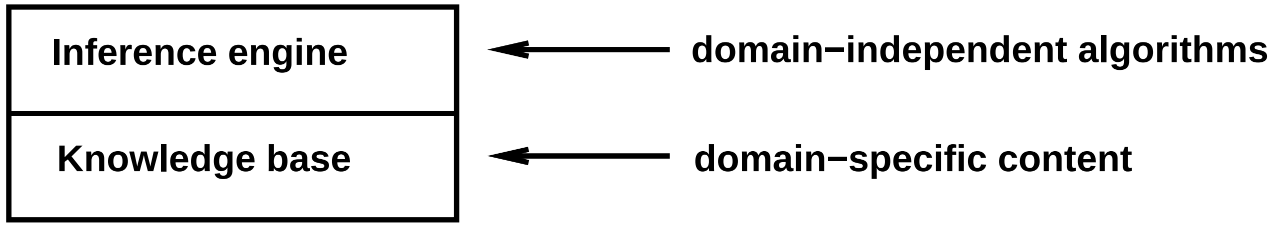
# Logical agents

## Chapter 7, Sections 1–5

# Outline

- ♢ Knowledge-based agents
- ♢ Example: The wumpus world
- ♢ Logic in general—models and entailment
- ♢ Propositional (Boolean) logic
- ♢ Equivalence, validity, satisfiability
- ♢ Inference rules and theorem proving
  - – forward chaining
  - – backward chaining
  - – resolution

# Knowledge bases




Knowledge base = set of sentences in a **formal** language

Declarative approach to building an agent (or other system):

