Quicksort
(Weiss chapter 8.6)

Recap of before Easter

We saw a load of sorting algorithms, including *mergesort*

To mergesort a list:

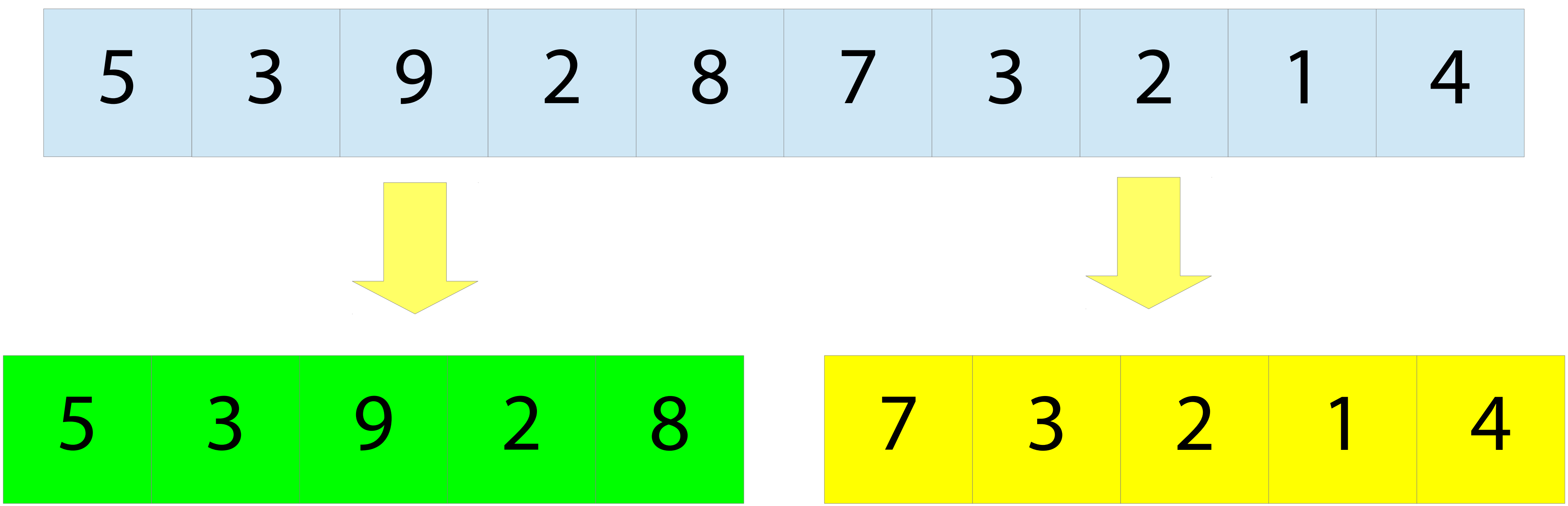
- *Split* the list into two halves
- *Recursively* mergesort the two halves
- *Merge* the two sorted halves into one

This is an instance of *divide and conquer*

Quicksort is also divide and conquer!

Mergesort

1. *Split* the list into two equal parts

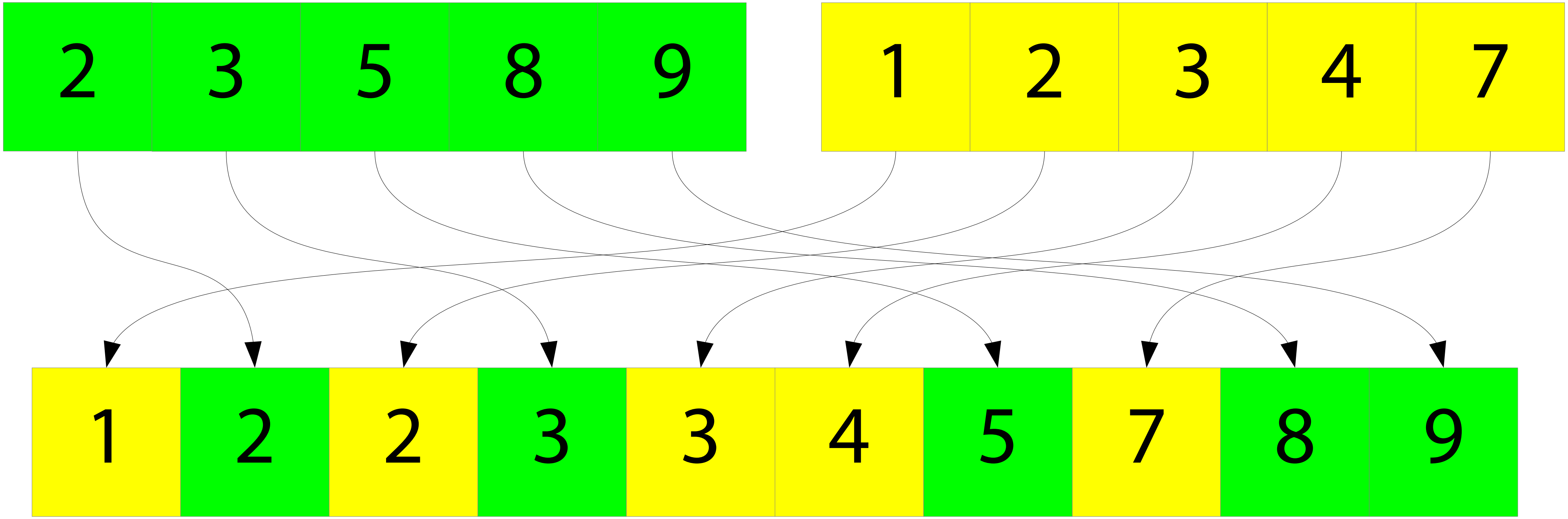


Mergesort

2. *Recursively* mergesort the two parts

Mergesort

3. *Merge* the two sorted lists together



Quicksort

Pick an element from the array, called the *pivot*

Partition the array:

- First come all the elements smaller than the pivot, then the pivot, then all the elements greater than the pivot

Recursively quicksort the two partitions

Quicksort

5	3	9	2	8	7	3	2	1	4

Say the pivot is 5.

Partition the array into: all elements less than 5, then 5, then all elements greater than 5

3	3	2	2	1	4	5	9	8	7

Less than the pivot

Greater than the pivot

Quicksort

Now recursively quicksort the two partitions!

3	3	2	2	1	4	5	9	8	7

Quicksort

Quicksort

1	2	2	3	3	4	5	7	8	9

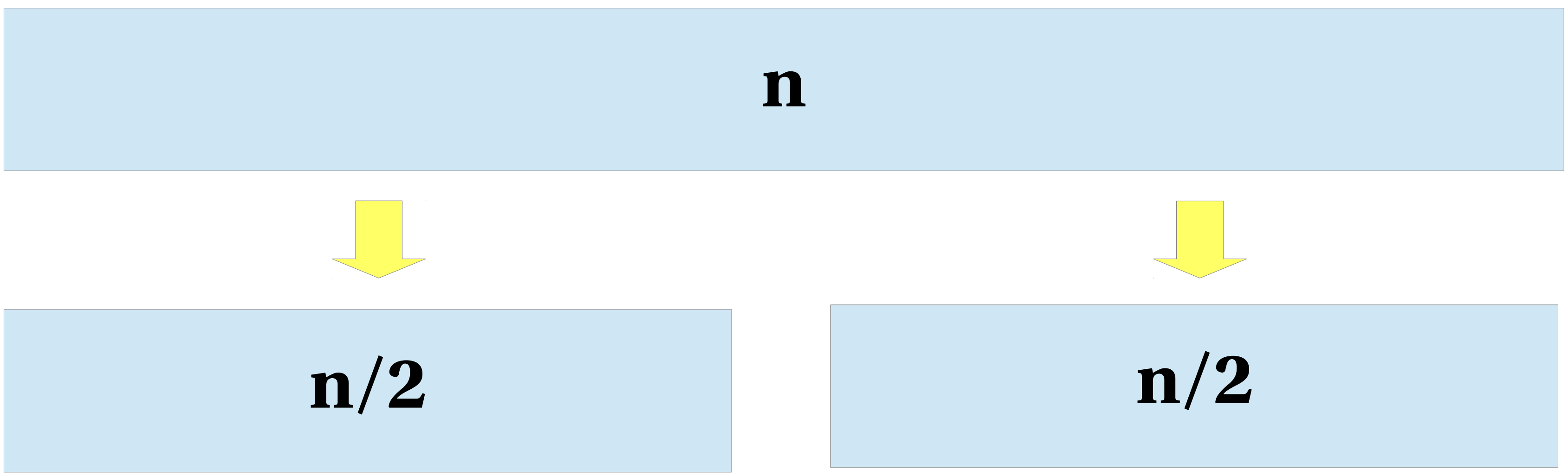
Pseudocode

```
// call as sort(a, 0, a.length-1);
void sort(int[] a, int low, int high) {
   if (low >= high) return;
   int pivot = partition(a, low, high);
      // assume that partition returns the
      // index where the pivot now is
   sort(a, low, pivot-1);
   sort(a, pivot+1, high);
}
```

