Software Engineering using Formal Methods

Reasoning about Programs with Dynamic Logic

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Part I

Where are we?

Where Are We?

before	specification of Java programs with JML
now	dynamic logic (DL) for resoning about Java programs
after that	generating DL from JML+Java
	+ verifying the resulting proof obligations

Motivation

Consider the method

```
public void doubleContent(int[] a) {
  int i = 0;
  while (i < a.length) {
    a[i] = a[i] * 2;
    i++;
  }
}
```

We want a logic/calculus allowing to express/prove properties like, e.g.:

If `a` $\neq$ `null`
then `doubleContent` terminates normally
and afterwards all elements of `a` are twice the old value

Motivation (contd.)

One such logic is dynamic logic (DL).
The above statemet in DL would be:

$$
\begin{array}{l}
\quad \mathtt{a} \neq \mathtt{null} \\
\wedge\ \mathtt{a} \neq \mathtt{b} \\
\wedge\ \forall \mathtt{int\ i}; ((0 \leq \mathtt{i} \wedge \mathtt{i} < \mathtt{a.length}) \rightarrow \mathtt{a[i]} = \mathtt{b[i]}) \\
\rightarrow \langle \mathtt{doubleContent(a);} \rangle \\
\quad \forall \mathtt{int\ i}; ((0 \leq \mathtt{i} \wedge \mathtt{i} < \mathtt{a.length}) \rightarrow \mathtt{a[i]} = 2 * \mathtt{b[i]})
\end{array}
$$

- DL combines first-order logic (FOL) with programs
- Theory of DL extends theory of FOL
- Necessary to look closer at FOL at first
- Then extend towards DL

Today

introducing dynamic logic for JAVA

- recap first-order logic (FOL)
- semantics of FOL
- dynamic logic = extending FOL with
  - dynamic interpretations
  - programs to describe state change

Repetition: First-Order Logic

Signature

A first-order signature Σ consists of

- a set T_Σ of types
- a set F_Σ of function symbols
- a set P_Σ of predicate symbols

Type Declarations

- $\tau\ x$; 'variable x has type τ'
- $p(\tau_1, \ldots, \tau_r)$; 'predicate p has argument types $\tau_1, \ldots, \tau_r$'
- $\tau\ f(\tau_1, \ldots, \tau_r)$; 'function f has argument types $\tau_1, \ldots, \tau_r$ and result type τ'

Part II

First-Order Semantics

First-Order Semantics

From propositional to first-order semantics

- In prop. logic, an interpretation of variables with $\{T, F\}$ sufficed
- In first-order logic we must assign meaning to:
  - function symbols (incl. constants)
  - predicate symbols
- Respect typing: `int i`, `List l` must denote different elements

What we need (to interpret a first-order formula)

1. A collection of typed universes of elements
2. A mapping from variables to elements
3. For each function symbol, a mapping from arguments to results
4. For each predicate symbol, a set of argument tuples where that predicate holds

First-Order Domains/Universes

1. A collection of typed universes of elements

Definition (Universe/Domain)

A non-empty set $\mathcal{D}$ of elements is a universe or domain.
Each element of $\mathcal{D}$ has a fixed type given by $\delta : \mathcal{D} \to T_\Sigma$

- Notation for the domain elements of type $\tau \in T_\Sigma$:
  $\mathcal{D}^\tau = \{d \in \mathcal{D} \mid \delta(d) = \tau\}$
- Each type $\tau \in T_\Sigma$ must 'contain' at least one domain element:
  $\mathcal{D}^\tau \neq \emptyset$

First-Order States

3. For each function symbol, a mapping from arguments to results

4. For each predicate symbol, a set of argument tuples where that predicate holds

Definition (First-Order State)

Let $\mathcal{D}$ be a domain with typing function δ.