TELL it what it needs to know

Then it can ASK itself what to do—answers should follow from the KB

Agents can be viewed at the knowledge level

i.e., **what they know**, regardless of how implemented

Or at the implementation level

i.e., data structures in KB and algorithms that manipulate them

# Wumpus world: PEAS description

## Performance measure

gold $+1000$, death $-1000$

$-1$ per step, $-10$ for using the arrow

## Environment

Squares adjacent to wumpus are smelly

Squares adjacent to pit are breezy

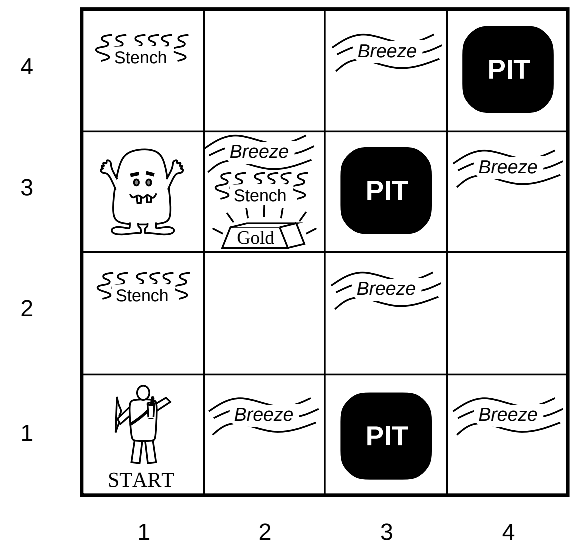
Glitter iff gold is in the same square

Shooting kills wumpus if you are facing it

Shooting uses up the only arrow

Grabbing picks up gold if in same square

Releasing drops the gold in same square




## Actuators

Left turn, Right turn, Forward, Grab, Release, Shoot

## Sensors

Breeze, Glitter, Smell

# Wumpus world characterization

Observable?? No—only local perception

Deterministic?? Yes—outcomes exactly specified

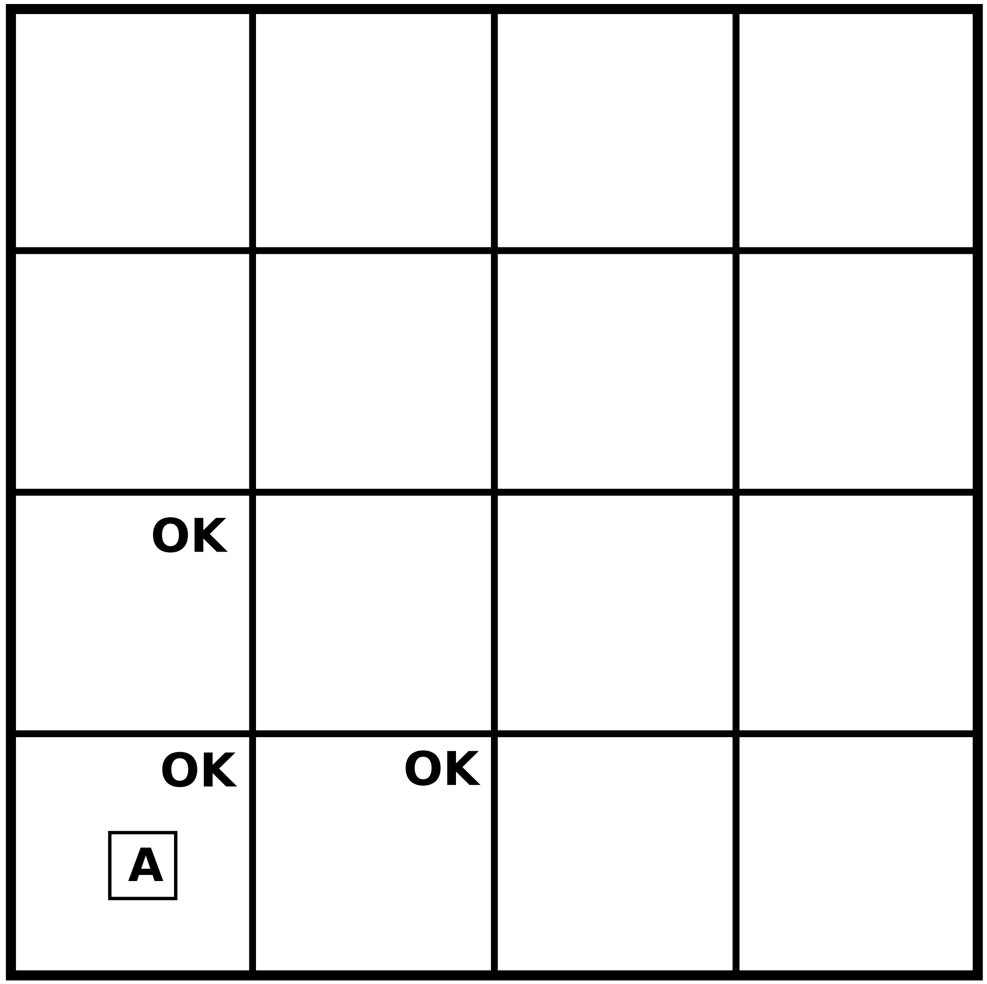
Episodic?? No—sequential at the level of actions

Static?? Yes—Wumpus and Pits do not move

Discrete?? Yes

Single-agent?? Yes—Wumpus is essentially a natural feature

# Exploring a wumpus world



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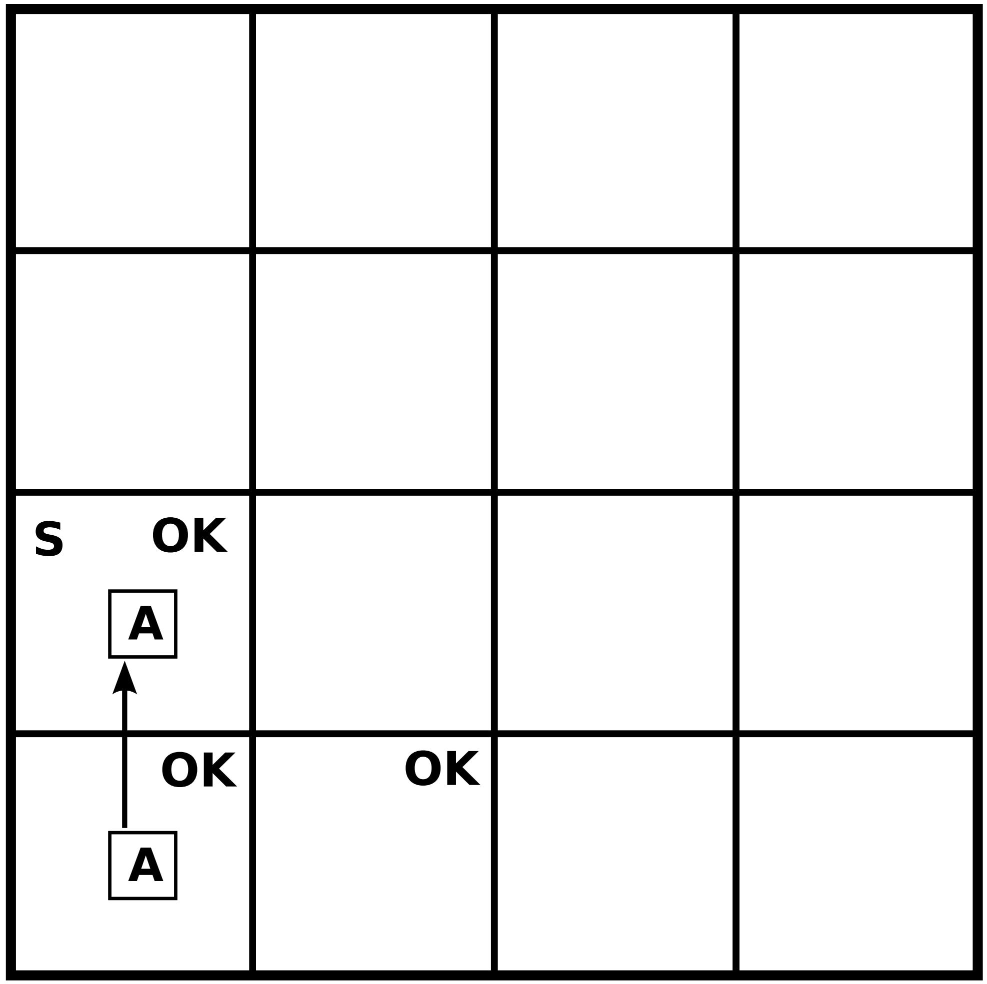