Common optimisation: switch to insertion sort when the input array is small

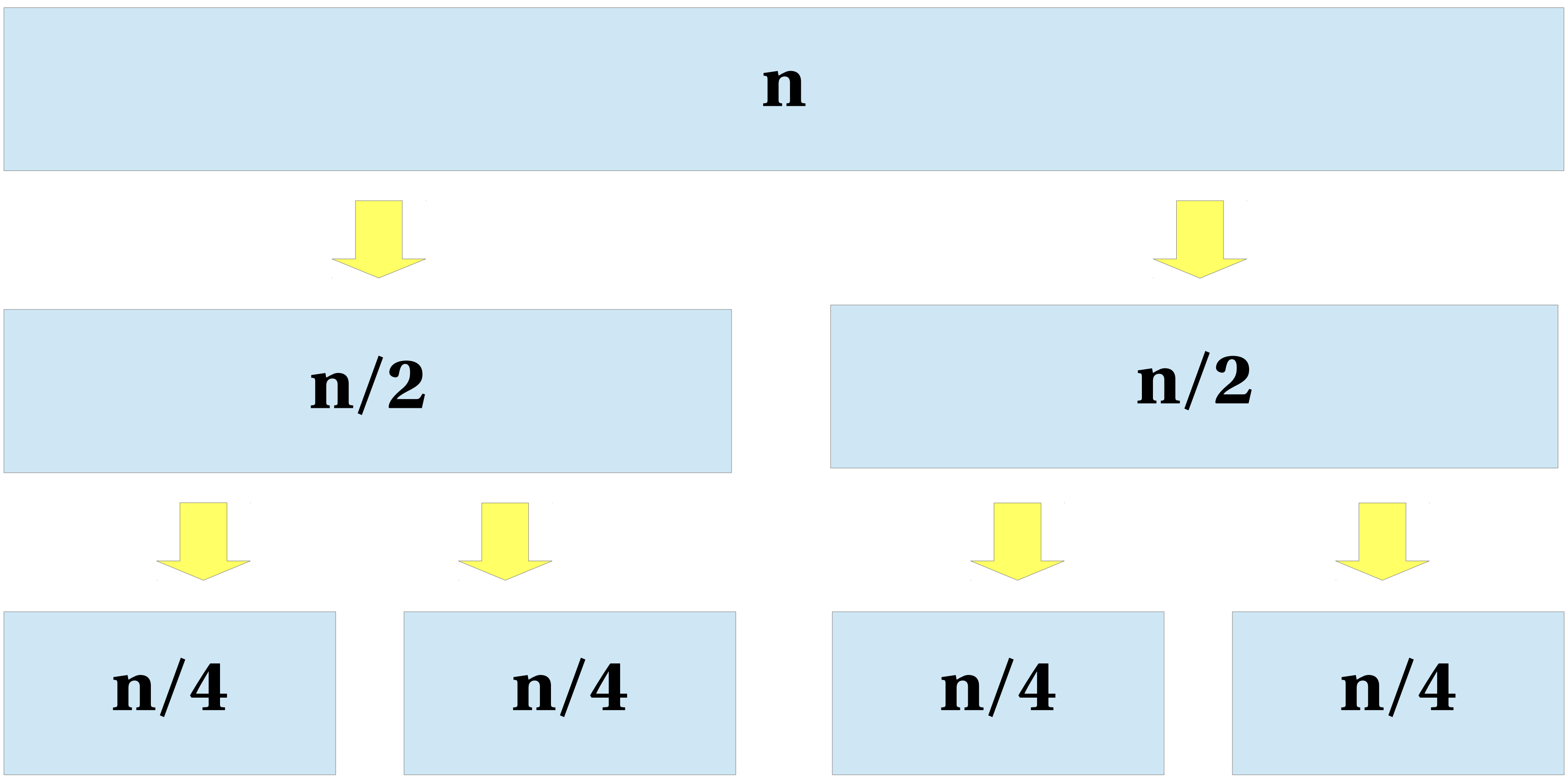
Complexity of quicksort

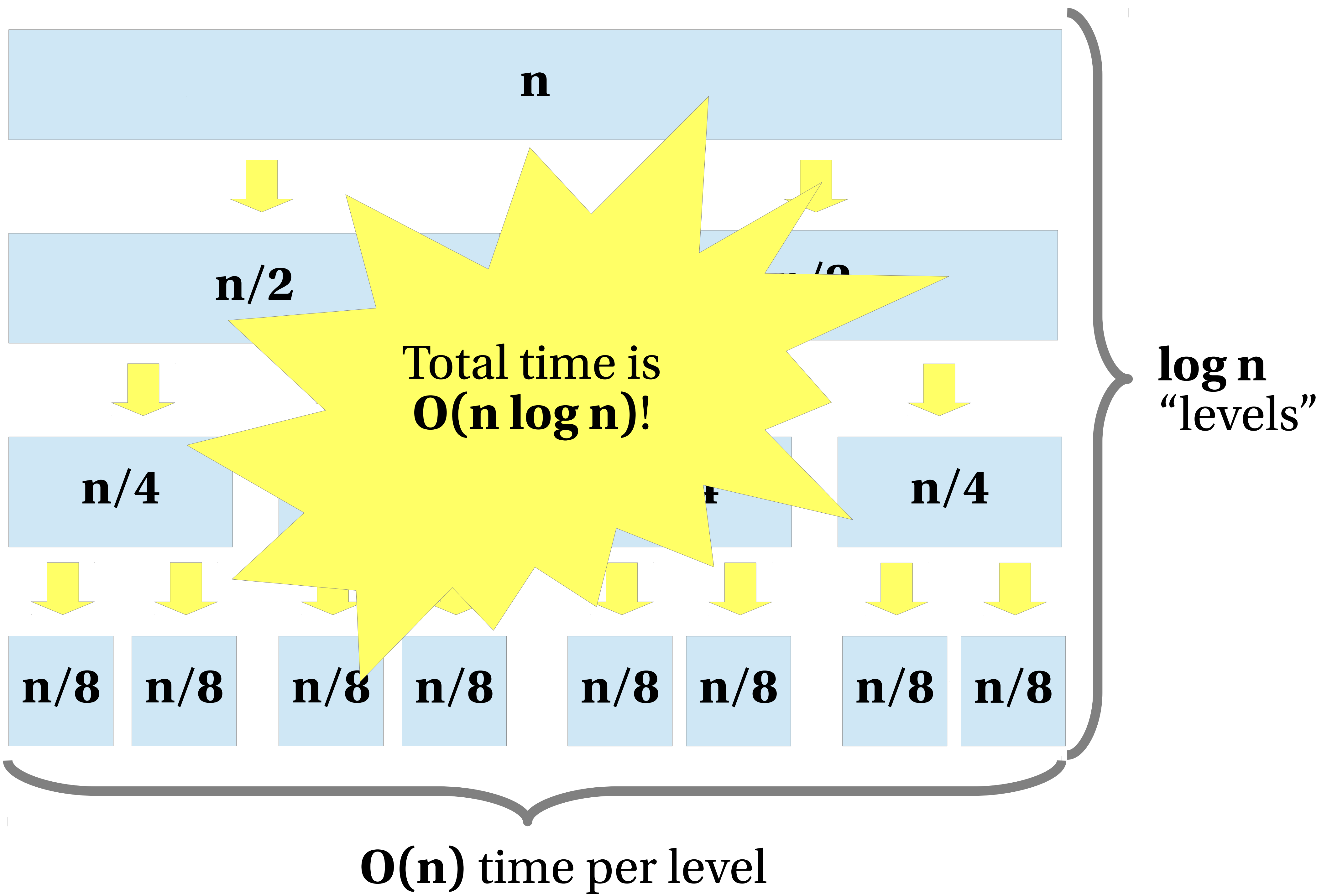
In the best case, partitioning splits an array of size n into two halves of size n/2:



Complexity of quicksort

The recursive calls will split these arrays into four arrays of size n/4:



