Linear Types for Controlled Array Fusion

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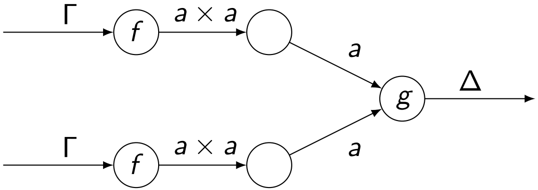
Fusion

- Elimination of intermediate data structures
- Example: $\mathit{foldr}(+)0 \circ \mathit{map}(*n)$
- Similar to partial evaluation
- Crucial for performance (laziness seems to help, but it is useless in case of parallelism)
- But, it is treated as an *optimisation*

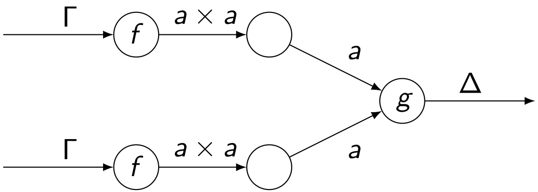
Fusion attempt: starting point

$$\xrightarrow{\ \Gamma\ } f \xrightarrow{\ a \times a\ } g \xrightarrow{\ \Delta\ }$$

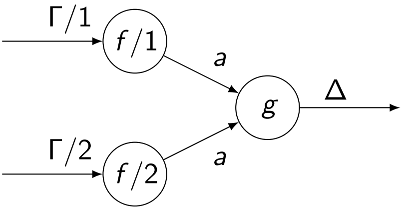
Fusion attempt...



Fusion attempt...



One can *hope* for:



Remember, all I'm offering is the truth. Nothing more.

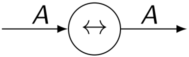
Duality

- Every type a has a *dual*, written $a^{\perp}$
- Dualisation is involutive: $(a^{\perp})^{\perp} = a$
- Returning a value of type a is the same as taking an argument of type $a^{\perp}$
- $a \to b$ becomes $a \times b^{\perp} \to \perp$
- NB: $\perp$ indicates termination rather than non-termination

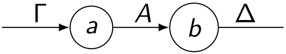
Linearity

Conservation principle

$$\frac{}{x : A, y : A^{\perp} \vdash x \leftrightarrow y}\ \textsc{Ax}$$

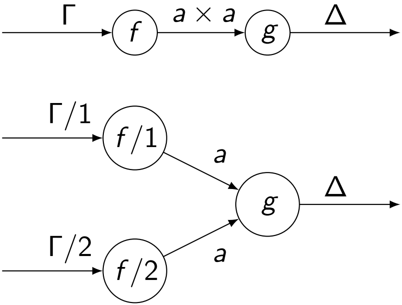


$$\frac{\Gamma, x : A^{\perp} \vdash a \qquad y : A, \Delta \vdash b}{\Gamma, \Delta \vdash \mathsf{cut}\ \{x : A^{\perp} \mapsto a; y : A \mapsto b\}}\ \textsc{Cut}$$



Back to the example

We would like to *guarantee* well-behaved fusion:



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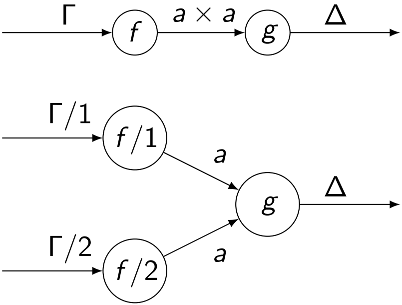


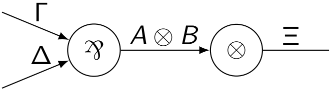
Need to use a tensor (⊗) product, which guarantees that its components are produced independently.

Technically:

- $(A \otimes B)^\perp = A^\perp \parr B^\perp$
- $$\frac{\Gamma, x : A, y : B \vdash a}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ x, y = z; a}\otimes$$
- $$\frac{\Gamma, x : A \vdash a \qquad y : B, \Delta \vdash b}{\Gamma, z : A \parr B, \Delta \vdash \mathsf{connect}\ z\ \mathsf{to}\{x \mapsto a; y \mapsto b\}}\parr$$

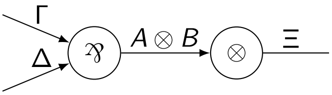
Fusion works!

$$\dfrac{\dfrac{\Gamma, A^{\perp} \vdash \qquad \Delta, B^{\perp} \vdash}{\Gamma, \Delta, A^{\perp} \parr B^{\perp} \vdash}\,\parr \qquad \dfrac{A, B, \Xi \vdash}{A \otimes B, \Xi \vdash}\,\otimes}{\Gamma, \Delta, \Xi \vdash}\,\textsc{Cut}$$