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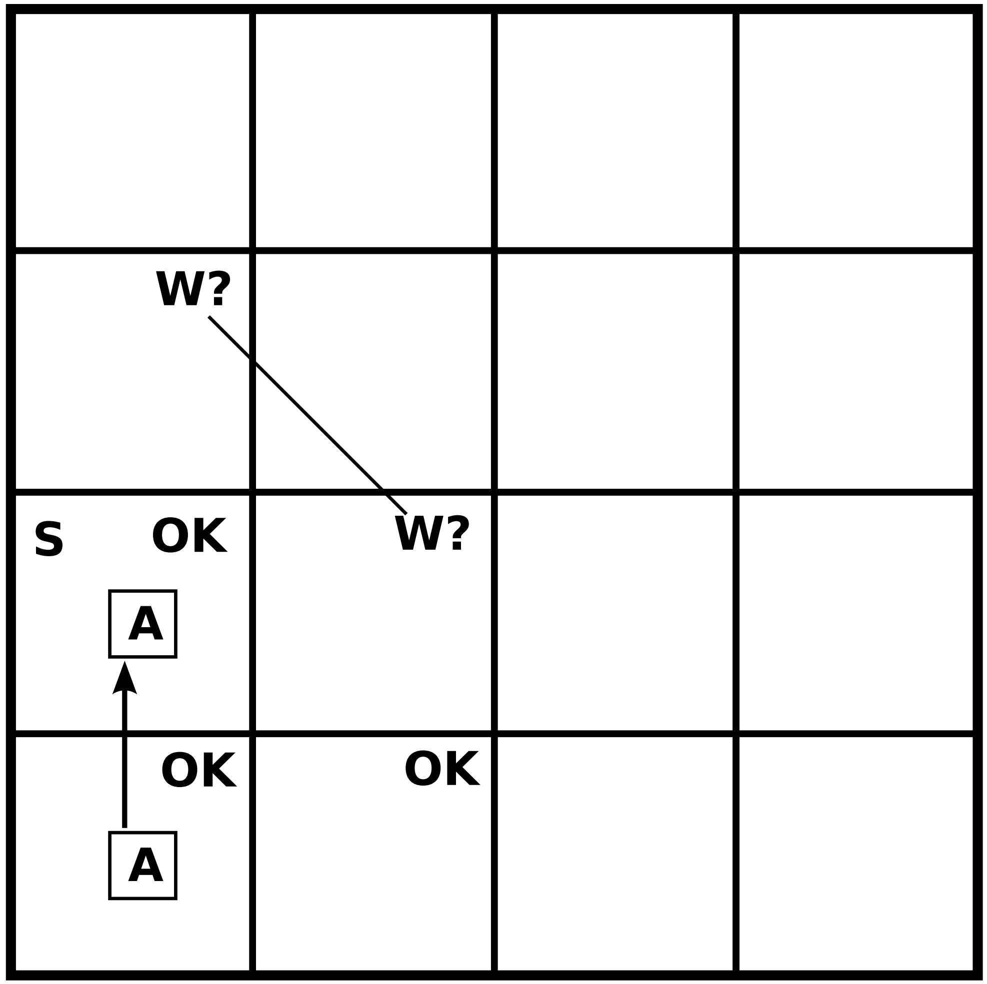

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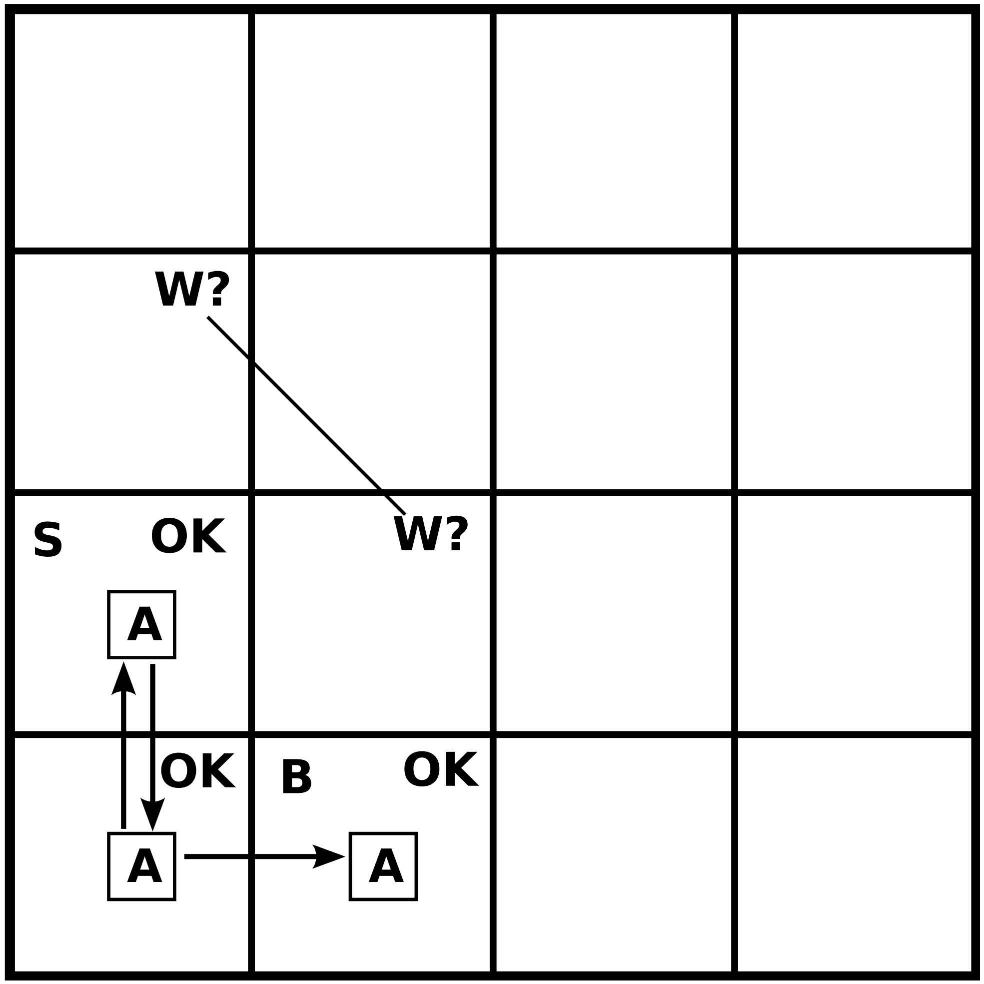