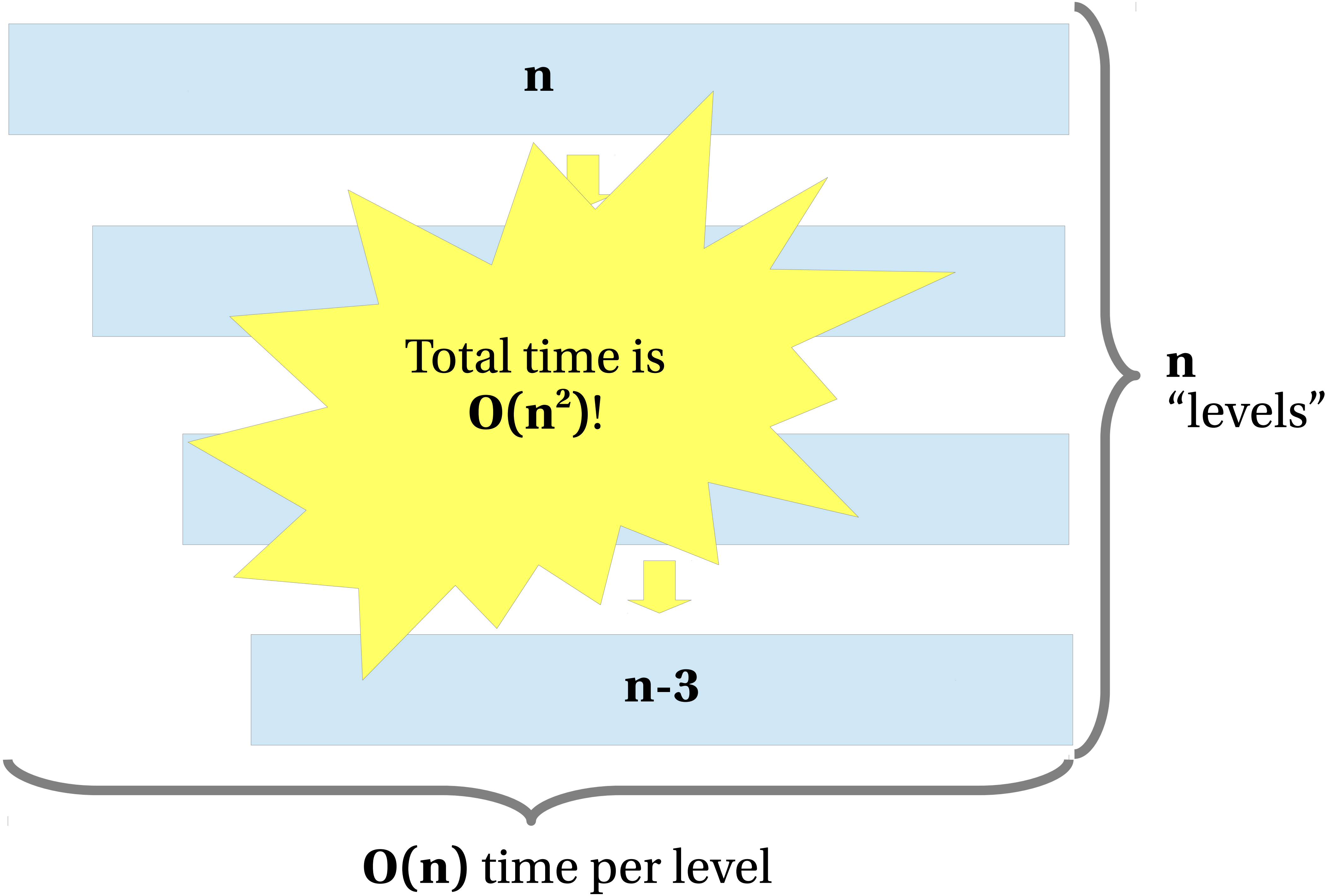

n
n/2
n/2
n/4
n/4
n/4
n/8
n/8
n/8
n/8
n/8
n/8
n/8
n/8
Total time is
O(n log n)!
log n
“levels”
O(n) time per level

Complexity of quicksort

But that's the best case!

In the worst case, everything is greater than the pivot (say)

- The recursive call has size n-1
- Which in turn recurses with size n-2, etc.
- Amount of time spent in partitioning:
  $n + (n-1) + (n-2) + \ldots + 1 = \mathbf{O(n^2)}$


n
Total time is
O(n²)!
n-3
n
"levels"
O(n) time per level

Worst cases

When we pick the first element as the pivot, we get this worst case for:

- Sorted arrays
- Reverse-sorted arrays

Complexity of quicksort

Quicksort works well when the pivot splits the array into roughly equal parts

- Median-of-three: pick first, middle and last element of the array and pick the median of those three
- Pick pivot at random: gives O(n log n) *expected* (probabilistic) complexity

Introsort: detect when we get into the $O(n^2)$ case and switch to a different algorithm (e.g. heapsort)

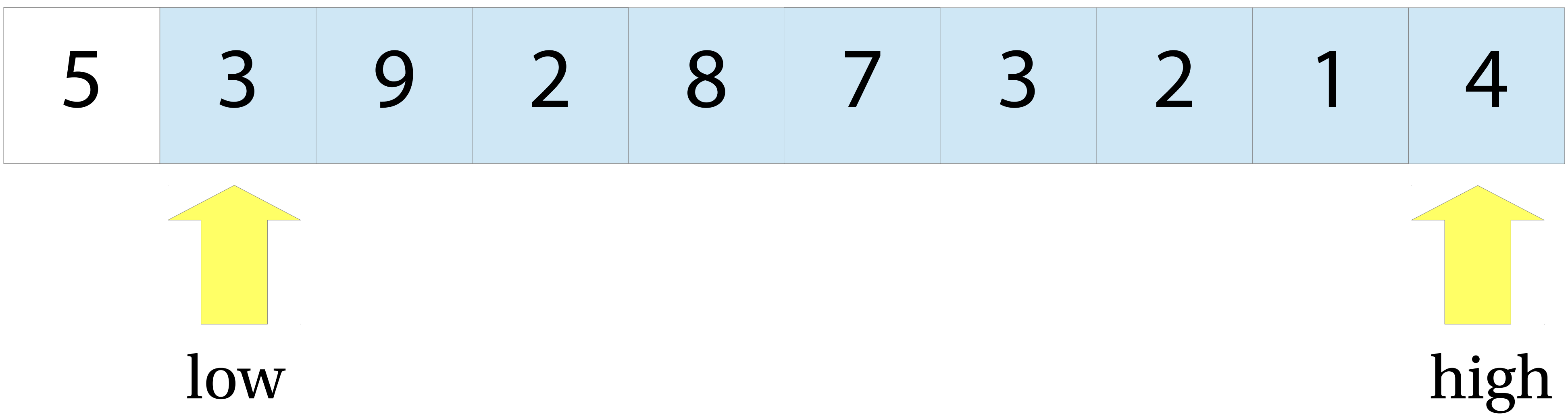
Partitioning algorithm

1. Pick a pivot (here 5)

5	3	9	2	8	7	3	2	1	4

Partitioning algorithm

2. Set two indexes, low and high



Idea: everything to the left of low is less than the pivot (coloured yellow), everything to the right of high is greater than the pivot (green)

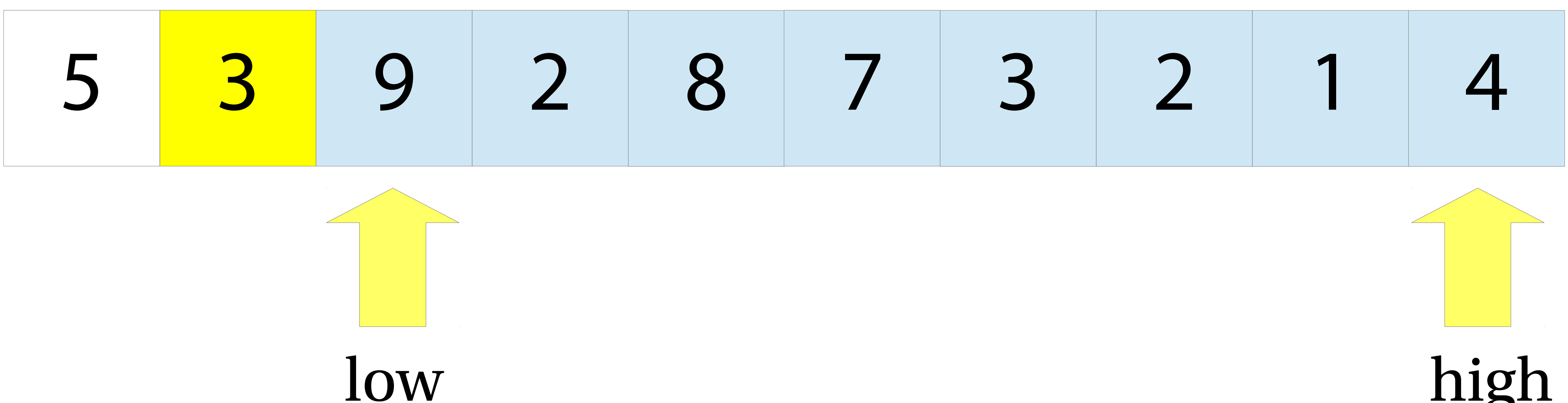
Partitioning algorithm

3. Move low right until you find something greater than the pivot

5	3	9	2	8	7	3	2	1	4

Partitioning algorithm

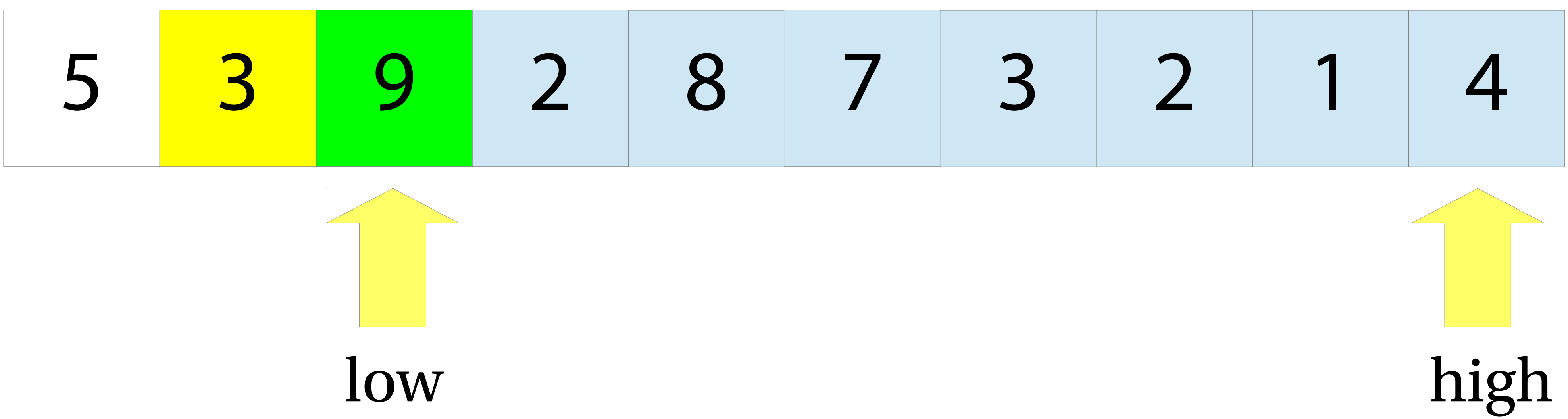
3. Move low right until you find something greater or equal to the pivot




```
while (a[low] < pivot) low++;
```