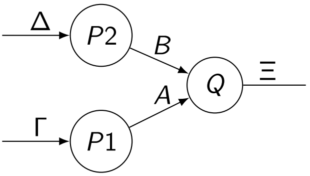


Fusion works!

$$\dfrac{\dfrac{\Gamma, A^\perp \vdash \qquad \Delta, B^\perp \vdash}{\Gamma, \Delta, A^\perp \parr B^\perp \vdash}\,\parr \qquad \dfrac{A, B, \Xi \vdash}{A \otimes B, \Xi \vdash}\,\otimes}{\Gamma, \Delta, \Xi \vdash}\,\textsc{Cut}$$



$$\dfrac{\Gamma, A^\perp \vdash \qquad \dfrac{\Delta, B^\perp \vdash \qquad B, A, \Xi \vdash}{A, \Delta, \Xi \vdash}\,\textsc{Cut}}{\Gamma, \Delta, \Xi \vdash}\,\textsc{Cut}$$



Crucial question: who is in control?

Two kinds of "products":

- $A \otimes B$: the consumer has control
- $A \parr B$: the producer has control

Example:

- sallad $\otimes$ pizza vs
- sallad $\parr$ pizza

NB: in control means that the order is chosen dynamically, and some order must be chosen.

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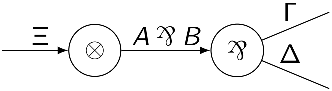
NB: in control means that the order is chosen dynamically, and some order must be chosen.

- Possible slogan: *strategies in the types*

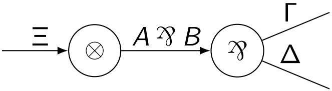
Fusion when the consumer is in control:

$$\dfrac{\dfrac{\Xi, A^{\perp}, B^{\perp} \vdash}{\Xi, A^{\perp} \otimes B^{\perp} \vdash}\,\otimes \qquad \dfrac{A, \Gamma \vdash \qquad B, \Delta \vdash}{A \parr B, \Gamma, \Delta \vdash}\,\parr}{\Gamma, \Delta, \Xi \vdash}\,\textsc{Cut}$$

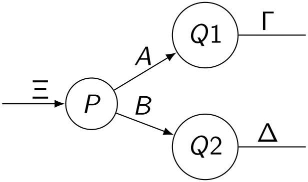


Fusion when the consumer is in control:

$$\dfrac{\dfrac{\Xi, A^{\perp}, B^{\perp} \vdash}{\Xi, A^{\perp} \otimes B^{\perp} \vdash}\otimes \qquad \dfrac{A, \Gamma \vdash \quad B, \Delta \vdash}{A \parr B, \Gamma, \Delta \vdash}\parr}{\Gamma, \Delta, \Xi \vdash}\textsc{Cut}$$



$$\dfrac{\Gamma, A \vdash \qquad \dfrac{\Delta, B \vdash \quad B^{\perp}, A^{\perp}, \Xi \vdash}{A^{\perp}, \Delta, \Xi \vdash}\textsc{Cut}}{\Gamma, \Delta, \Xi \vdash}\textsc{Cut}$$



remark

(a $\multimap$ b is a defined to be $a^{\perp} \parr b$)

Conversions between $\parr$ and $\otimes$.

- alloc : $a \parr b \multimap a \otimes b$

Conversions between ⅋ and $\otimes$.

- alloc : $a \parr b \multimap a \otimes b$
- par : $a \otimes b \multimap a \parr b$

Conversions between $\parr$ and $\otimes$.

- alloc : $a \parr b \multimap a \otimes b$
- par : $a \otimes b \multimap a \parr b$
- seqR : $a \otimes b \multimap a \parr b$
- seqL : $a \otimes b \multimap a \parr b$

Generalisation to Arrays (simplified)

$$\frac{\Gamma, x : A^m \vdash a}{\Gamma, z : \bigotimes_m A \vdash \mathsf{let}\ x = \mathsf{slice}\ z; a} \otimes$$

$$\frac{\Gamma, x : A \vdash a}{\Gamma^n, z : \bigparr_n A \vdash \mathsf{coslice}\ z\{x \mapsto a\}} \parr$$

Generalisation to Arrays (not so simplified)

$$\frac{\Gamma, x : A^m \vdash a}{\Gamma, z : \bigotimes_m A \vdash \text{let } x = \text{slice } z; a} \otimes$$

$$\frac{\Gamma, x : A \vdash a \qquad \Delta, y : A \vdash b}{\Gamma^n, \Delta^m, z : \bigparr_{n+m} A \vdash \text{coslice } z\{x \mapsto a; y \mapsto b\}} \parr$$

Simple examples

Only one argument/result can be in control.