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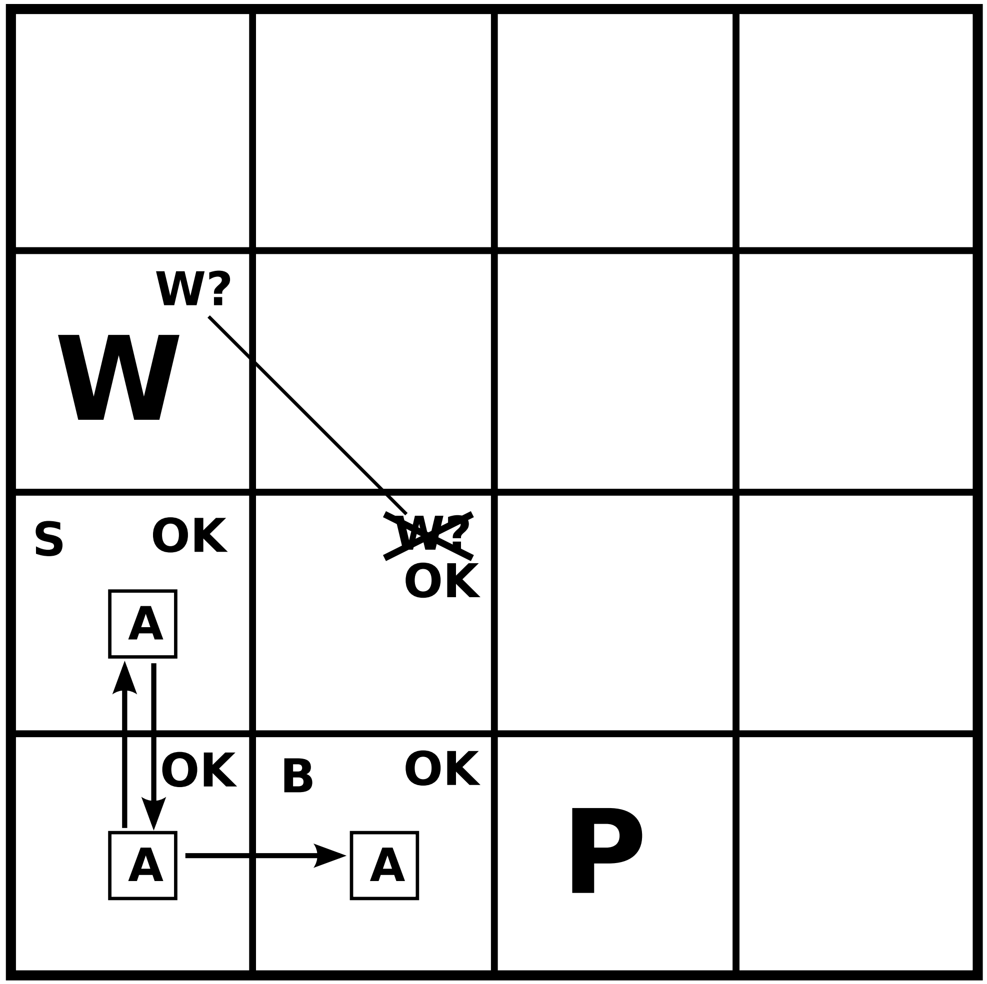