For each f be declared as $\tau\ f(\tau_1, \ldots, \tau_r)$;

and each p be declared as $p(\tau_1, \ldots, \tau_r)$;

$\mathcal{I}(f)$ is a mapping $\mathcal{I}(f) : \mathcal{D}^{\tau_1} \times \cdots \times \mathcal{D}^{\tau_r} \rightarrow \mathcal{D}^{\tau}$

$\mathcal{I}(p)$ is a set $\mathcal{I}(p) \subseteq \mathcal{D}^{\tau_1} \times \cdots \times \mathcal{D}^{\tau_r}$

Then $\mathcal{S} = (\mathcal{D}, \delta, \mathcal{I})$ is a first-order state

First-Order States Cont'd

Example

Signature: `int i; int j; int f(int); Object obj; <(int,int);`
$\mathcal{D} = \{17, 2, o\}$
The following $\mathcal{I}$ is a possible interpretation:

$$\mathcal{I}(i) = 17$$
$$\mathcal{I}(j) = 17$$
$$\mathcal{I}(\texttt{obj}) = o$$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\mathbf{int}} \times \mathcal{D}^{\mathbf{int}}$	in $\mathcal{I}(<)$?
$(2, 2)$	*no*
$(2, 17)$	*yes*
$(17, 2)$	*no*
$(17, 17)$	*no*

One of uncountably many possible first-order states!

Semantics of Reserved Signature Symbols

Definition

Reserved predicate symbol for equality: $=$

Interpretation is fixed as $\mathcal{I}(=) = \{(d, d) \mid d \in \mathcal{D}\}$

Exercise: write down all elements of the set $\mathcal{I}(=)$ for example domain

Signature Symbols vs. Domain Elements

- Domain elements different from the terms representing them
- First-order formulas and terms have no access to domain

Example

Signature: `Object obj1, obj2;`
Domain: $\mathcal{D} = \{o\}$

In this state, necessarily $\mathcal{I}(\texttt{obj1}) = \mathcal{I}(\texttt{obj2}) = o$

Variable Assignments

2. A mapping from variables to domain elements

Definition (Variable Assignment)

A variable assignment β maps variables to domain elements
It respects the variable type, i.e., if x has type τ then $\beta(x) \in \mathcal{D}^\tau$

Semantic Evaluation of Terms

Given a first-order state $\mathcal{S}$ and a variable assignment β
it is possible to evaluate first-order terms under $\mathcal{S}$ and β

Definition (Valuation of Terms)

$val_{\mathcal{S},\beta}$: Term $\rightarrow \mathcal{D}$ such that $val_{\mathcal{S},\beta}(t) \in \mathcal{D}^{\tau}$ for $t \in \text{Term}_{\tau}$:

- $val_{\mathcal{S},\beta}(x) = \beta(x)$
- $val_{\mathcal{S},\beta}(f(t_1, \ldots, t_r)) = \mathcal{I}(f)(val_{\mathcal{S},\beta}(t_1), \ldots, val_{\mathcal{S},\beta}(t_r))$

Semantic Evaluation of Terms Cont'd

Example

Signature: **int** `i`; **int** `j`; **int** `f(`**int**`)`;

$\mathcal{D} = \{17, 2, o\}$ Variables: `Object obj`; **int** `x`;

$\mathcal{I}(\mathtt{i}) = 17$
$\mathcal{I}(\mathtt{j}) = 17$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(\mathtt{f})$
2	17
17	2

Var	β
obj	o
x	17

- $val_{\mathcal{S},\beta}(\mathtt{f(f(i))})$?
- $val_{\mathcal{S},\beta}(\mathtt{f(f(x))})$?
- $val_{\mathcal{S},\beta}(\mathtt{obj})$?

