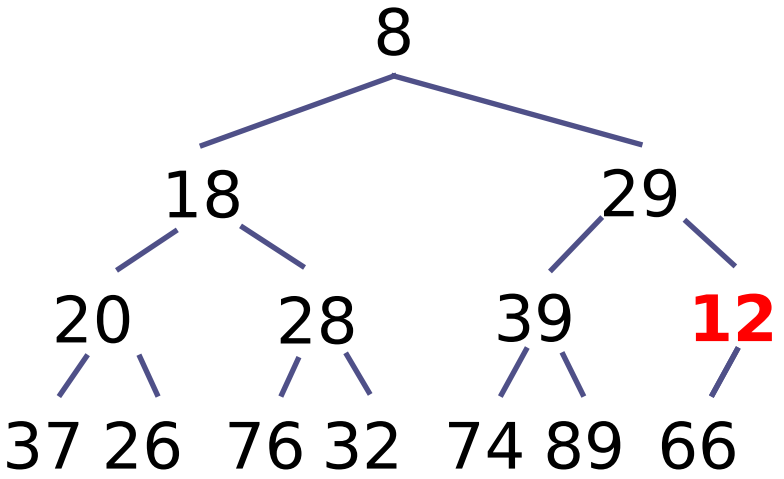


Now 12 is in its right place, and the invariant is restored. (Think about why this algorithm restores the invariant.)

Why this works

At every step, the heap property almost holds *except* that the new element might be less than its parent

After swapping the element and its parent, still only the new element can be in the wrong place (why?)



Removing the minimum element

To remove the minimum element, we are going to follow a similar scheme as for insertion:

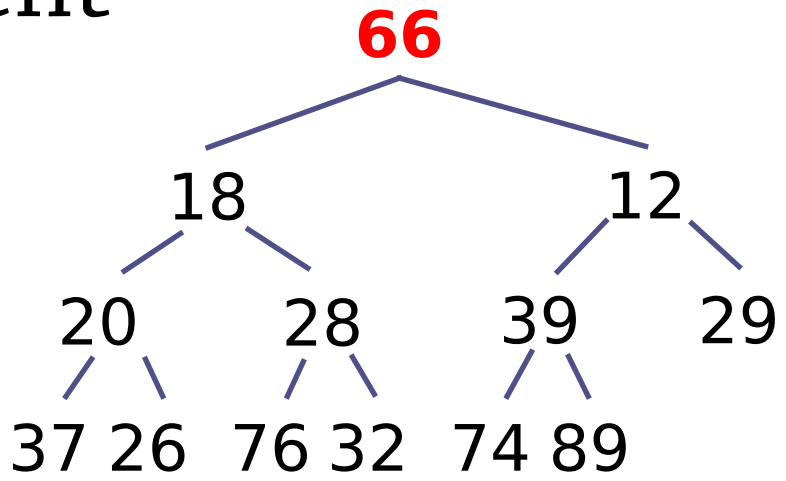
- First remove the minimum (root) element from the tree somehow, breaking the invariant in the process
- Then repair the invariant

Because of *completeness*, we can only really remove the *last* (bottom-right) element from the tree

- Solution: first *swap* the root element with the last element, then remove the last element

Removing the minimum element

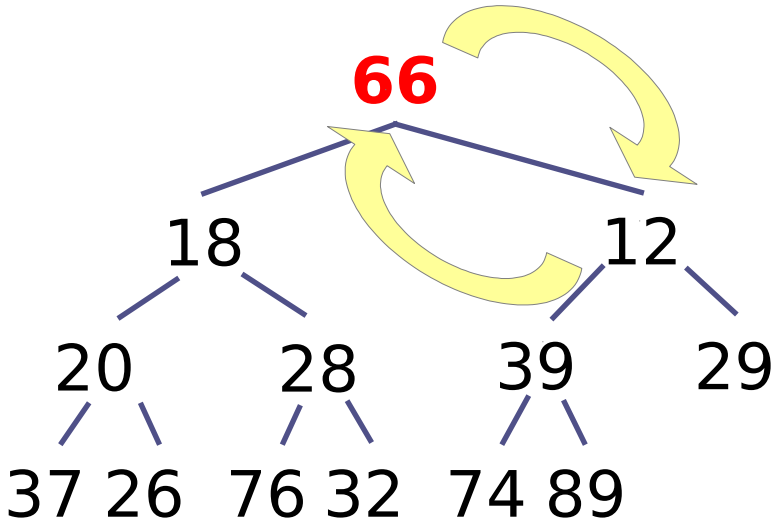
Step 1: replace the root element with the *last element* in the tree, and remove the last element