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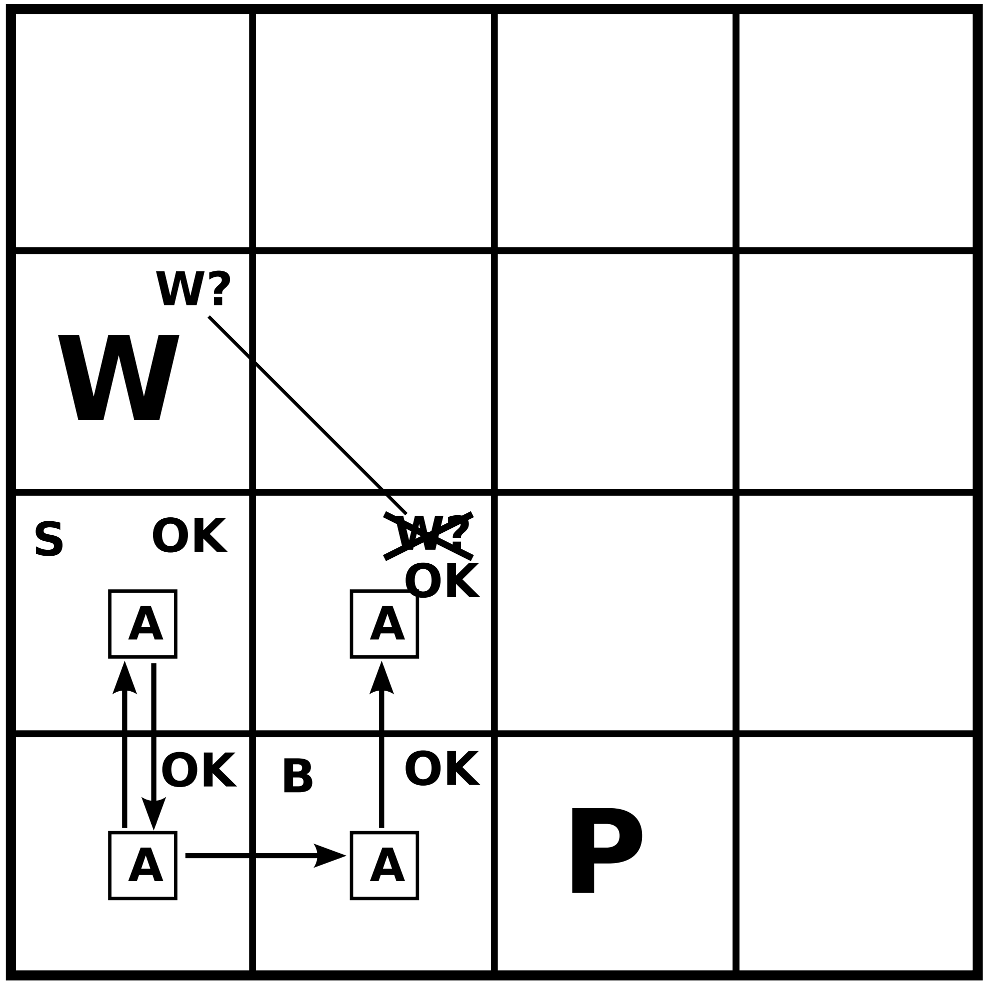

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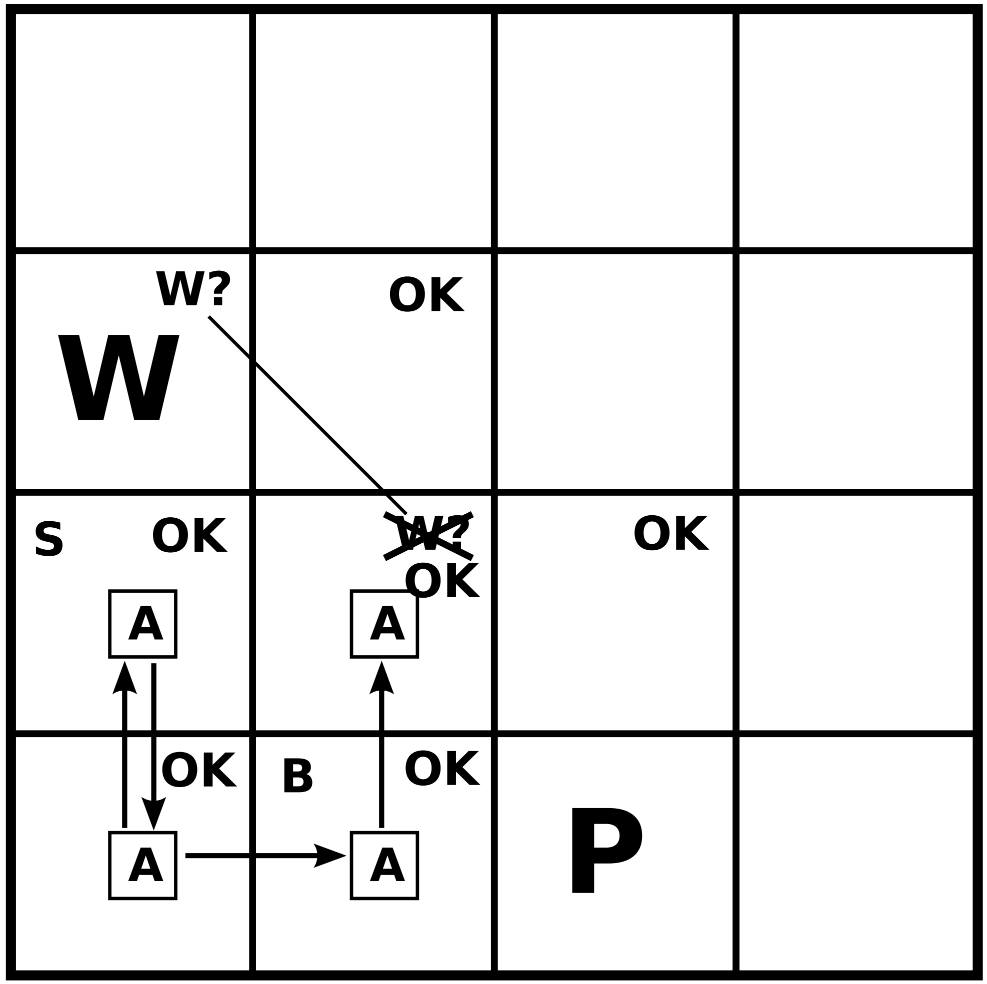
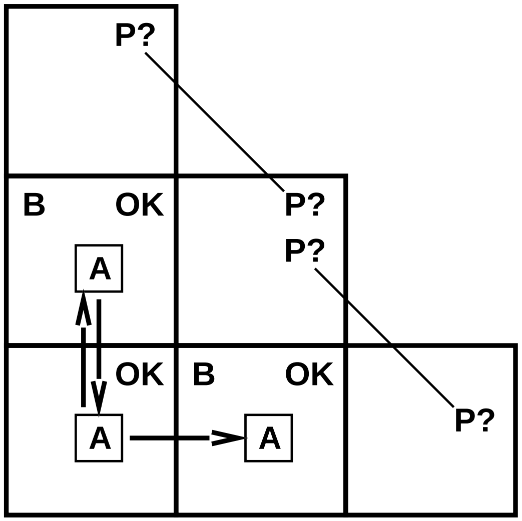
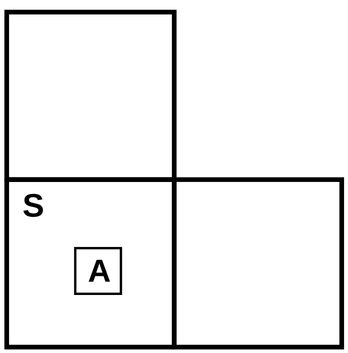