- map ::

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- map :: $\bigotimes_n (a \multimap b) \multimap \bigotimes_n a \multimap \bigotimes_n b$
- map' ::

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- map' :: $\bigotimes_n (a \multimap b) \multimap \bigparr_n a \multimap \bigparr_n b$
- zipWith ::

Simple examples

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- map :: $\bigotimes_n (a \multimap b) \multimap \bigotimes_n a \multimap \bigotimes_n b$
- map' :: $\bigotimes_n (a \multimap b) \multimap \bigparr_n a \multimap \bigparr_n b$
- zipWith :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigotimes_n b \multimap \bigotimes_n c$
- zipWith' ::

Simple examples

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- map :: $\bigotimes_n (a \multimap b) \multimap \bigotimes_n a \multimap \bigotimes_n b$
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- zipWith :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigotimes_n b \multimap \bigotimes_n c$
- zipWith' :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigparr_n b \multimap \bigparr_n c$
- append ::

Simple examples

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- map :: $\bigotimes_n (a \multimap b) \multimap \bigotimes_n a \multimap \bigotimes_n b$
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- zipWith' :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigparr_n b \multimap \bigparr_n c$
- append :: $\bigotimes_n a \multimap \bigotimes_m a \multimap \bigotimes_{n+m} a$
- append' ::

Simple examples

Only one argument/result can be in control.

- map :: $\bigotimes_n (a \multimap b) \multimap \bigotimes_n a \multimap \bigotimes_n b$
- map' :: $\bigotimes_n (a \multimap b) \multimap \bigparr_n a \multimap \bigparr_n b$
- zipWith :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigotimes_n b \multimap \bigotimes_n c$
- zipWith' :: $\bigotimes_n (a \multimap b \multimap c) \multimap \bigotimes_n a \multimap \bigparr_n b \multimap \bigparr_n c$
- append :: $\bigotimes_n a \multimap \bigotimes_m a \multimap \bigotimes_{n+m} a$
- append' :: $\bigparr_n a \multimap \bigparr_m a \multimap \bigparr_{n+m} a$

Conversions between $\parr$ and $\otimes$.

- alloc : $\bigparr_n a \multimap \bigotimes_n a$

Conversions between $\parr$ and $\otimes$.

- alloc : $\parr_n\, a \multimap \bigotimes_n a$
- parallel : $\bigotimes_n a \multimap \parr_n\, a$
- sequential : $\bigotimes_n a \multimap \parr_n\, a$

Allocation

If A is a data type:

$$\frac{\Gamma, {A^{\perp}}^{n} \vdash \qquad A^{n}, \Delta \vdash}{\Gamma, \Delta \vdash}\ \textsc{freeze}$$

Loops

- Loops are realisation of parallel opportunities
- They can either be sequential or parallel
- Example of a parallel loop:
  - $$\dfrac{B[0]^{\perp}, \Delta \vdash \quad B[k], B[k], B[k+1]^{\perp} \vdash \quad B[n], \Gamma \vdash}{n : \mathsf{Ix}; \Gamma, \Delta^{2^n} \vdash}\ \textsc{FoldMap}$$
- See lecture 9 (skeletons) and in particular map-reduce (lecture 10)

Full example: dot product (0)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\,\parr}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr}{mul : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}}{mul : a^\perp \parr a^\perp \parr a; a^{2^n}, a^{2^n}, \bigparr_{2^n} a^\perp \vdash}\,\parr}{mul : a^\perp \parr a^\perp \parr a; a^{2^n}, \bigotimes_{2^n} a, \bigparr_{2^n} a^\perp \vdash}\,\otimes$$

$$\dfrac{mul : a^\perp \parr a^\perp \parr a; a^{2^n}, \bigotimes_{2^n} a, \bigparr_{2^n} a^\perp \vdash}{mul : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, \bigparr_{2^n} a^\perp \vdash}\,\otimes$$

$$\dfrac{\dfrac{\dfrac{}{a^\perp, a \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\,\parr}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr}{plus : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load} \quad \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}}{plus : a^\perp \parr a^\perp \parr a; a^{2^n}, a^\perp \vdash}\,\textsc{foldmap}}{plus : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, a^\perp \vdash}\,\otimes$$

$$\dfrac{mul : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, \bigparr_{2^n} a^\perp \vdash \quad plus : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, a^\perp \vdash}{plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^\perp \vdash}\,\textsc{Cut}$$

Full example: dot product (1)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^{\perp}, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^{\perp}, a^{\perp} \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$\mathit{mul} : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$\mathit{mul} : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, a^{2^n}, \bigparr_{2^n} a^{\perp} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\mathit{mul} : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, \bigotimes_{2^n} a, \bigparr_{2^n} a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^{\perp}, a \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^{\perp}, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^{\perp}, a^{\perp} \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$\mathit{plus} : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^{\perp} \vdash$}
\RightLabel{\textsc{foldmap}}
\TrinaryInfC{$\mathit{plus} : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, a^{\perp} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\mathit{plus} : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{2^n} a, a^{\perp} \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$\mathit{plus} : a^{\perp} \parr a^{\perp} \parr a, \mathit{mul} : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, \bigotimes_{2^n} a, a^{\perp} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\mathit{plus} : a^{\perp} \parr a^{\perp} \parr a, \mathit{mul} : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^{\perp} \vdash$}
\end{prooftree}
$$

