Finite Automata Theory and Formal Languages
TMV027/DIT321 – LP4 2014

Recap Exercises on Logic, Sets, Relations and Functions

1 Logic

1. Let p, q, r be the following propositions:
   p: "it is raining"
   q: "the sun is shining"
   r: "there are clouds in the sky".

   Translate the following into logical notation, using p, q, r and logical connectives.

   (a) It is raining and the sun is shining;
   (b) If it is raining then there are clouds in the sky;
   (c) If it is not raining then the sun is not shining and there are clouds in the sky;
   (d) The sun is shining if and only if it is not raining;
   (e) If there are no clouds in the sky then the sun is shining.

2. Let p, q, r be as in exercise 1). Translate the following into English sentences.

   (a) $(p \wedge q) \Rightarrow r$;
   (b) $(p \Rightarrow r) \Rightarrow q$;
   (c) $\neg p \Leftrightarrow (q \vee r)$;
   (d) $\neg(p \Leftrightarrow (q \vee r))$;
   (e) $\neg(p \vee q) \wedge r$.

3. Give the truth value of the propositions in exercises 1) and 2).

4. Which of the following propositions is logically equivalent to $p \Rightarrow q$:

   $$\neg p \Rightarrow \neg q, \qquad q \Rightarrow p, \qquad \neg q \Rightarrow \neg p, \qquad \neg q \vee p, \qquad \neg p \vee q, \qquad p \wedge \neg q, \qquad q \wedge \neg p.$$

5. Construct the truth tables for:

   (a) $(p \Rightarrow q) \Rightarrow ((p \vee \neg q) \Rightarrow (p \vee q))$;
   (b) $((p \vee q) \wedge r) \Rightarrow (p \wedge \neg q)$;
   (c) $((p \Leftrightarrow q) \vee (p \Rightarrow r)) \Rightarrow (\neg q \wedge p)$.

6. Suppose that $p \Rightarrow q$ is known to be false. Give the truth values for

   $$p \wedge q, \qquad p \vee q, \qquad q \Rightarrow p.$$

7. Write down the negation of the following statement: "for every number x there is a number y such that $y < x$". Find an equivalent formulation without negation.

8. Find an equivalent formulation to $\neg\forall x.(P(x) \Rightarrow Q(x))$ which does not contain a negation at the front nor an implication inside.

9. Consider the statement "everybody loves someone sometime". Let $L(x, y, z)$ be a proposition stating that x loves y at time z. Using this notation, express the original statement using quantifiers.

10. Let $F(x, y)$ be the proposition "you can fool person x at time y". Using this notation, write a quantified statement to formalise Abraham Lincoln's statement: "you can fool all the people some of the time, you can fool some people all the time, but you cannot fool all people all the time".

11. Consider the following universes:

$$(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\} \text{ and } [0, 1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}.$$

Determine whether the statements below are true or false in each of these universes.

(a) $\forall x.\exists y.x > y$;
(b) $\forall x.\exists y.x \geqslant y$;
(c) $\exists x.\forall y.x > y$;
(d) $\exists x.\forall y.x \geqslant y$.

2 Sets

1. Write the following sets in enumerated form:

(a) The set of all vowels;
(b) $\{x \in \mathbb{N} \mid 10 \leqslant x \leqslant 20 \text{ and } x \text{ is divisible by } 3\}$;
(c) The set of all natural numbers that leave a remainder of 1 after division by 5.

2. Write the following sets using a characteristic property:

(a) $\{4, 8, 12, 16, 20\}$;
(b) $\{000, 001, 010, 011, 100, 101, 110, 111\}$;
(c) $\{1, 4, 9, 16, 25, \ldots\}$.

3. Let $A = \{a, b, c\}$ and $B = \{p, q\}$. Write down the following sets in enumeration form:

$$A \times B, \qquad A^2, \qquad B^3.$$

4. Let $A = \{1, \{1\}, \{2\}, 3\}$. Identify which of the following statements are true or false.

   (a) $\emptyset \in A$, $\emptyset \subseteq A$;
   (b) $1 \in A$, $1 \subseteq A$;
   (c) $\{1\} \in A$, $\{1\} \subseteq A$;
   (d) $\{\{1\}\} \subseteq A$;
   (e) $2 \in A$;
   (f) $\{2\} \in A$, $\{2\} \subseteq A$;
   (g) $\{3\} \in A$, $\{3\} \subseteq A$.