Partitioning algorithm

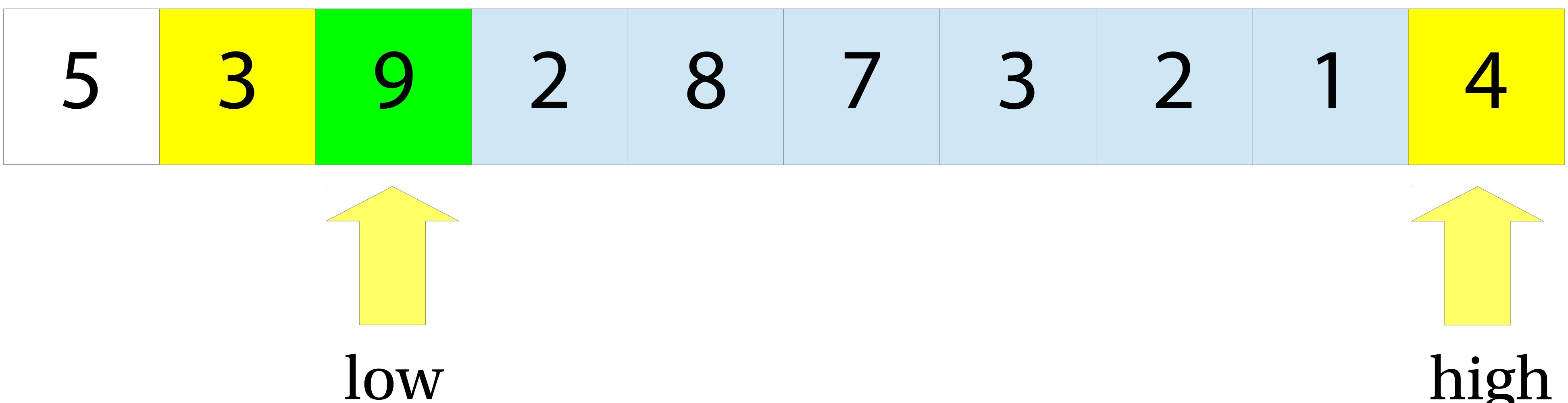
3. Move low right until you find something greater than the pivot




```
while (a[low] < pivot) low++;
```

Partitioning algorithm

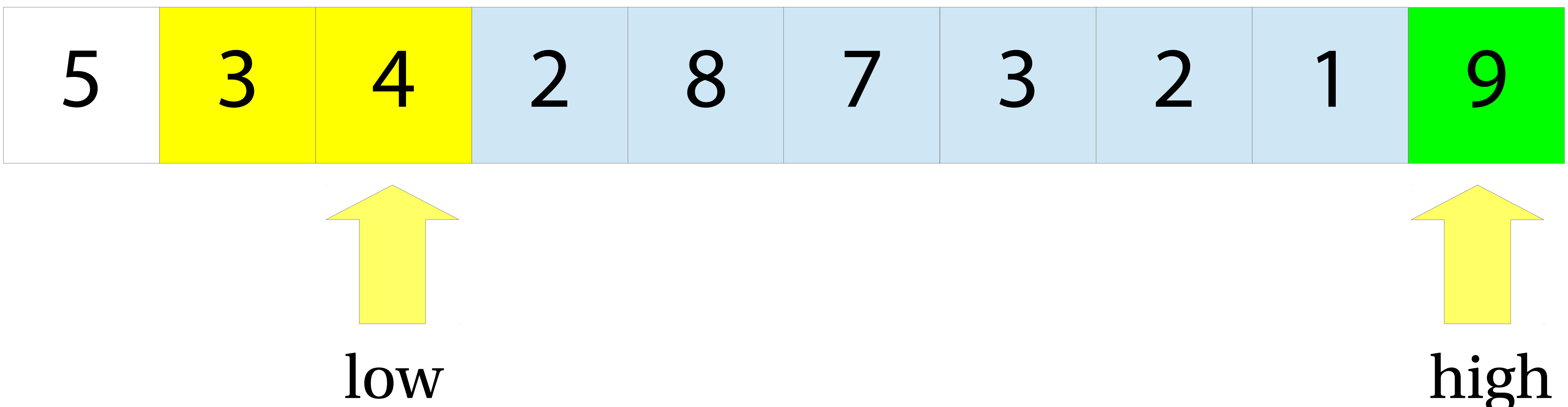
3. Move high left until you find something less than the pivot




```
while (a[high] < pivot) high--;
```

Partitioning algorithm

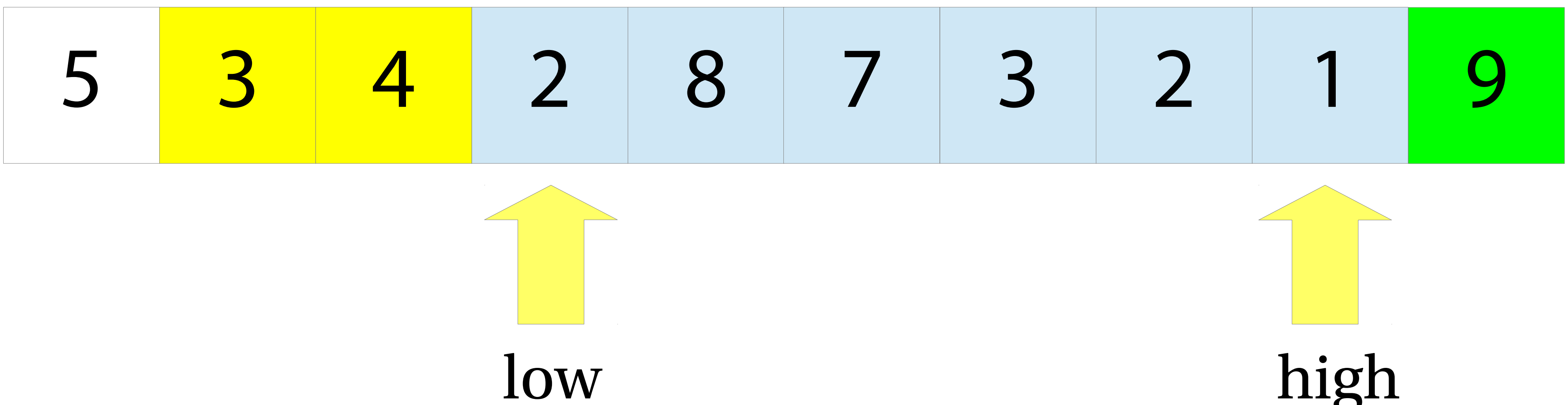
4. Swap them!




```
swap(a[low], a[high]);
```

Partitioning algorithm

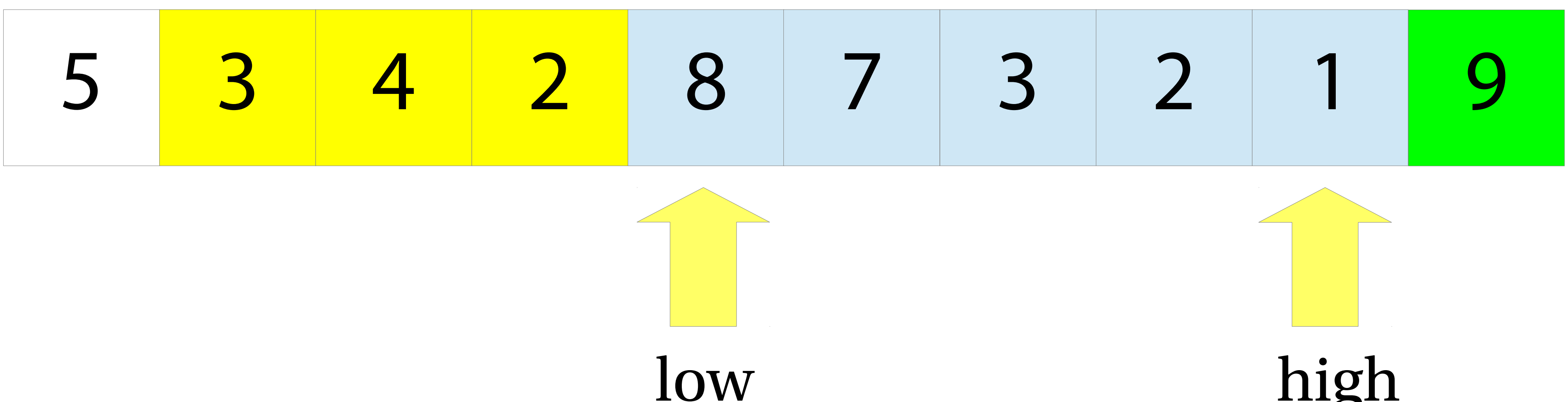
5. Advance low and high and repeat




```
low++; high--;
```