Full example: dot product (2)

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          \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
          \quad
          \dfrac{
            \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
            \quad
            \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}
          }{a, a^\perp, a^\perp \parr a \vdash}\,\parr
        }{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr
      }{\mathit{mul} : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}
    }{\mathit{mul} : a^\perp \parr a^\perp \parr a; a^{2^n}, a^{2^n}, \parr_{2^n} a^\perp \vdash}\,\parr
    \quad
    \dfrac{
      \dfrac{
        \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}
        \quad
        \dfrac{
          \dfrac{
            \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
            \quad
            \dfrac{
              \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
              \quad
              \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}
            }{a, a^\perp, a^\perp \parr a \vdash}\,\parr
          }{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr
        }{\mathit{plus} : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}
        \quad
        \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
      }{\mathit{plus} : a^\perp \parr a^\perp \parr a; a^{2^n}, a^\perp \vdash}\,\textsc{foldmap}
    }{\mathit{plus} : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, a^\perp \vdash}\,\otimes
  }{
    \dfrac{
      \dfrac{\mathit{plus} : a^\perp \parr a^\perp \parr a, \mathit{mul} : a^\perp \parr a^\perp \parr a; a^{2^n}, a^{2^n}, a^\perp \vdash}{\mathit{plus} : a^\perp \parr a^\perp \parr a, \mathit{mul} : a^\perp \parr a^\perp \parr a; a^{2^n}, \bigotimes_{2^n} a, a^\perp \vdash}\,\otimes
    }{\mathit{plus} : a^\perp \parr a^\perp \parr a, \mathit{mul} : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^\perp \vdash}\,\otimes
  }
}{}\,\textsc{Cut}
$$

Full example: dot product (3)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$mul : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$plus : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{foldmap}
\TrinaryInfC{$plus : a^\perp \parr a^\perp \parr a; a^{2^n}, a^\perp \vdash$}
\RightLabel{Cut_{2^n}}
\BinaryInfC{$plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; a^{2^n}, a^{2^n}, a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; a^{2^n}, \bigotimes_{2^n} a, a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^\perp \vdash$}
\end{prooftree}
$$

Full example: dot product (4)

$$
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\dfrac{
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
\quad
\dfrac{
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
\quad
\dfrac{}{a^\perp, a \vdash}\textsc{Ax}
}{a, a^\perp, a^\perp \parr a \vdash}\parr
}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\parr
}{mul : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\textsc{Load}
\quad
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
}{mul : a^\perp \parr a^\perp \parr a; a^\perp, a, a \vdash}\textsc{Cut}
\quad
\dfrac{
\dfrac{
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
\quad
\dfrac{
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
\quad
\dfrac{}{a^\perp, a \vdash}\textsc{Ax}
}{a, a^\perp, a^\perp \parr a \vdash}\parr
}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\parr
}{plus : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\textsc{Load}
\quad
\dfrac{}{a, a^\perp \vdash}\textsc{Ax}
}{
\dfrac{
\dfrac{plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; a^{2^n}, a^{2^n}, a^\perp \vdash}{plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; a^{2^n}, \bigotimes_{2^n} a, a^\perp \vdash}\otimes
}{plus : a^\perp \parr a^\perp \parr a, mul : a^\perp \parr a^\perp \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^\perp \vdash}\otimes
}\textsc{foldmap}
$$

Full example: dot product (5)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^{\perp}, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^{\perp}, a, a^{\perp} \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^{\perp}, a, a, a^{\perp} \parr a^{\perp} \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$mul : a^{\perp} \parr a^{\perp} \parr a; a^{\perp}, a, a \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^{\perp}, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^{\perp}, a^{\perp} \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$plus : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^{\perp} \vdash$}
\RightLabel{foldmap}
\TrinaryInfC{$plus : a^{\perp} \parr a^{\perp} \parr a, mul : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, a^{2^n}, a^{\perp} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$plus : a^{\perp} \parr a^{\perp} \parr a, mul : a^{\perp} \parr a^{\perp} \parr a; a^{2^n}, \bigotimes_{2^n} a, a^{\perp} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$plus : a^{\perp} \parr a^{\perp} \parr a, mul : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{2^n} a, \bigotimes_{2^n} a, a^{\perp} \vdash$}
\end{prooftree}
$$

Done!