5. Let $\{x \in \mathbb{N} \mid x \leqslant 12\}$ be our universe. Let $A = \{x \mid x \text{ is odd}\}$, $B = \{x \mid x > 7\}$ and $C = \{x \mid x \text{ is divisible by } 3\}$. Write down the following sets in enumerated form:

   (a) $A \cap B$;
   (b) $B \cup C$;
   (c) $\overline{A}$;
   (d) $(A \cup \overline{B}) \cap C$;
   (e) $\overline{A \cup C} \cup \overline{C}$.

6. Show that $\overline{\overline{A} \cap B} = A \cup \overline{B}$ using the laws of sets.

7. Show that

   (a) Difference of sets is not commutative, that is, $A - B = B - A$ can fail.
   (b) Difference of sets is not associative, that is, $A - (B - C) = (A - B) - C$ can fail.

8. Prove the following properties on sets A, B, C:

   (a) $A - B = A \cap \overline{B}$;
   (b) $A \subseteq B$ if and only of $A - B = \emptyset$;
   (c) $A - (A - B) = A \cap B$;
   (d) $A \cap B \subseteq (A \cap B) \cup (B \cap \overline{C})$;
   (e) $(A \cup C) \cap (B \cup \overline{C}) \subseteq A \cup B$;
   (f) $A \cap B = \emptyset$ if and only if $A \subseteq \overline{B}$ if and only if $B \subseteq \overline{A}$;
   (g) $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$;
   (h) $(A \cup B) - (A \cup C) \subseteq B - C$;
   (i) $(A \cup B) - C = (A - C) \cup (B - C)$;
   (j) $A - (B - C) = (A - B) \cup (A \cap C)$.

3 Relations

1. Determine which of these relations are reflexive, symmetric, antisymmetric and transitive.
   (a) "is a sibling of", on the set of all people;
   (b) "is the son of", on the set of all people;
   (c) "is greater than", on the set of real numbers;
   (d) "has the same integer part", on the set of real numbers;
   (e) "is a multiple of", on the set of natural numbers;
   (f) The relation R on the set of real numbers defined by $x\,R\,y$ if $x^2 = y^2$.
2. Describe the equivalence classes of those relations in exercise 1) which are equivalence relations.
3. Prove that logical equivalence is an equivalence relation on the set of all propositions formulas in a fixed set of atoms.
4. Let $R \subseteq \mathbb{Z} \to \mathbb{Z}$ such that $x\,R\,y$ if $x - y$ is divisible by 4. Show that R is an equivalence relation and describe its equivalence classes.
5. Prove, by supplying a counterexample, that no two of reflexivity, symmetry, and transitivity imply the third.
6. Define the relation R on the set S of students at a school such that for $x, y \in S, x\,R\,y$ if and only if x and y have a class together. Determine whether R is an equivalence relation.
7. The inclusion relation $\subseteq$ is a relation on the power set $\mathcal{P}(S)$ of a set S. Write the elements of such relation for $S = \{1, 2, 3\}$.
8. Let R_1 and R_2 be relations on a set S.
   (a) Show that $R_1 \cap R_2$ is reflexive if R_1 and R_2 are;
   (b) Show that $R_1 \cap R_2$ is symmetric if R_1 and R_2 are,
   (c) Show that $R_1 \cap R_2$ is transitive if R_1 and R_2 are;
   (d) Must $R_1 \cup R_2$ is reflexive if R_1 and R_2 are?
   (e) Must $R_1 \cup R_2$ is symmetric if R_1 and R_2 are?
   (f) Must $R_1 \cup R_2$ is transitive if R_1 and R_2 are?
9. Three relations are given on the set of all nonempty subsets of $\mathbb{N}$. In each case, determine whether the relation is reflexive, symmetric and transitive. Would these answers change if we would consider the set of *all* subsets of $\mathbb{N}$. Justify.
   (a) $A\,R\,B$ if and only of $A \subseteq B$;
   (b) $A\,R\,B$ if and only of $A \cap B \neq \emptyset$;
   (c) $A\,R\,B$ if and only of $1 \in A \cap B$.

10. For the following relations on $S = \{0, 1, 2, 3\}$, write each relation as a set of ordered pairs. In addition, determine which of these relations are reflexive, symmetric, antisymmetric and transitive.