The invariant is broken, because 66 is greater than its children

Removing the minimum element

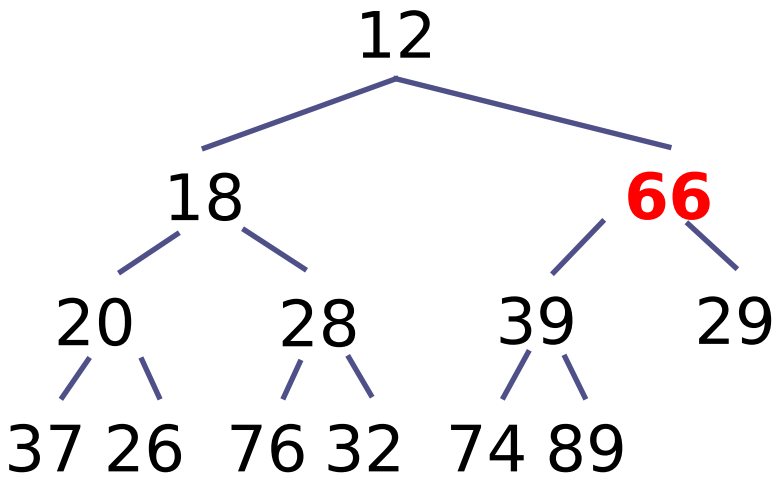
Step 2: if the moved element is greater than its children, swap it with its *least child*



(Why the least child in particular?)

Removing the minimum element

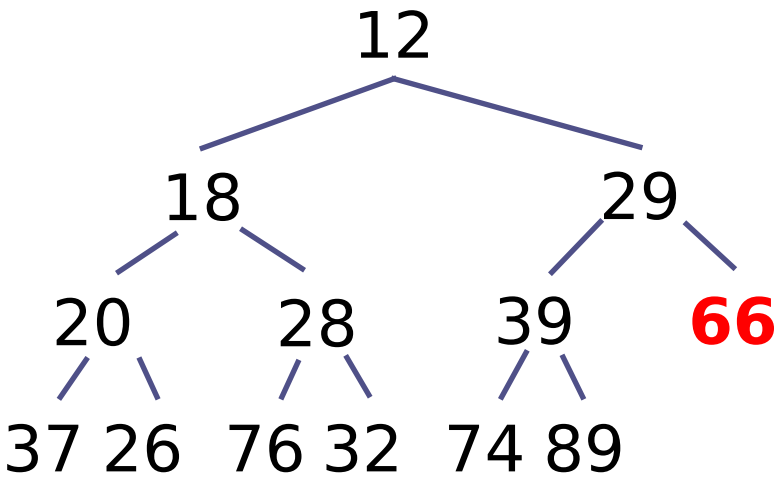
Step 2: if the moved element is greater than its children, swap it with its *least child*



(Why the least child in particular?)

Removing the minimum element

Step 3: repeat until the moved element is less than or equal to its children



Sifting

Two useful operations we can extract from all this

Sift up: if an element might be less than its parent, i.e. needs "moving up" (used in insert)

- Repeatedly swap the element with its parent

Sift down: if an element might be greater than its children, i.e. needs "moving down" (used in removing the minimum element)

- Repeatedly swap the element with its least child

The book says *percolate* instead of *sift*

Binary heaps – summary so far

Implementation of priority queues

- *Heap property* – means smallest value is always at root
- *Completeness* – means tree is always balanced

Complexity:

- *find minimum* – **O(1)**
- *insert*, *delete minimum* –
  O(height of tree), **O(log n)** because tree is balanced

Today

Main topic was binary heaps, but it was also about *how to design data structures*

- The main task is not *how to implement the operations*, but choosing the right representation and invariant
- These are the main design decisions – once you choose them, lots of stuff falls into place
- Understanding them is the best way to understand a data structure, and checking invariants is a very good way of avoiding bugs!

But you also need lots of existing data structures to get inspiration from!

- Many of these in the rest of the course