# Other tight spots




Breeze in (1,2) and (2,1)
$\Rightarrow$ no safe actions
Assuming pits uniformly distributed,
(2,2) has pit w/ prob 0.86, vs. 0.31




Smell in (1,1)
$\Rightarrow$ cannot move
Can use a strategy of coercion:
shoot straight ahead
wumpus was there $\Rightarrow$ dead $\Rightarrow$ safe
wumpus wasn't there $\Rightarrow$ safe

# Logic in general

Logics are formal languages for representing information
such that conclusions can be drawn

Syntax defines the sentences in the language

Semantics define the "meaning" of sentences;
i.e., define truth of a sentence in a world

E.g., the language of arithmetic

$x + 2 \geq y$ is a sentence; $x2 + y >$ is not a sentence

$x + 2 \geq y$ is true iff the number $x + 2$ is no less than the number $y$

$x + 2 \geq y$ is true in a world where $x = 7,\ y = 1$
$x + 2 \geq y$ is false in a world where $x = 0,\ y = 6$

# Entailment

Entailment means that one thing **follows from** another:

$$KB \models \alpha$$

A knowledge base $KB$ entails a sentence $\alpha$
if and only if
$\alpha$ is true in all worlds where $KB$ is true

E.g., the KB containing "There's a pit ahead" and "There's gold to the left" entails "Either there's a pit ahead or gold to the left"

E.g., $x + y = 4$ entails $4 = x + y$

Entailment is a relationship between sentences (i.e., **syntax**) that is based on **semantics**

# Models

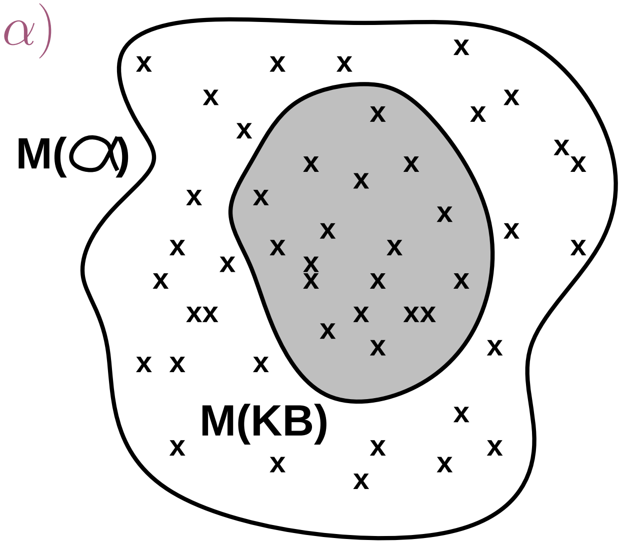
Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated

We say $m$ is a model of a sentence $\alpha$ if $\alpha$ is true in $m$

$M(\alpha)$ is the set of all models of $\alpha$

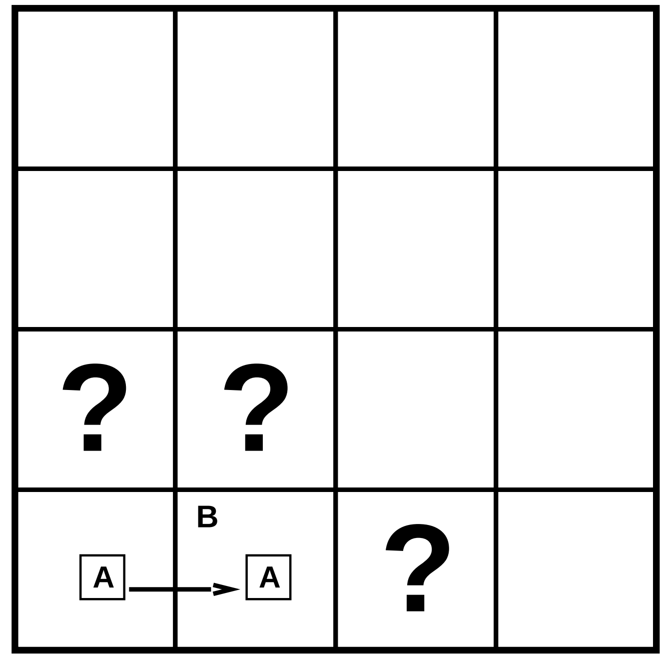
Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$

E.g. $KB =$ { there's a pit ahead, there's gold to the left }
$\alpha =$ there's gold to the left



# Entailment in the wumpus world

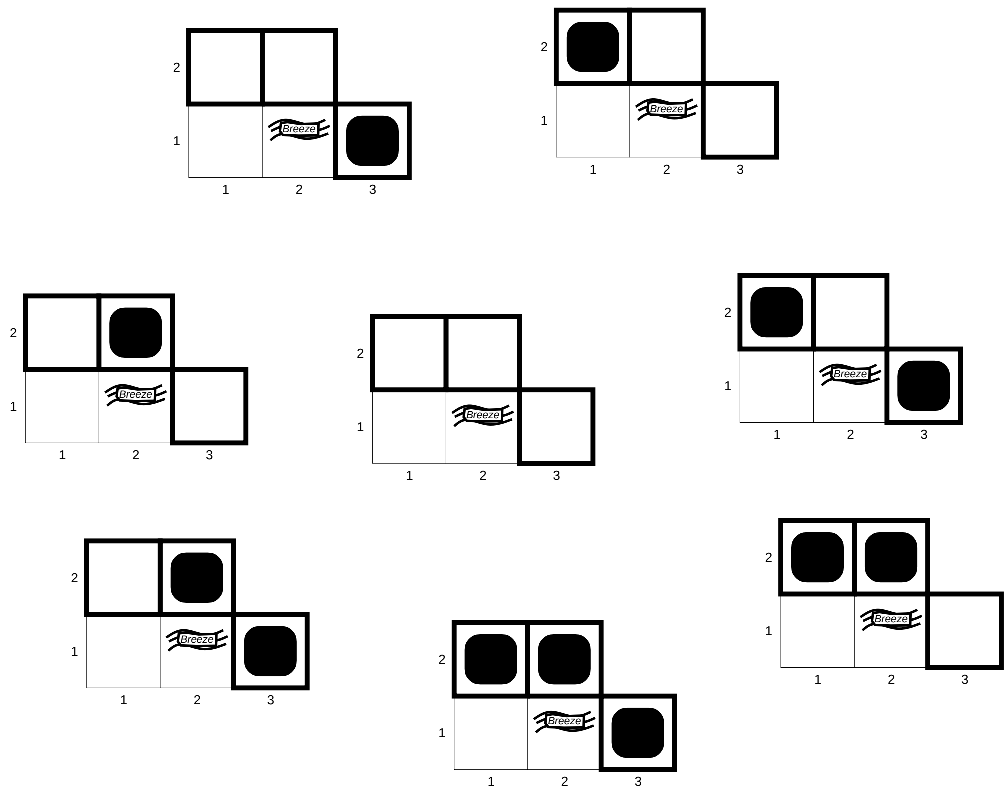
Situation after detecting nothing in [1,1], moving right, breeze in [2,1]