Partitioning algorithm

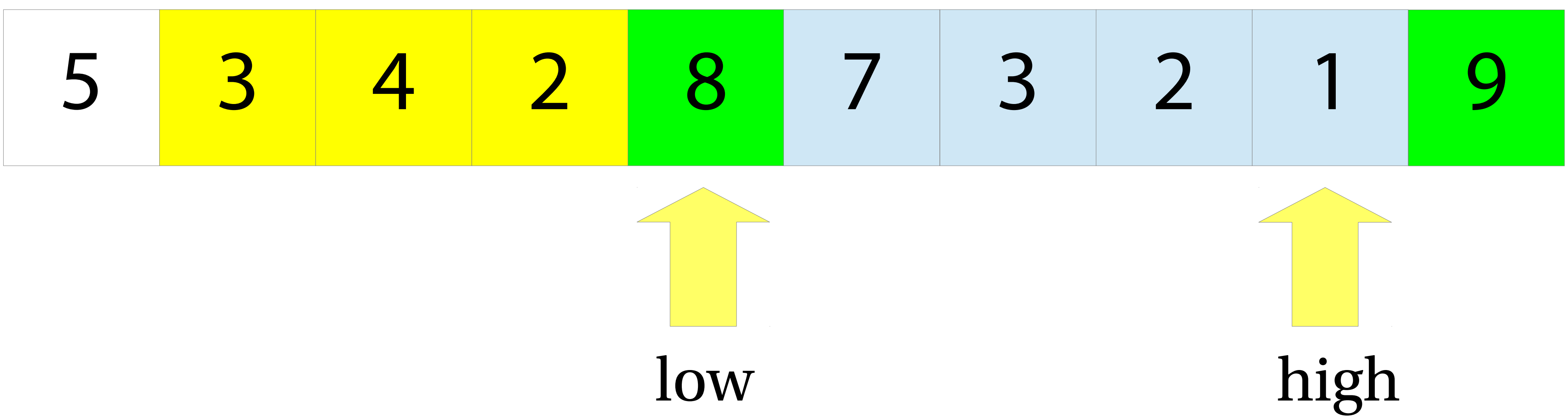
5. Advance low and high and repeat




```
while (a[low] < pivot) low++;
```

Partitioning algorithm

5. Advance low and high and repeat



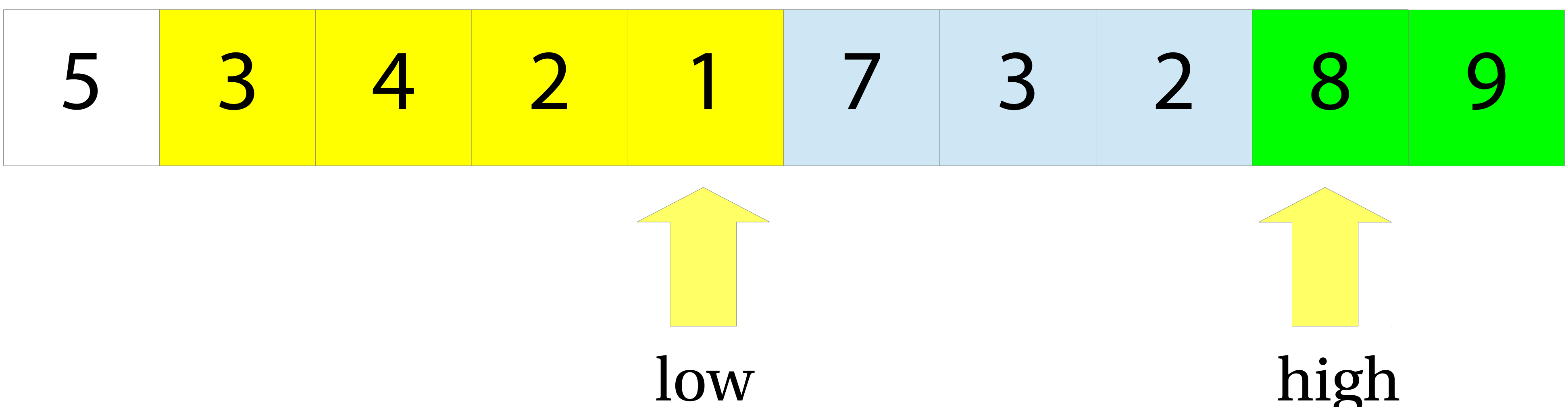
Partitioning algorithm

5. Advance low and high and repeat


```
while (a[high] < pivot) high++;
```

Partitioning algorithm

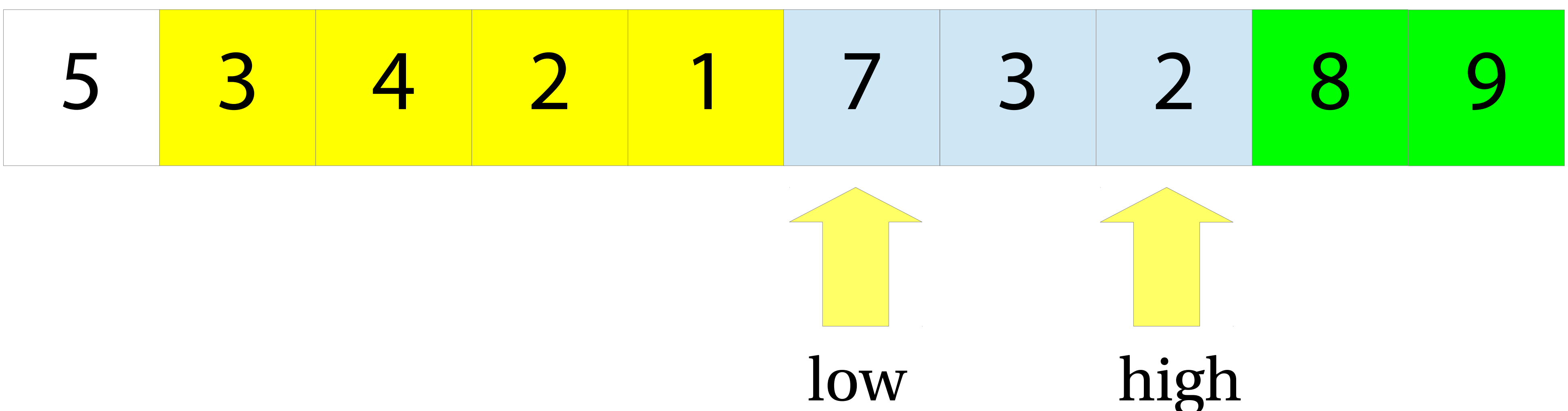
5. Advance low and high and repeat




```
swap(a[low], a[high]);
```