Preparing for Semantic Evaluation of Formulas

Definition (Modified Variable Assignment)

Let y be variable of type τ, β variable assignment, $d \in \mathcal{D}^\tau$:

$$\beta_y^d(x) := \begin{cases} \beta(x) & x \neq y \\ d & x = y \end{cases}$$

Semantic Evaluation of Formulas

Definition (Valuation of Formulas)

$val_{\mathcal{S},\beta}(\phi)$ for $\phi \in For$

- $val_{\mathcal{S},\beta}(p(t_1, \ldots, t_r)) = T$ iff $(val_{\mathcal{S},\beta}(t_1), \ldots, val_{\mathcal{S},\beta}(t_r)) \in \mathcal{I}(p)$
- $val_{\mathcal{S},\beta}(\phi \land \psi) = T$ iff $val_{\mathcal{S},\beta}(\phi) = T$ and $val_{\mathcal{S},\beta}(\psi) = T$
- ...as in propositional logic
- $val_{\mathcal{S},\beta}(\forall\, \tau\ x;\ \phi) = T$ iff $val_{\mathcal{S},\beta_x^d}(\phi) = T$ for all $d \in \mathcal{D}^\tau$
- $val_{\mathcal{S},\beta}(\exists\, \tau\ x;\ \phi) = T$ iff $val_{\mathcal{S},\beta_x^d}(\phi) = T$ for at least one $d \in \mathcal{D}^\tau$

Semantic Evaluation of Formulas Cont'd

Example

Signature: `int j; int f(int); Object obj; <(int,int);`
$\mathcal{D} = \{17,\ 2,\ o\}$, $\mathcal{D}^{\mathbf{int}} = \{17,\ 2\}$, $\mathcal{D}^{\mathtt{Object}} = \{o\}$

$\mathcal{I}(j) = 17$
$\mathcal{I}(\mathtt{obj}) = o$

$\mathcal{D}^{\mathbf{int}}$	$\mathcal{I}(f)$
2	2
17	2

$\mathcal{D}^{\mathbf{int}} \times \mathcal{D}^{\mathbf{int}}$	in $\mathcal{I}(<)$?
$(2, 2)$	F
$(2, 17)$	T
$(17, 2)$	F
$(17, 17)$	F

- $val_{\mathcal{S},\beta}(f(j) < j)$?
- $val_{\mathcal{S},\beta}(\exists\, \mathbf{int}\ x;\ f(x) = x)$?
- $val_{\mathcal{S},\beta}(\forall\, \mathtt{Object}\ o1;\ \forall\, \mathtt{Object}\ o2;\ o1 = o2)$?

Semantic Notions

Definition (Satisfiability, Truth, Validity)

$val_{\mathcal{S},\beta}(\phi) = T$			($\mathcal{S}, \beta$ **satisfies** ϕ)
$\mathcal{S} \models \phi$	iff	for all β : $val_{\mathcal{S},\beta}(\phi) = T$	(ϕ is **true** in $\mathcal{S}$)
$\models \phi$	iff	for all $\mathcal{S}$: $\mathcal{S} \models \phi$	(ϕ is **valid**)

Example

- $f(j) < j$ is true in $\mathcal{S}$
- $\exists\,\mathbf{int}\ x;\ i = x$ is valid
- $\exists\,\mathbf{int}\ x;\ \neg(x = x)$ is not satisfiable

Part III

Towards Dynamic Logic

Type Hierarchy

First, we *refine the type system* of FOL:

Definition (Type Hierarchy)

- T_Σ is set of types
- Given subtype relation '$\sqsubseteq$', with top element '*any*'
- $\tau \sqsubseteq any$ for all $\tau \in T_\Sigma$

Example (A Minimal Type Hierarchy)

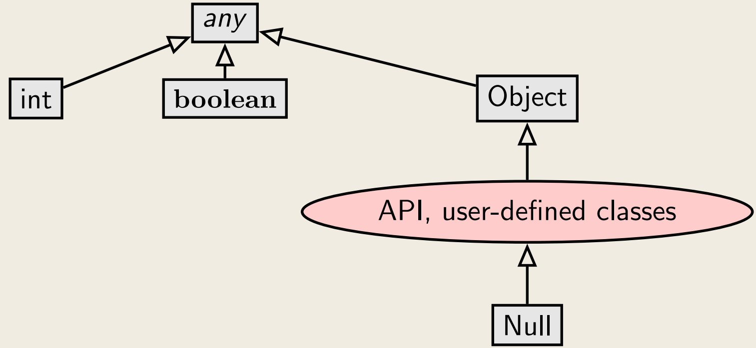
$\mathcal{T} = \{any\}$
All signature symbols have same type *any*.