More complicated example: stencil (1)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\bigotimes_n a, a, \parr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\bigotimes_n a, a, \parr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^{n+1}, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^{n+1}, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_{n+1} a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (2)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\bigotimes_n a, a, \parr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \parr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^{n+1}, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^{n+1}, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_{n+1} a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (3)

$$
\dfrac{
\dfrac{
\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax}
\quad
\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax}
}{a^{n}, a, \bigparr_{n+1} a^{\perp} \vdash}\,\parr
}{\bigotimes_n a, a, \bigparr_{n+1} a^{\perp} \vdash}\,\otimes
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax}
\quad
\dfrac{
\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax}
\quad
\dfrac{}{a^{\perp}, a \vdash}\,\textsc{Ax}
}{a^{\perp}, a, a^{\perp} \parr a \vdash}\,\parr
}{a, a^{\perp}, a, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; a, a^{\perp}, a \vdash}\,\textsc{Load}
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, \bigparr_{n+1} a^{\perp}, a^{n+1} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, \bigparr_{n+1} a^{\perp}, \bigotimes_{n+1} a \vdash}\,\otimes
\quad
\dfrac{
\dfrac{}{a^{\perp}, a \vdash}\,\textsc{Ax}
\quad
\dfrac{}{a^{\perp}, a \vdash}\,\textsc{Ax}
}{\bigparr_{n+1} a^{\perp}, a^{n}, a \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, a^{n}, a, \bigparr_{n+1} a^{\perp} \vdash}\,\textsc{Cut}
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n+1} a, a^{n}, a, \bigparr_{n+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \bigparr_{n+1} a^{\perp} \vdash}\,\otimes
}{
\dfrac{
\dfrac{
\dfrac{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \bigparr_{n+1} a^{\perp} \vdash}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^{\perp} \vdash}\,\textsc{Split}
}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_n a)^{k+k}, a^{k}, a^{k}, \bigparr_k \bigparr_{n+1} a^{\perp} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{k+k} \bigotimes_n a, a^{k}, a^{k}, \bigparr_k \bigparr_{n+1} a^{\perp} \vdash}\,\otimes
}\,\textsc{Cut}
$$

More complicated example: stencil (4)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\bigotimes_n a, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^{n+1}, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Merge}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^n, a^{n+1}, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^{n+1}, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\textsc{Cut}_n$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (5)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$\bigotimes_n a, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^n, a, a^n, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^n, a^{n+1}, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^{n+1}, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}$_n$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (6)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax}}{a^{n}, a, \parr_{n+1} a^{\perp} \vdash}\,\parr}{\bigotimes_{n} a, a, \parr_{n+1} a^{\perp} \vdash}\,\otimes \quad \dfrac{\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\textsc{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\textsc{Load} \quad \dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\textsc{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\textsc{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\textsc{Load}}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n}, a, a^{n}, a, \parr_{n+1} a^{\perp} \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n}, a^{n+1}, a, \parr_{n+1} a^{\perp} \vdash}\,\textsc{Split}}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, a^{n}, a, \parr_{n+1} a^{\perp} \vdash}\,\textsc{Cut}_{n}}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n+1} a, a^{n}, a, \parr_{n+1} a^{\perp} \vdash}\,\otimes}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n+1} a, \bigotimes_{n} a, a, \parr_{n+1} a^{\perp} \vdash}\,\otimes}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n} a, \bigotimes_{n} a, a, a, \parr_{n+1} a^{\perp} \vdash}\,\textsc{Cut}}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n} a)^{1+1}, a, a, \parr_{n+1} a^{\perp} \vdash}\,\textsc{Split}}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n} a)^{k+k}, a^{k}, a^{k}, \parr_{k} \parr_{n+1} a^{\perp} \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{k+k} \bigotimes_{n} a, a^{k}, a^{k}, \parr_{k} \parr_{n+1} a^{\perp} \vdash}\,\otimes
$$

More complicated example: stencil (7)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
      \quad
      \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
    }{a^n, a, \parr_{n+1} a^\perp \vdash}\,\parr
  }{\bigotimes_n a, a, \parr_{n+1} a^\perp \vdash}\,\otimes
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
              \quad
              \dfrac{
                \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
                \quad
                \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}
              }{a, a^\perp, a^\perp \parr a \vdash}\,\parr
            }{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr
          }{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}
          \quad
          \dfrac{
            \dfrac{
              \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
              \quad
              \dfrac{
                \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
                \quad
                \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}
              }{a, a^\perp, a^\perp \parr a \vdash}\,\parr
            }{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr
          }{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}
        }{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \parr_{n+1} a^\perp \vdash}\,\parr
      }{s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \parr_{n+1} a^\perp \vdash}\,\textsc{Split}
    }{s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \parr_{n+1} a^\perp \vdash}\,\otimes
  }{s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \parr_{n+1} a^\perp \vdash}\,\otimes
}{
  \dfrac{
    \dfrac{
      \dfrac{s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash}\,\textsc{Split}
    }{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash}\,\parr
  }{s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash}\,\otimes
}\,\textsc{Cut}
$$