Partitioning algorithm

5. Advance low and high and repeat

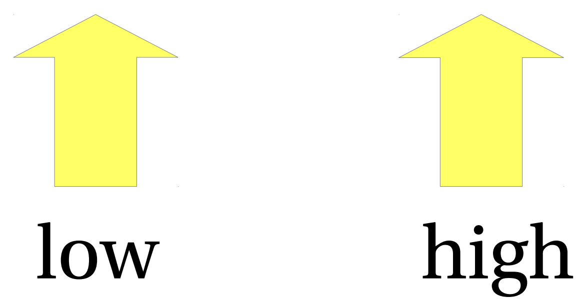



```
low++; high--;
```

Partitioning algorithm

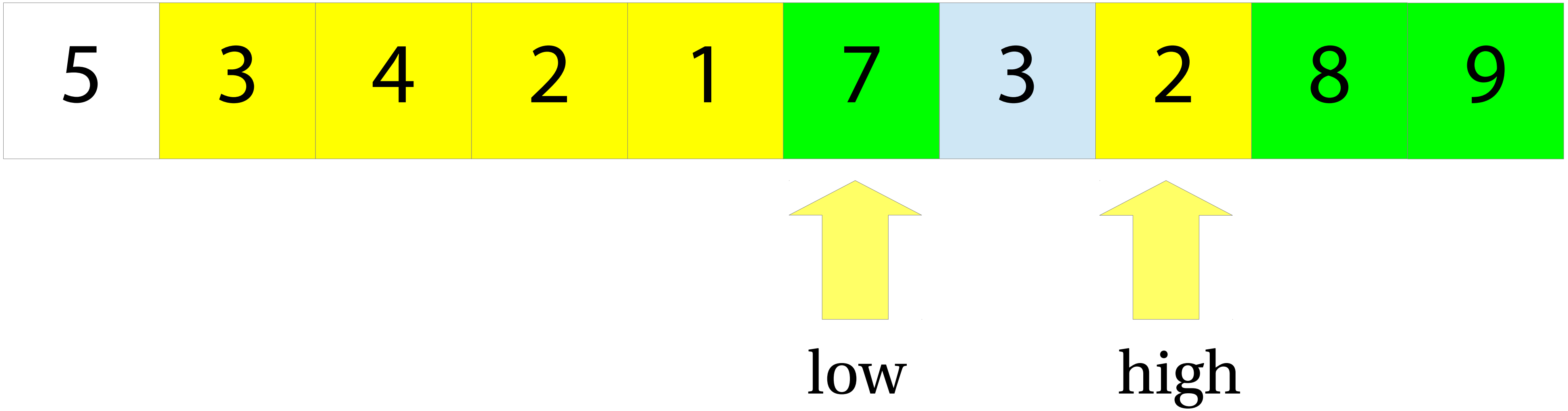
5. Advance low and high and repeat

5	3	4	2	1	7	3	2	8	9



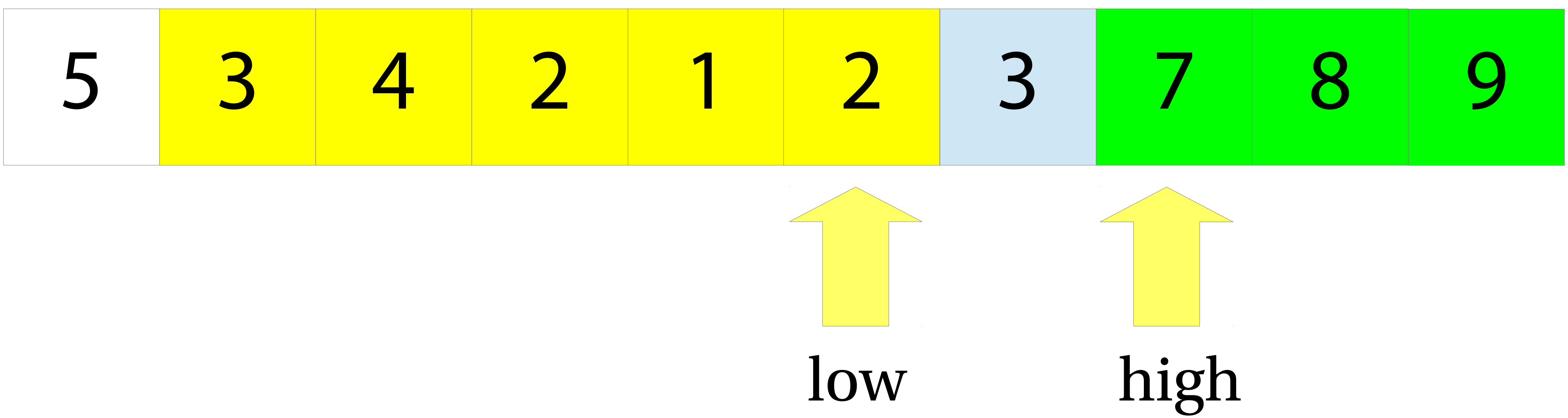
Partitioning algorithm

5. Advance low and high and repeat



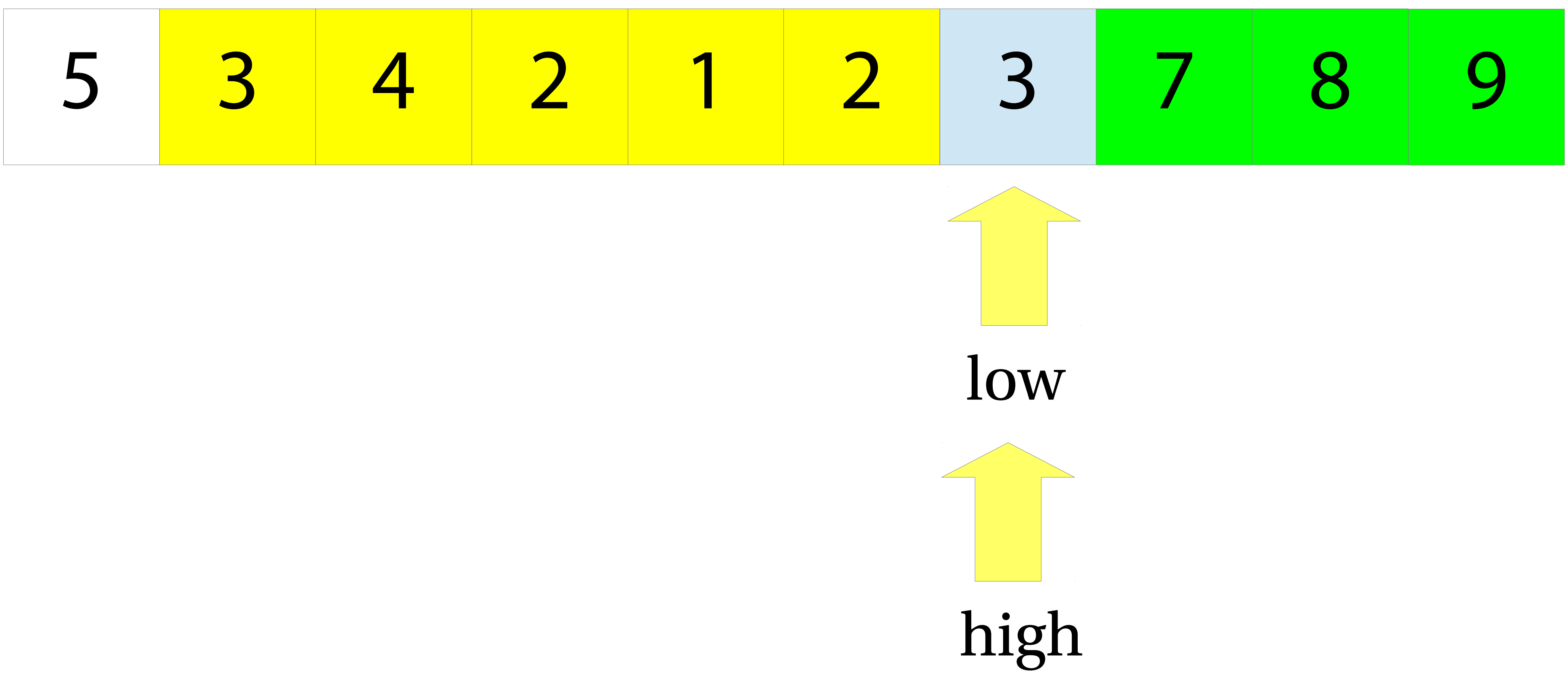
Partitioning algorithm

5. Advance low and high and repeat



Partitioning algorithm

5. Advance low and high and repeat



Partitioning algorithm

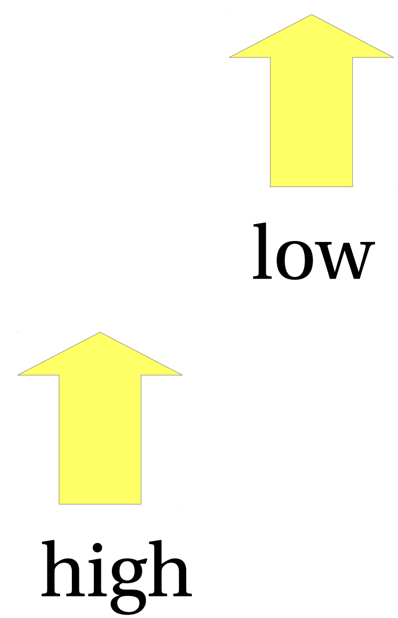
5. Advance low and high and repeat

Partitioning algorithm

6. When low and high have crossed, we are finished!

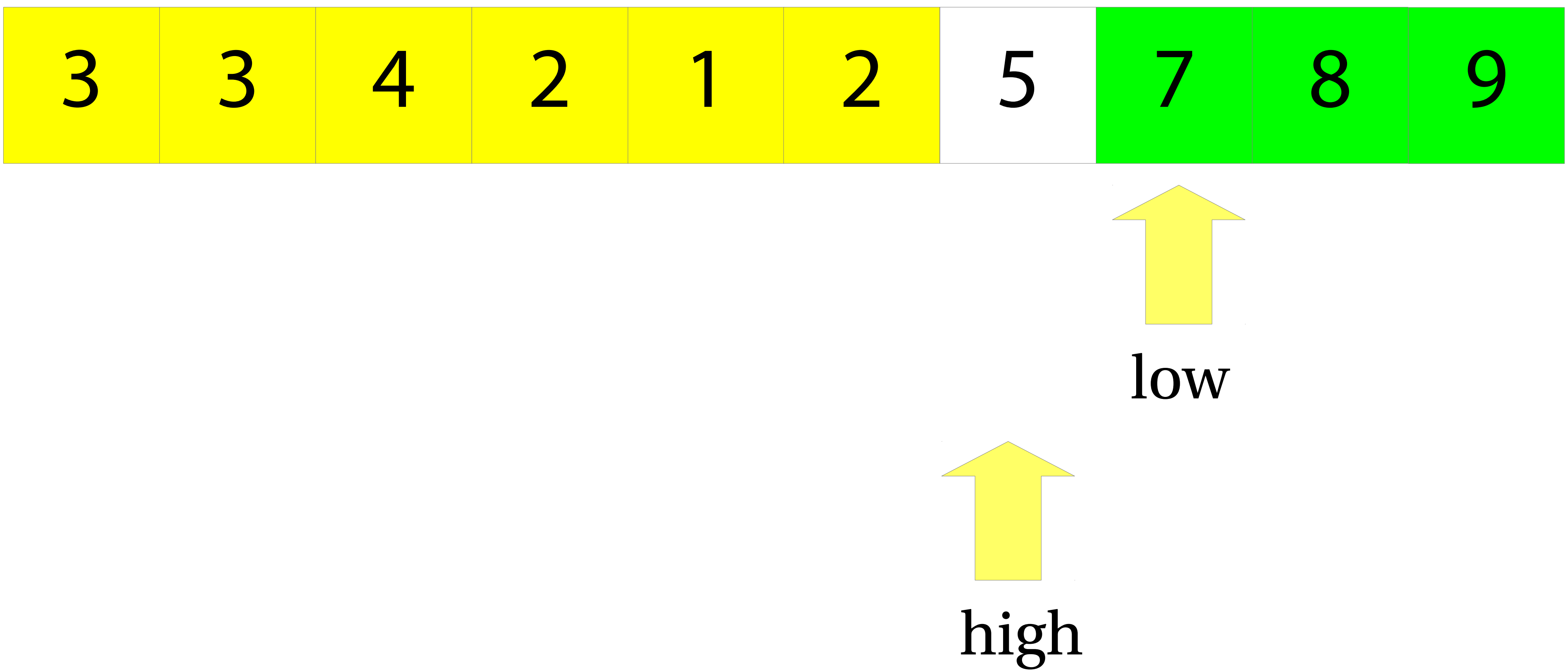
5	3	4	2	1	2	3	7	8	9

But the pivot is in the wrong place.



Partitioning algorithm

7. Last step: swap pivot with high



Details

1. What to do if we want to use a different element (not the first) for the pivot?

- Swap the pivot with the first element before starting partitioning!

Details

2. What happens if the array contains many duplicates?

- Notice that we only advance a[low] as long as a[low] < pivot
- If a[low] == pivot we stop, same for a[high]
- If the array contains just one element over and over again, low and high will advance at the same rate
- Hence we get equal-sized partitions

Pivot

Which pivot should we pick?

- First element: gives $O(n^2)$ behaviour for already-sorted lists
- Median-of-three: pick first, middle and last element of the array and pick the median of those three