Example (Type Hierarchy for Java)

(see next slide)

Modelling Java in FOL: Fixing a Type Hierarchy

Signature based on Java's type hierarchy (simplified)



Each class in API and target program is a type, with appropriate subtyping.

Modelling Classes and Fields in FOL

Modeling instance fields

Person
int age int id
int setAge(int newAge) int getId()

- domain of all Person objects: $\mathcal{D}^{\mathsf{Person}}$
- each $o \in \mathcal{D}^{\mathsf{Person}}$ has associated age value
- $\mathcal{I}(\mathsf{age})$ is mapping from $\mathcal{D}^{\mathsf{Person}}$ to $\mathcal{D}^{\mathbf{int}}$
- for each class C with field τ a:
  FSym declares function τ a(C);

Field Access

Signature FSym: **int** age(Person); Person p;

Java/JML expression `p.age >= 0`

Typed FOL age(p)>=0

KeY postfix notation for FOL `p.age >= 0`

Navigation expressions in KeY look exactly as in JAVA/JML

Dynamic View

Only static properties expressable in typed FOL, e.g.,

- Values of fields in a certain range
- Property (invariant) of a subclass implies property of a superclass
- ...

Considers only one state at a time.

Goal: Express functional properties of a program, e.g.

> If method `setAge` is called on an object *o* of type `Person`
> and the method argument `newAge` is positive
> then afterwards field `age` has same value as `newAge`.

Observation

Need a logic that allows us to

- relate different program states, i.e., before and after execution, within one formula
- program variables/fields represented by constant/function symbols that depend on program state

Dynamic Logic meets the above requirements.

Dynamic Logic

(JAVA) Dynamic Logic

Typed FOL

- $+$ programs $\mathtt{p}$
- $+$ modalities $\langle \mathtt{p} \rangle \phi$, $[\mathtt{p}]\phi$ ($\mathtt{p}$ program, ϕ DL formula)
- $+$ ... (later)

An Example

$$\mathtt{i} > 5 \ \rightarrow \ [\mathtt{i = i + 10;}]\mathtt{i} > 15$$

Meaning?

If program variable `i` is greater than 5, then after executing `i = i + 10;`, `i` is greater than 15.

Program Variables

Dynamic Logic = Typed FOL + ...

$$\mathtt{i} > 5 \rightarrow [\mathtt{i\ =\ i\ +\ 10;}]\mathtt{i} > 15$$

Program variable `i` refers to different values before and after execution of a program.

- Program variables like `i` are state-dependent constant symbols.
- Value of state dependent symbols changeable by program.

Three words one meaning: flexible, state-dependent, non-rigid

Rigid versus Flexible Symbols

Signature of dynamic logic defined as in FOL, but:
In addition there are flexible symbols

Rigid versus Flexible

- Rigid symbols, same interpretation in all program states
  - First-order variables (aka logical variables)
  - Built-in functions and predicates such as `0,1,...,+,*,...,<,...`
- Flexible (or non-rigid) symbols, interpretation depends on state

  Capture side effects on state during program execution
  - Functions modeling program variables and fields are flexible

Any term containing at least one flexible symbol is also flexible

Signature of Dynamic Logic

Definition (Dynamic Logic Signature)

$\Sigma = (\mathsf{PSym}_r, \mathsf{FSym}_r, \mathsf{FSym}_f, \alpha), \quad \mathsf{FSym}_r \cap \mathsf{FSym}_f = \emptyset$

Rigid Predicate Symbols	$\mathsf{PSym}_r = \{>, >=, \ldots\}$
Rigid Function Symbols	$\mathsf{FSym}_r = \{+, -, *, 0, 1, \ldots\}$
Flexible Function Symbols	$\mathsf{FSym}_f = \{i, j, k, \ldots\}$

Standard typing: **boolean** `TRUE`; `<(`**int**`,`**int**`)`; etc.