More complicated example: stencil (8)

$$\frac{}{a, a^\perp \vdash}\,\textsc{Ax} \qquad \frac{}{a, a^\perp \vdash}\,\textsc{Ax} \qquad \frac{}{a^\perp, a \vdash}\,\textsc{Ax}$$

$$\frac{a, a^\perp \vdash \qquad a^\perp, a \vdash}{a, a^\perp, a^\perp \parr a \vdash}\,\parr$$

$$\frac{a, a^\perp \vdash \qquad a, a^\perp, a^\perp \parr a \vdash}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr$$

$$\frac{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}$$

(the same derivation is repeated on the right)

$$\frac{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash \qquad s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \parr_{n+1} a^\perp \vdash}\,\parr$$

$$\frac{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \parr_{n+1} a^\perp \vdash}\,\textsc{Split}$$

$$\frac{s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \parr_{n+1} a^\perp \vdash}\,\otimes$$

$$\frac{s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \parr_{n+1} a^\perp \vdash}\,\otimes$$

$$\frac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \qquad \dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}}{a^n, a, \parr_{n+1} a^\perp \vdash}\,\parr$$

$$\frac{a^n, a, \parr_{n+1} a^\perp \vdash \qquad s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash}\,\textsc{Cut}$$

$$\frac{s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash}\,\otimes$$

$$\frac{s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash}\,\textsc{Split}$$

$$\frac{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash}\,\parr$$

$$\frac{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash}\,\otimes$$

More complicated example: stencil (9)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^\perp, a \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a, \parr_{n+1} a^\perp, a, a^n \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a, \parr_{n+1} a^\perp, a^{n+1} \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a, \parr_{n+1} a^\perp, \bigotimes_{n+1} a \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$\parr_{n+1} a^\perp, a^n, a \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \parr_k \parr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (10)

$$
\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax}
\qquad
\dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\,\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\,\parr
$$

$$
\dfrac{\dfrac{a, a^\perp \vdash \qquad a, a^\perp, a^\perp \parr a \vdash}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\,\parr}{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\,\textsc{Load}
$$

$$
\dfrac{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash \qquad s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \bigparr_{n+1} a^\perp \vdash}\,\parr
$$

$$
\dfrac{\dfrac{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \bigparr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a^{n+1}, a^n, a, \bigparr_{n+1} a^\perp \vdash}\,\textsc{Split}}{s : a^\perp \parr a^\perp \parr a; a^n, a, a^n, a, \bigparr_{n+1} a^\perp \vdash}\,\textsc{Merge}_n
$$

$$
\dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \qquad \dfrac{\dfrac{}{a, a^\perp \vdash}\,\textsc{Ax} \quad s : a^\perp \parr a^\perp \parr a; a^n, a, a^n, a, \bigparr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \bigparr_{n+1} a^\perp \vdash}\,\textsc{Cut}_n}{s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \bigparr_{n+1} a^\perp \vdash}\,\textsc{Cut}
$$

$$
\dfrac{\dfrac{\dfrac{s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \bigparr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash}\,\otimes}{s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash}\,\otimes}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash}\,\textsc{Split}
$$

$$
\dfrac{\dfrac{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash}\,\parr}{s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash}\,\otimes
$$

More complicated example: stencil (11)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Merge}$_{n-1}$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{\textsc{Cut}$_n$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^n, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (12)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{\textsc{Load}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\TrinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a, a^{n-1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}$_{n-1}$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a, a^{n}, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n}, a, a^{n}, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{\textsc{Cut}$_n$}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n}, a^{n}, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{\textsc{Ax}}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n}, a^{n}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n}, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{\textsc{Split}}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (13)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}
\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}
\TrinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^{n-1}, a^{n-1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{Split$_{n-1}$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^{n-1}, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{Split}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^n, a^n, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{Cut}
\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{Split}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\parr$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}
\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash$}
\end{prooftree}
$$

More complicated example: stencil (14)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\parr}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\parr}{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\textsc{Load} \quad \dfrac{\dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\parr}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\parr}{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\textsc{Load} \quad \dfrac{\dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{\dfrac{}{a, a^\perp \vdash}\textsc{Ax} \quad \dfrac{}{a^\perp, a \vdash}\textsc{Ax}}{a, a^\perp, a^\perp \parr a \vdash}\parr}{a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash}\parr}{s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash}\textsc{Load}}{s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a^{n-1}, a, a, a, \bigparr_{n+1} a^\perp \vdash}\parr}{s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a^{n}, a, a, \bigparr_{n+1} a^\perp \vdash}\textsc{Split}_{n-1}}{s : a^\perp \parr a^\perp \parr a; a^{n}, a^{n}, a, a, \bigparr_{n+1} a^\perp \vdash}\textsc{Split}}{s : a^\perp \parr a^\perp \parr a; a^{n}, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash}\otimes}{s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, a, \bigparr_{n+1} a^\perp \vdash}\otimes}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, a, a, \bigparr_{n+1} a^\perp \vdash}\textsc{Split}}{s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash}\parr}{s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k} \bigotimes_n a, a^k, a^k, \bigparr_k \bigparr_{n+1} a^\perp \vdash}\otimes
$$

Done!

