Simply Typed Lambda-Calculus

Types, Programs and Computations

Types

$$A, B, T \ ::= \ \mathsf{Bool} \mid T \to T$$

Contexts

$$\Gamma, \Delta \ ::= \ () \mid \Gamma.A$$

Terms

$$t \ ::= \ \lambda\, T\; t \mid t\; t \mid i \mid bv \qquad\qquad i \ ::= \ 0 \mid i+1$$

Typed Environments

$$\rho \ ::= \ () \mid (\rho, v : T)$$

Values

$$v \ ::= \ bv \mid (\lambda\, T\; t)\rho \qquad\qquad bv \ ::= \ \mathsf{true} \mid \mathsf{false}$$

Computations

$$u \ ::= \ bv \mid t\rho \mid u\; u$$

Operational semantics

$$\frac{u \to u'}{u\; u_1 \to u'\; u_1} \qquad \frac{u_1 \to u_1'}{v\; u_1 \to v\; u_1'} \qquad \frac{}{(\lambda\, T\; t)\rho\; v \to t(\rho, v : T)}$$

$$\frac{}{\mathsf{true}\; \rho \to \mathsf{true}} \qquad \frac{}{\mathsf{false}\; \rho \to \mathsf{false}}$$

$$\frac{}{(t\; t_1)\rho \to t\rho\; (t_1\rho)}$$

$$\frac{}{0(\rho, v : T) \to v} \qquad \frac{i\rho \to v}{(i+1)(\rho, v' : T) \to v}$$

Typing

Typing of terms

$$\frac{\Gamma \vdash t : A_1 \to A \qquad \Gamma \vdash t_1 : A_1}{\Gamma \vdash t\; t_1 : A} \qquad \frac{\Gamma.A \vdash t : B}{\Gamma \vdash \lambda\, A\; t : A \to B}$$

$$\frac{}{\Gamma.A \vdash 0 : A} \qquad \frac{\Gamma \vdash i : B}{\Gamma.A \vdash i+1 : B}$$

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{Bool}} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{Bool}}$$

Typing of computations

$$\frac{u : A_1 \to A \qquad u_1 : A_1}{u\; u_1 : A} \qquad \frac{\rho : \Gamma \qquad \Gamma \vdash t : T}{t\rho : T}$$

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{Bool}} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{Bool}}$$

Typing of environments

$$\frac{}{() : ()} \qquad \frac{\rho : \Gamma \qquad v : A}{(\rho, v : A) : \Gamma.A}$$

Main Properties

Lemma 0.1 *If* v : Bool *then* $v = \mathsf{true}$ *or* $v = \mathsf{false}$*. If* $v : A \to B$ *then there exists* Γ, ρ, t *such that* $\Gamma.A \vdash t : B$ *and* $\rho : \Gamma$ *and* $v = (\lambda\ A\ t)\rho$*.*

Theorem 0.2 *(***Progress***) If* $u : T$ *then either* u *is a value or* $(\exists u')\ u \to u'$

Theorem 0.3 *(***Preservation***) If* $u : T$ *and* $u \to u'$ *then* $u' : T$

Theorem 0.4 *(***Normalization***) If* $u : T$ *then there exists a value* v *such that* $u \to^* v$*. Furthermore, by Preservation, we have* $v : T$*.*