Flexible constant/function symbols FSym_f used to model

- program variables (flexible constants) and
- fields (flexible unary functions)

Dynamic Logic Signature - KeY input file

```
\sorts {
  // only additional sorts (predefined:  int/boolean/any)
}
\functions {
 // only additional rigid functions
 // (arithmetic functions like +,- etc.  predefined)
}
\predicates {  /* same as for functions */  }

\programVariables { // flexible functions
   int i, j;
   boolean b;
}
```

Empty sections can be left out.

Variables

Logical Variables

Typed logical variables (rigid), declared locally in quantifiers as `T x;`

Program Variables

Flexible constants **int** `i;` **boolean** `p;` used as program variables

Dynamic Logic Programs

Dynamic Logic = Typed FOL + programs . . .

Programs here: any legal sequence of JAVA statements.

Example

Signature for $FSym_f$: `int r; int i; int n;`

Signature for $FSym_r$: `int 0; int +(int,int); int -(int,int);`

Signature for $PSym_r$: `<(int,int);`

```
i=0;
r=0;
while (i<n) {
  i=i+1;
  r=r+i;
}
r=r+r-n;
```

Which value does the program compute in `r`?

Relating Program States: Modalities

DL extends FOL with two additional (mix-fix) operators:

- $\langle \mathrm{p} \rangle \phi$ (diamond)
- $[p]\phi$ (box)

with p a program, ϕ another DL formula

Intuitive Meaning

- $\langle \mathrm{p} \rangle \phi$: p terminates and formula ϕ holds in final state (total correctness)
- $[p]\phi$: If p terminates then formula ϕ holds in final state (partial correctness)

Attention: JAVA programs are deterministic, i.e., if a JAVA program terminates then exactly one state is reached from a given initial state.

Dynamic Logic - Examples

Let `i`, `j`, `old_i`, `old_j` denote program variables.
Give the meaning in natural language:

1. $\texttt{i} = \texttt{old\_i} \rightarrow \langle \texttt{i = i + 1;} \rangle \texttt{i} > \texttt{old\_i}$

   If `i = i + 1;` is executed in a state where `i` and `old_i` have the same value, then the program terminates and in its final state the value of `i` is greater than the value of `old_i` .
2. $\texttt{i} = \texttt{old\_i} \rightarrow [\texttt{while(true)}\{\texttt{i = old\_i - 1;}\}]\texttt{i} > \texttt{old\_i}$

   If the program is executed in a state where `i` and `old_i` have the same value and if the program terminates then in its final state the value of `i` is greater than the value of `old_i`.
3. $\forall\, x.\ (\ \langle \texttt{p} \rangle\ \texttt{i} = x\ \leftrightarrow\ \langle \texttt{q} \rangle\ \texttt{i} = x\ )$

   `p` and `q` are equivalent concerning termination and the final value of `i`.

Dynamic Logic - KeY input file

KeY

```
\programVariables {  // Declares global program variables
       int i, j;
       int old_i, old_j;
}


\problem {  // The problem to verify is stated here
       i = old_i -> \<{   i = i + 1;  }\> i > old_i
}
```

KeY

Visibility: Program variables declared

- global can be accessed anywhere in the formula.
- inside modality like *pre* $\rightarrow$ ⟨`int j; p`⟩*post* only visible in `p`

Dynamic Logic Formulas

Definition (Dynamic Logic Formulas (DL Formulas))

- Each FOL formula is a DL formula
- If p is a program and ϕ a DL formula then $\left\{ \begin{array}{l} \langle \mathrm{p} \rangle \phi \\ [\mathrm{p}]\phi \end{array} \right\}$ is a DL formula
- DL formulas closed under FOL quantifiers and connectives

- Program variables are flexible constants: never bound in quantifiers
- Program variables need not be declared or initialized in program
- Programs contain no logical variables
- Modalities can be arbitrarily nested

Dynamic Logic Formulas Cont'd

Example (Well-formed? If yes, under which signature?)