Even more complicated example: stencil fusion (0)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
\quad
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
\quad
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
}{s : a^{\perp} \parr a^{\perp} \parr a; a, a^{n-1}, a^{n-1}, a, a, a, \parr_{n+1} a^{\perp} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; a, a^{n-1}, a^{n}, a, a, \parr_{n+1} a^{\perp} \vdash}\,\text{Split}_{n-1}
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n}, a^{n}, a, a, \parr_{n+1} a^{\perp} \vdash}\,\text{Split}
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n}, \bigotimes_{n} a, a, a, \parr_{n+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n} a, \bigotimes_{n} a, a, a, \parr_{n+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n} a)^{1+1}, a, a, \parr_{n+1} a^{\perp} \vdash}\,\text{Split}
}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n} a)^{k+k+k+k}, a^{k+k}, a^{k+k}, \parr_{k+k} \parr_{n+1} a^{\perp} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{k+k+k+k} \bigotimes_{n} a, a^{k+k}, a^{k+k}, \parr_{k+k} \parr_{n+1} a^{\perp} \vdash}\,\otimes
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
\quad
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
\quad
\dfrac{\dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{\dfrac{}{a, a^{\perp} \vdash}\,\text{Ax} \quad \dfrac{}{a^{\perp}, a \vdash}\,\text{Ax}}{a, a^{\perp}, a^{\perp} \parr a \vdash}\,\parr}{a, a, a^{\perp}, a^{\perp} \parr a^{\perp} \parr a \vdash}\,\parr}{s : a^{\perp} \parr a^{\perp} \parr a; a, a, a^{\perp} \vdash}\,\text{Load}
}{s : a^{\perp} \parr a^{\perp} \parr a; a, a^{n}, a^{n}, a, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; a, a^{n}, a^{n+1}, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\text{Split}_{n}
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, a^{n+1}, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\text{Split}
}{s : a^{\perp} \parr a^{\perp} \parr a; a^{n+1}, \bigotimes_{n+1} a, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{n+1} a, \bigotimes_{n+1} a, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n+1} a)^{1+1}, a, a, \parr_{n+1+1} a^{\perp} \vdash}\,\text{Split}
}{s : a^{\perp} \parr a^{\perp} \parr a; (\bigotimes_{n+1} a)^{k+k}, a^{k}, a^{k}, \parr_{k} \parr_{n+1+1} a^{\perp} \vdash}\,\parr
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{k+k} \bigotimes_{n+1} a, a^{k}, a^{k}, \parr_{k} \parr_{n+1+1} a^{\perp} \vdash}\,\otimes
}{s : a^{\perp} \parr a^{\perp} \parr a; \bigotimes_{k+k+k+k} \bigotimes_{n} a, a^{k+k}, a^{k+k}, a^{k}, a^{k}, \parr_{k} \parr_{n+1+1} a^{\perp} \vdash}\,\text{Cut}
$$

Even more complicated example: stencil fusion (30)