   (a) $m \, R_1 \, n$ if $m + n = 3$;
   (b) $m \, R_2 \, n$ if $m \leqslant n$;
   (c) $m \, R_3 \, n$ if $\mathsf{max}\{m, n\} = 3$;
   (d) $m \, R_4 \, n$ if $m - n$ is even;
   (e) $m \, R_5 \, n$ if $m + n \leqslant 4$.

4 Functions

1. Let $f : A \to B$.

   (a) Show that the relation R defined as $x \, R \, y$ if $f(x) = f(y)$ is an equivalence relation;
   (b) Describe the equivalence classes when both A and B are the set of real numbers and $f(x) = x^2$;
   (c) Suppose A has n elements and B has m elements.
      i. If f is an injective function (and not necessarily surjective), how many equivalence classes are there?
      ii. If f is an surjective function (and not necessarily injective), how many equivalence classes are there?

2. Determine which of the following functions are injective and surjective.

   (a) $f : S \to S$ for S a finite set of nonempty strings and $f(s)$ returns the reverse of the string s;
   (b) $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(x, y) = x + y$;
   (c) $s : \mathbb{N} \to \mathbb{N}$ such that $s(n) = n + 1$;
   (d) $h : \{\text{English words}\} \to \{\text{letters}\}$ such that $h(w)$ returns the first letter of the word w;
   (e) $|\_| : \mathcal{P}(A) \to \mathbb{N}$ such that $|X|$ is the cardinality of the set $X \subseteq A$, for A any given finite set.

3. Let $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ be defined as $f(n) = 3n \ \mathsf{mod}\ 5$. Determine all the pairs that are related by f. State whether f is injective and surjective.

4. Consider the following functions from $\mathbb{N}$ to $\mathbb{N}$:

$$id(n) = n, \qquad f(n) = 3n, \qquad g(n) = n + (-1)^n,$$
$$h(n) = \mathsf{min}\{n, 100\}, \qquad k(n) = \mathsf{max}\{0, n - 5\}.$$

   (a) Which of these functions are injective?
   (b) Which of these functions are subjective?

5. Here are two "shift" functions mapping $\mathbb{N}$ to $\mathbb{N}$:

$$f(n) = n + 1 \qquad\qquad g(n) = \mathsf{max}\{0, n - 1\}$$

   (a) Calculate $f(n)$ for $n = 0, 1, 2, 3, 4, 73$;
   (b) Calculate $g(n)$ for $n = 0, 1, 2, 3, 4, 73$;
   (c) Show that f is injective but not surjective;
   (d) Show that g is surjective but not injective.
   (e) Show that $g \circ f(n) = n$ for all n, but $f \circ g(n) = n$ does not hold for all n.

6. Find the inverse of each of the following functions, or explain why no inverse exists.
   (a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x + 2$;
   (b) $|\_| : \mathbb{R} \to \mathbb{R}$, the absolute value of a real numbers;
   (c) $g : \mathbb{N} \to \mathbb{N}$ where $g(x) = \begin{cases} n + 1 \text{ if } n \text{ is odd} \\ n - 1 \text{ if } n \text{ is even} \end{cases}$;
   (d) $h : S \to S$ for S a finite set of nonempty strings and h the function that moves the last character of the string to the beginning, for example $h(\texttt{abcd}) = \texttt{dabc}$;
   (e) $k : \mathbb{R} \to \mathbb{R}$ such that $k(x) = x^3 - 2$.

7. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$. Show that f is a bijection and find a formula for f^{-1}.

8. Let A, B, C, D be sets, and let $f : A \to B$, $g :\to C$ and $h :\to D$. Prove that composition of functions is associative.

9. Prove that a function cannot have more than one inverse.
   Hint: Assume inverses are not unique and try to deduce a contradiction using exercise 8).

10. Let $f : S \to T$ and $g : T \to U$ be invertible functions, that is, have inverse functions.
   (a) Show that f^{-1} is invertible and that $(f^{-1})^{-1} = f$;
   (b) Show that $g \circ f$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

11. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be as follows:

$$f(x) = 4x - 3, \qquad g(x) = x^2 + 1, \qquad h(x) = \begin{cases} 1 \text{ if } x \geqslant 0 \\ 0 \text{ if } x < 0 \end{cases}.$$

   Find an expression defining the following functions:
   (a) $f \circ f$, $g \circ f$, $h \circ f$;
   (b) $f \circ g$, $h \circ g$;
   (c) $f \circ h$, $g \circ h$.