- $\forall\, \mathbf{int}\ y; ((\langle \texttt{x = 2;} \rangle \texttt{x} = y) \leftrightarrow (\langle \texttt{x = 1; x++;} \rangle \texttt{x} = y))$

  Well-formed if FSym_f contains **int** `x;`
- $\exists\, \mathbf{int}\ x; [x = 1;](x = 1)$

  Not well-formed, because logical variable occurs in program
- $\langle \texttt{x = 1;} \rangle ([\mathbf{while}\ (\mathbf{true})\ \{\}]\mathbf{false})$

  Well-formed if FSym_f contains **int** `x;`
  program formulas can be nested

Dynamic Logic Semantics: States

First-order state can be considered as program state

- Interpretation of flexible symbols can vary from state to state (eg, program variables, field values)
- Interpretation of rigid symbols is the same in all states (eg, built-in functions and predicates)

Program states as first-order states

From now, consider program state s as first-order state $(\mathcal{D}, \delta, \mathcal{I})$

- Only interpretation $\mathcal{I}$ of flexible symbols in FSym_f can change
- *States* is set of all states s

Kripke Structure

Definition (Kripke Structure)

Kripke structure or Labelled transition system $K = (States, \rho)$

- State (=first-order model) $s = (\mathcal{D}, \delta, \mathcal{I}) \in States$
- Transition relation $\rho : Program \rightarrow (States \rightharpoonup States)$

$$\rho(\mathrm{p})(s1) = s2$$

iff.

program p executed in state $s1$ terminates and its final state is $s2$, otherwise undefined.

- ρ is the semantics of programs $\in Program$
- $\rho(\mathrm{p})(s)$ can be undefined ('$\rightharpoonup$'):
  p may not terminate when started in s
- Our programs are deterministic (unlike PROMELA):
  $\rho(\mathrm{p})$ is a function (at most one value)

Semantic Evaluation of Program Formulas

Definition (Validity Relation for Program Formulas)

- $s \models \langle \mathrm{p} \rangle \phi$ iff $\rho(\mathrm{p})(s)$ is defined and $\rho(\mathrm{p})(s) \models \phi$

  (p terminates and ϕ is true in the final state after execution)
- $s \models [\mathrm{p}]\phi$ iff $\rho(\mathrm{p})(s) \models \phi$ whenever $\rho(\mathrm{p})(s)$ is defined

  (If p terminates then ϕ is true in the final state after execution)

- Duality: $\langle \mathrm{p} \rangle \phi$ iff $\neg[\mathrm{p}]\neg\phi$
  Exercise: justify this with help of semantic definitions
- Implication: if $\langle \mathrm{p} \rangle \phi$ then $[\mathrm{p}]\phi$
  Total correctness implies partial correctness
  - converse is false
  - holds only for deterministic programs

More Examples

valid?
meaning?

Example

$\forall\, \tau\ y; ((\langle \mathrm{p} \rangle \mathrm{x} = y) \leftrightarrow (\langle \mathrm{q} \rangle \mathrm{x} = y))$

Not valid in general

Programs p behave q equivalently on variable τ x

Example

$\exists\, \tau\ y; (\mathrm{x} = y \rightarrow \langle \mathrm{p} \rangle \mathbf{true})$

Not valid in general

Program p terminates if initial value of x is suitably chosen

Semantics of Programs

In labelled transition system $K = (States, \rho)$:
$\rho : Program \to (States \rightharpoonup States)$ is semantics of programs $\mathtt{p} \in Program$

ρ defined recursively on programs

Example (Semantics of assignment)

States s interpret flexible symbols f with $\mathcal{I}_s(f)$

$\rho(\mathtt{x=t;})(s) = s'$ where s' identical to s except $\mathcal{I}_{s'}(x) = val_s(t)$

Very tedious task to define ρ for JAVA. $\Rightarrow$ Not in this course.
Next lecture, we go directly to calculus for program formulas!

Literature for this Lecture

KeY Book Verification of Object-Oriented Software (see course web page), Chapter 10: Using KeY

KeY Book Verification of Object-Oriented Software (see course web page), Chapter 3: Dynamic Logic (Sections 3.1, 3.2, 3.4, 3.5, 3.6.1, 3.6.3, 3.6.4)