- Pick pivot at random: gives $O(n \log n)$ *expected* (probabilistic) complexity

Quicksort

Typically the fastest sorting algorithm…
…but very sensitive to details!

- Must choose a good pivot to avoid $O(n^2)$ case
- Must take care with duplicates
- Switch to insertion sort for small arrays to get better constant factors

Mergesort vs quicksort

Quicksort:

- In-place
- $O(n \log n)$ but $O(n^2)$ if you are not careful
- Works on arrays only (random access)

Compared to mergesort:

- Not in-place
- $O(n \log n)$
- Only requires sequential access to the list – this makes it good in functional programming

Both the best in their fields!

- Quicksort best imperative algorithm
- Mergesort best functional algorithm

Complexity of recursive functions

(Weiss 7.5)

Calculating complexity

Let T(n) be the time mergesort takes on a list of size n

Mergesort does O(n) work to split the list in two, two recursive calls of size n/2 and O(n) work to merge the two halves together, so...

$$T(n) = O(n) + 2T(n/2)$$

Time to sort a list of size n

Linear amount of time spent in splitting + merging

Plus two recursive calls of size n/2

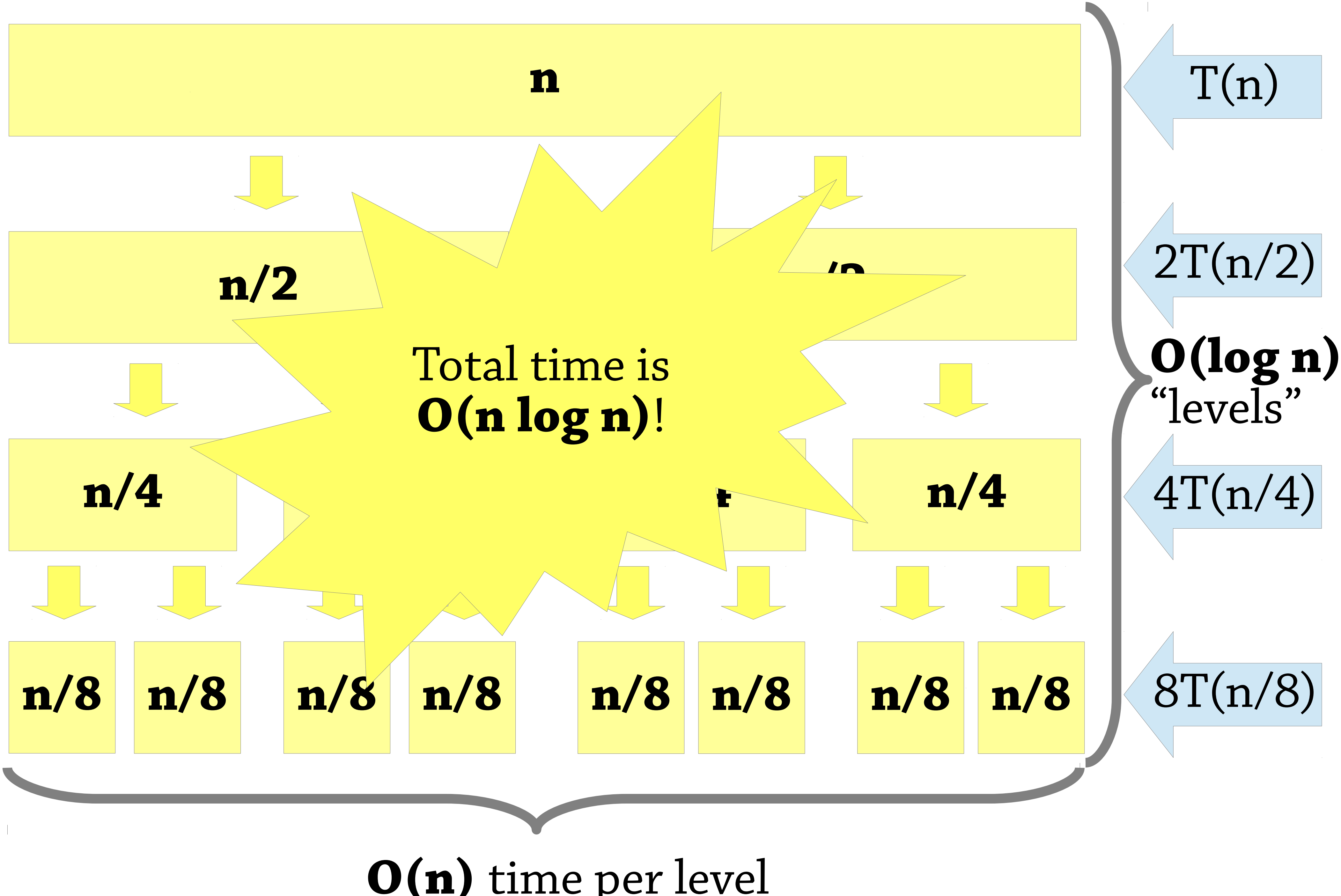
Calculating complexity

Procedure for calculating complexity of a recursive algorithm:

- Write down a *recurrence relation*
  e.g. $T(n) = O(n) + 2T(n/2)$
- *Solve* the recurrence relation to get a formula for $T(n)$ (difficult!)

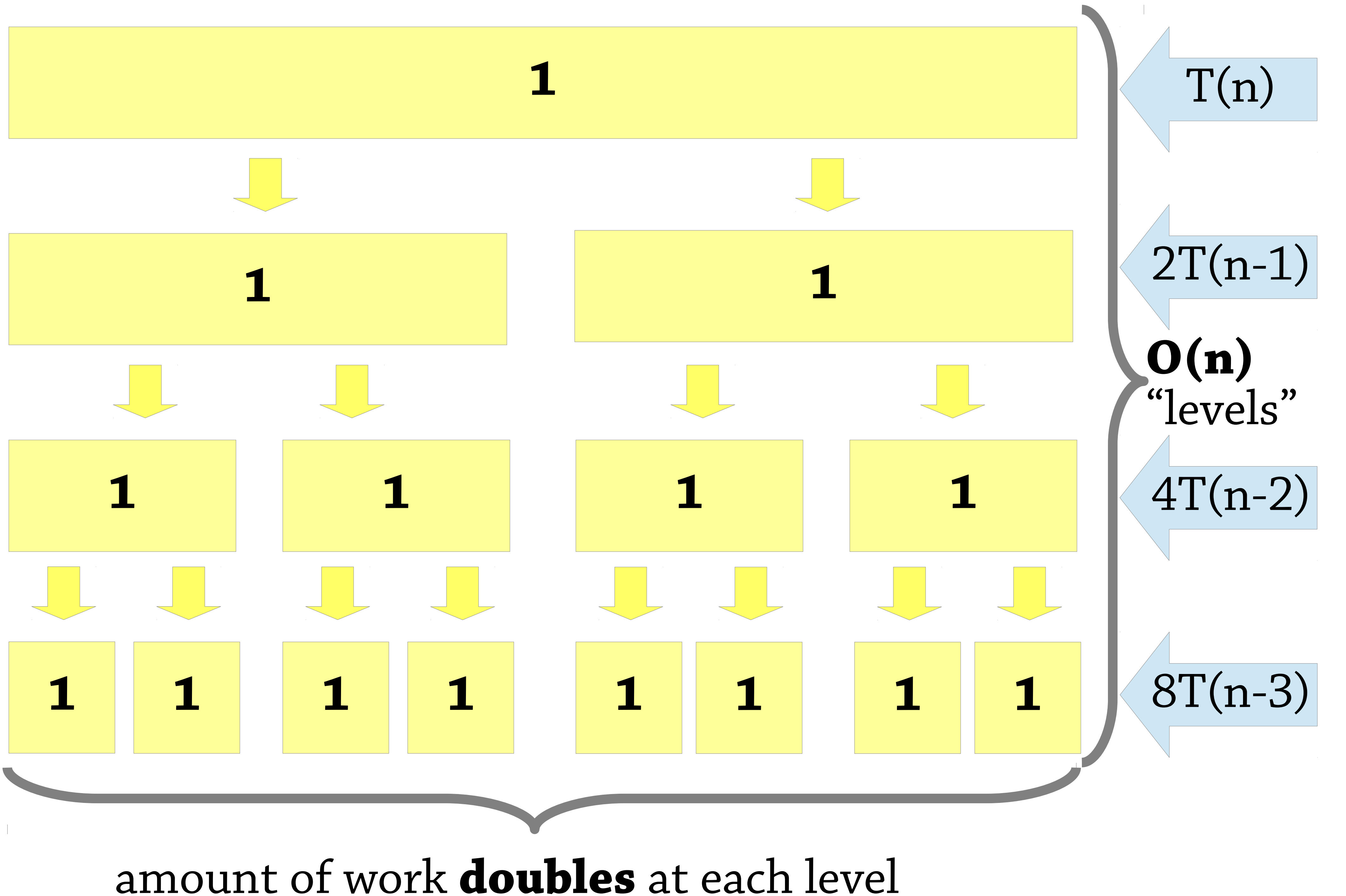
There isn't a general way of solving *any* recurrence relation – we'll just see a few families of them