$$
\begin{prooftree}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp \parr a, a^\perp \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp \parr a^\perp \parr a, a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a, a, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a, a^\perp, a^\perp \parr a^\perp \parr a, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a^\perp, a^\perp \parr a, a \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a^\perp, a, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a^\perp, a, a, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Cut}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a, a, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a, a^\perp, a^\perp \parr a^\perp \parr a, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}\QuaternaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a, a^{n-1}, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp, (a^\perp \parr a^\perp \parr a)^{n-1}, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^n, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp, (a^\perp \parr a^\perp \parr a)^{n-1}, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n+1}, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp, (a^\perp \parr a^\perp \parr a)^{n-1}, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n+1}, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp, (a^\perp \parr a^\perp \parr a)^{n-1} \vdash$}
\RightLabel{Load_{n-1}}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n+1}, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{$\vdots$}
\noLine\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n+1}, a^{n-1}, a^{n-1}, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Cut}\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; a^{n+1}, a, a^{n-1}, a^{n-1}, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a, a^{n-1}, a^{n-1}, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split_{n-1}}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a, a^{n-1}, a^n, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, a^n, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, a^n, \bigotimes_n a, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, \bigotimes_n a, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a, a^\perp \vdash$}
\AxiomC{}\RightLabel{Ax}\UnaryInfC{$a^\perp, a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a^\perp, a^\perp \parr a \vdash$}
\RightLabel{$\parr$}\BinaryInfC{$a, a, a^\perp, a^\perp \parr a^\perp \parr a \vdash$}
\RightLabel{Load}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a, a^\perp \vdash$}
\RightLabel{$\parr$}\TrinaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a^{n-1}, a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{Split_{n-1}}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a, a^{n-1}, a^n, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, a^n, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; a^n, \bigotimes_n a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, a, a, a, \bigparr_{n+1} a^\perp \vdash$}
\AxiomC{$\vdots$}
\noLine\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{n+1} a, \bigotimes_n a, \bigotimes_n a, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Cut}\BinaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, a, a, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, a^{1+1}, a, a, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, \bigotimes_n a, a^{1+1}, a^{1+1}, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_n a, \bigotimes_n a, (\bigotimes_n a)^{1+1}, a^{1+1}, a^{1+1}, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{1+1}, (\bigotimes_n a)^{1+1}, a^{1+1}, a^{1+1}, a, a, \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{$\parr$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k}, (\bigotimes_n a)^{k+k}, a^{k+k}, a^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{Split_{k+k}}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; (\bigotimes_n a)^{k+k+k+k}, a^{k+k}, a^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1+1} a^\perp \vdash$}
\RightLabel{$\otimes$}\UnaryInfC{$s : a^\perp \parr a^\perp \parr a; \bigotimes_{k+k+k+k} \bigotimes_n a, a^{k+k}, a^{k+k}, a^k, a^k, \bigparr_k \bigparr_{n+1+1} a^\perp \vdash$}
\end{prooftree}
$$

Even more complicated example: stencil fusion (60)

Even more complicated example: stencil fusion (90)

Even more complicated example: stencil fusion (120)

Why, oh why didn't I take the blue pill?

Why, oh why didn't I take the blue pill?

- $a^\perp = \mathrm{a} \rightarrow \mathrm{Effect}\ ()$
- $\bigotimes_n a = \mathrm{Int} \rightarrow \mathrm{a}$
- $\bigparr_n a = (\mathrm{Int} \rightarrow a^\perp) \rightarrow \mathrm{Effect}\ ()$

Conclusion

- Reliable Fusion is of Utmost Importance
- Two important questions: Who is in control? How many copies do we need?
- CPS gives answer to to the 1st question
- Linear Logic gives answers to both questions, and is realized with linear types.

$$\frac{}{A, A^{\perp} \vdash}\ \textsc{Ax} \qquad \frac{\Gamma, A^n \vdash \quad A^{\perp}, \Delta \vdash}{\Gamma, \Delta^n \vdash}\ \textsc{Cut}_n \qquad \frac{}{\bot \vdash}\ \bot$$

$$\frac{\Gamma \vdash}{\Gamma, 1 \vdash}\ 1 \qquad \frac{}{\Gamma, 0 \vdash}\ 0 \qquad \frac{\Gamma, A, B \vdash}{\Gamma, A \otimes B \vdash}\ \otimes \qquad \frac{\Gamma, A \vdash \quad B, \Delta \vdash}{\Gamma, A \parr B, \Delta \vdash}\ \parr$$

$$\frac{\Gamma, A, \Delta \vdash \quad \Gamma, B, \Delta \vdash}{\Gamma, A \oplus B, \Delta \vdash}\ \oplus \qquad \frac{\Gamma, A \vdash}{\Gamma, A \& B \vdash}\ \&_1 \qquad \frac{\Gamma, B \vdash}{\Gamma, A \& B \vdash}\ \&_2$$

$$\frac{\Gamma, A[B] \vdash}{\Gamma, \forall\alpha.A[\alpha] \vdash}\ \forall \qquad \frac{\Gamma, A[\beta] \vdash}{\Gamma, \exists\alpha.A[\alpha] \vdash}\ \exists \qquad \frac{\Gamma, A^n, A^m \vdash}{\Gamma, A^{n+m} \vdash}\ \textsc{Split}_n$$

$$\frac{A^{m+n}, \Gamma \vdash}{\Gamma, A^m, A^n \vdash}\ \textsc{Merge}_m \qquad \frac{\Gamma, A^m \vdash}{\Gamma, \bigotimes_m A \vdash}\ \bigotimes$$

$$\frac{\Gamma, A \vdash \quad \Delta, A \vdash}{\Gamma^n, \Delta^m, \parr_{n+m} A \vdash}\ \parr \qquad \frac{\Gamma, {A^{\perp}}^{n} \vdash \quad A^n, \Delta \vdash}{\Gamma, \Delta \vdash}\ \textsc{freeze}$$

$$\frac{B[0]^{\perp}, \Delta \vdash \quad B[k], B[k], B[k+1]^{\perp} \vdash \quad B[n], \Gamma \vdash}{n : \mathsf{I}x; \Gamma, \Delta^{2^n} \vdash}\ \textsc{foldmap}$$