Approach 1:
draw a diagram

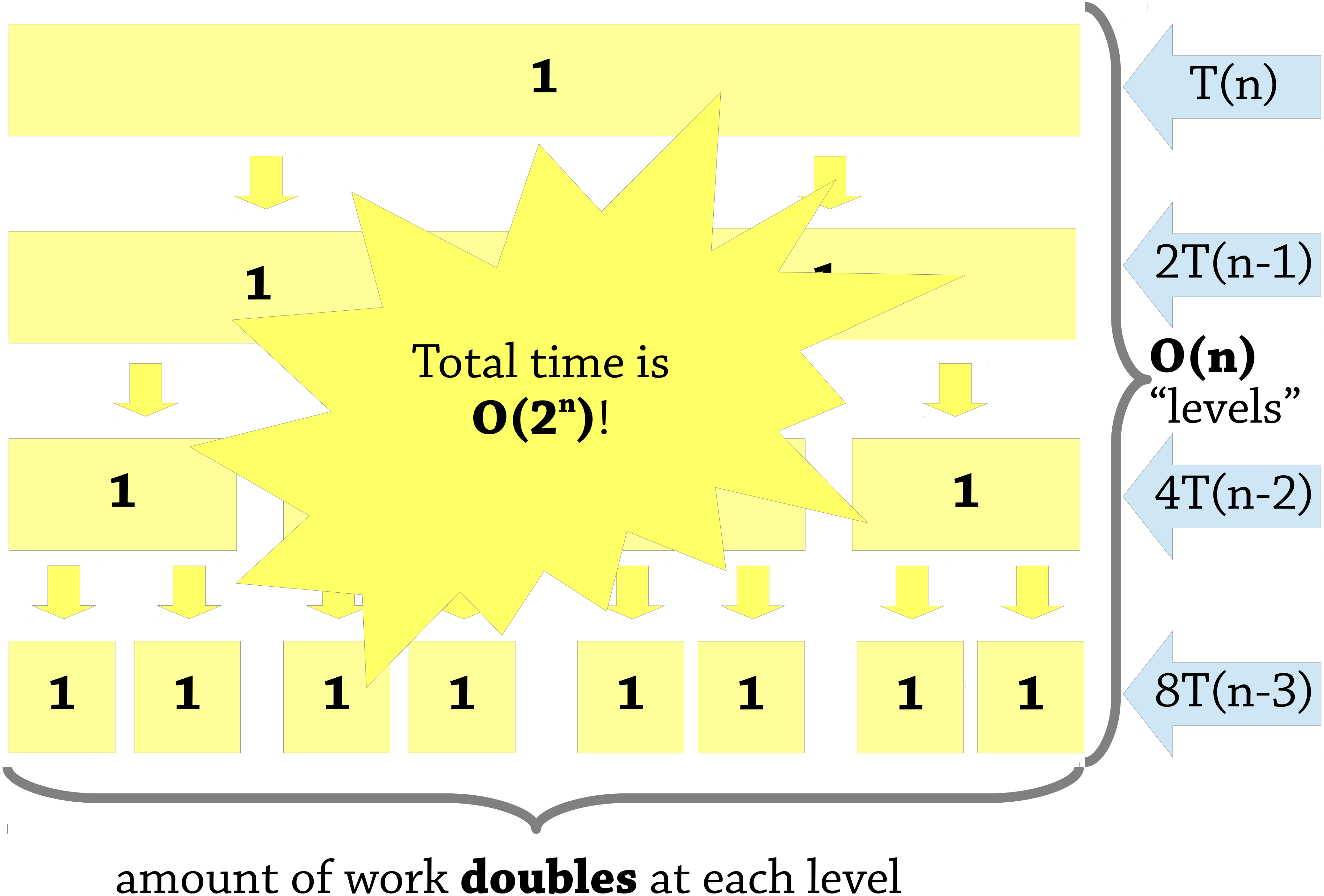

n
n/2
Total time is
O(n log n)!
n/4
n/4
n/8
n/8
n/8
n/8
n/8
n/8
n/8
n/8
T(n)
2T(n/2)
O(log n)
"levels"
4T(n/4)
8T(n/8)
O(n) time per level

Another example:

$$T(n) = O(1) + 2T(n-1)$$



1
1
1
1
1
1
1
1
1
1
1
1
1
1
1
T(n)
2T(n-1)
O(n)
"levels"
4T(n-2)
8T(n-3)
amount of work doubles at each level


1
1
1
1
1
1
1
1
1
1
1
1
1
Total time is
O(2ⁿ)!
T(n)
2T(n-1)
4T(n-2)
8T(n-3)
O(n)
"levels"
amount of work doubles at each level

This approach

Good for building an intuition

Maybe a bit error-prone

Approach 2: *expand out* the definition

Example: solving $T(n) = O(1) + T(n-1)$

Expanding out recurrence relations

$T(n) = 1 + T(n-1)$

$= 2 + T(n-2)$

$= 3 + T(n-3)$

$= \ldots$

$= n + T(0)$

$= O(n)$

T(0) is a constant,
so O(1)

Another example: $T(n) = O(n) + T(n-1)$

$$T(n) = n + T(n-1)$$

$$= n + (n-1) + T(n-2)$$

$$= n + (n-1) + (n-2) + T(n-3)$$

$$= \ldots$$

$$= n + (n-1) + (n-2) + \ldots + 1 + T(0)$$

$$= n(n+1) / 2 + T(0)$$

$$= O(n^2)$$

Another example: $T(n) = O(1) + T(n/2)$

$$
\begin{aligned}
T(n) &= 1 + T(n/2) \\
&= 2 + T(n/4) \\
&= 3 + T(n/8) \\
&= \ldots \\
&= \log n + T(1) \\
&= O(\log n)
\end{aligned}
$$

Another example: $T(n) = O(n) + T(n/2)$

$T(n) = n + T(n/2)$:

$T(n) = n + T(n/2)$

$= n + n/2 + T(n/4)$

$= n + n/2 + n/4 + T(n/8)$

$= \ldots$

$= n + n/2 + n/4 + \ldots$

$< 2n$

$= O(n)$

Functions that recurse once

$T(n) = O(1) + T(n-1)$: $T(n) = O(n)$

$T(n) = O(n) + T(n-1)$: $T(n) = O(n^2)$

$T(n) = O(1) + T(n/2)$: $T(n) = O(\log n)$

$T(n) = O(n) + T(n/2)$: $T(n) = O(n)$

An *almost-rule-of-thumb*:

- Solution is *maximum recursion depth* times *amount of work in one call*

(except that this rule of thumb would give $O(n \log n)$ for the last case)

Divide-and-conquer algorithms

$T(n) = O(n) + 2T(n/2)$: $T(n) = O(n \log n)$

- This is mergesort! There is a nice proof in the book (theorem 7.4).

$T(n) = 2T(n-1)$: $T(n) = O(2^n)$

- Because 2^n recursive calls of depth n

Other cases: *master theorem* (Wikipedia) or theorem 7.5 from book

- Kind of fiddly – best to just look it up if you need it

Complexity of recursive functions

Basic idea – recurrence relations

Easy enough to write down, hard to solve

- One technique: expand out the recurrence and see what happens
- Another rule of thumb: multiply work done per level with number of levels
- Drawing a diagram (like for quicksort) can help!

Master theorem for divide and conquer

Luckily, in practice you come across the same few recurrence relations, so you just need to know how to solve those