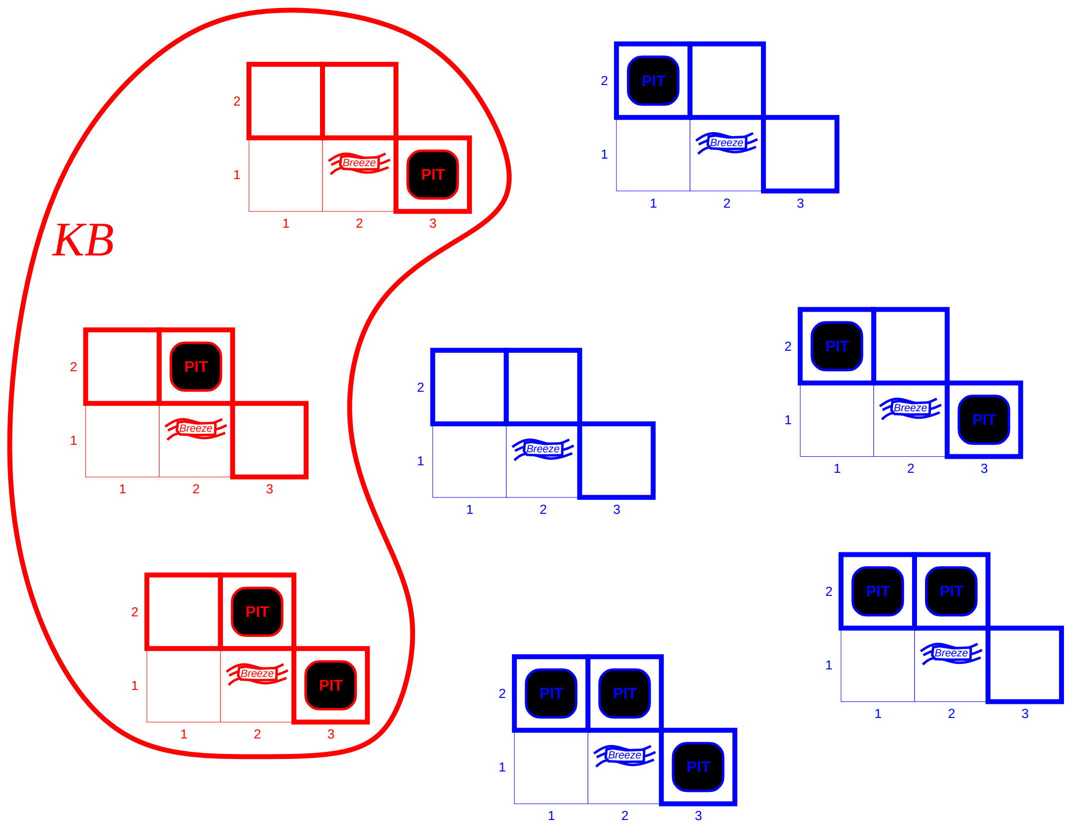

Consider possible models for ?s assuming only pits

3 Boolean choices $\Rightarrow$ 8 possible models

# Wumpus models

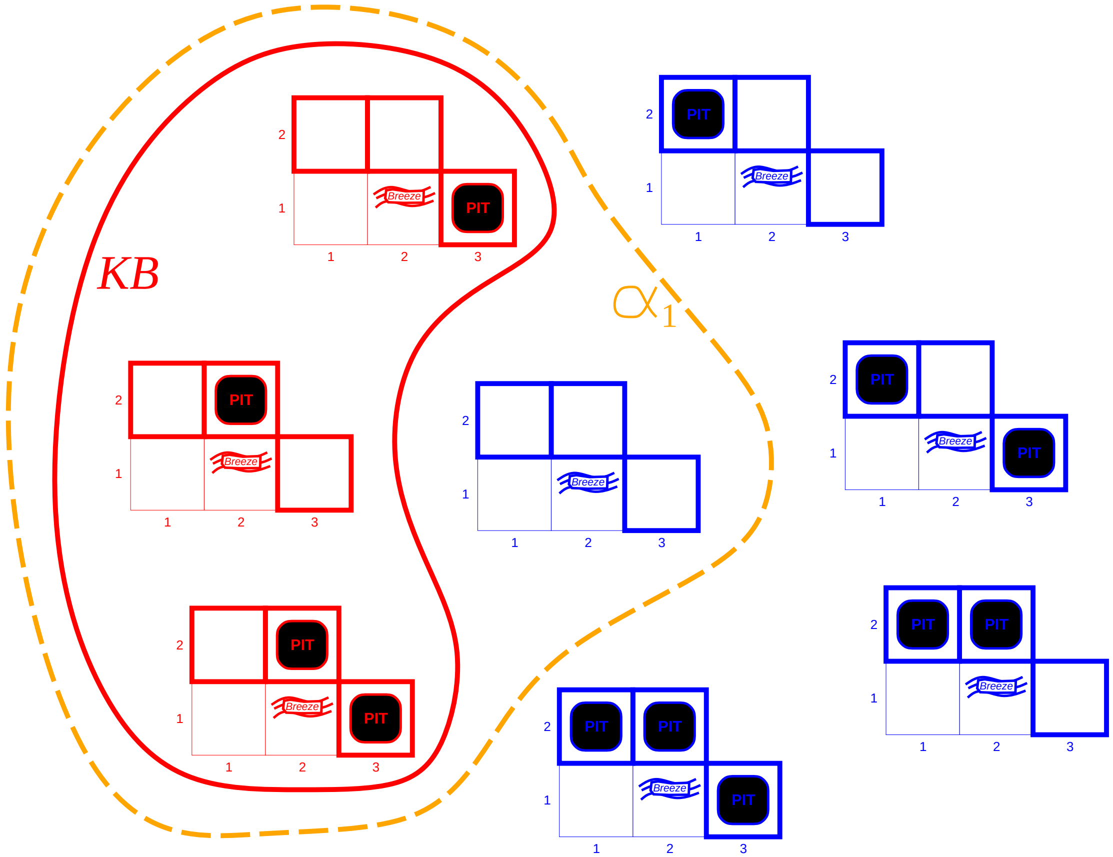


# Wumpus models




$KB$ = wumpus-world rules + observations

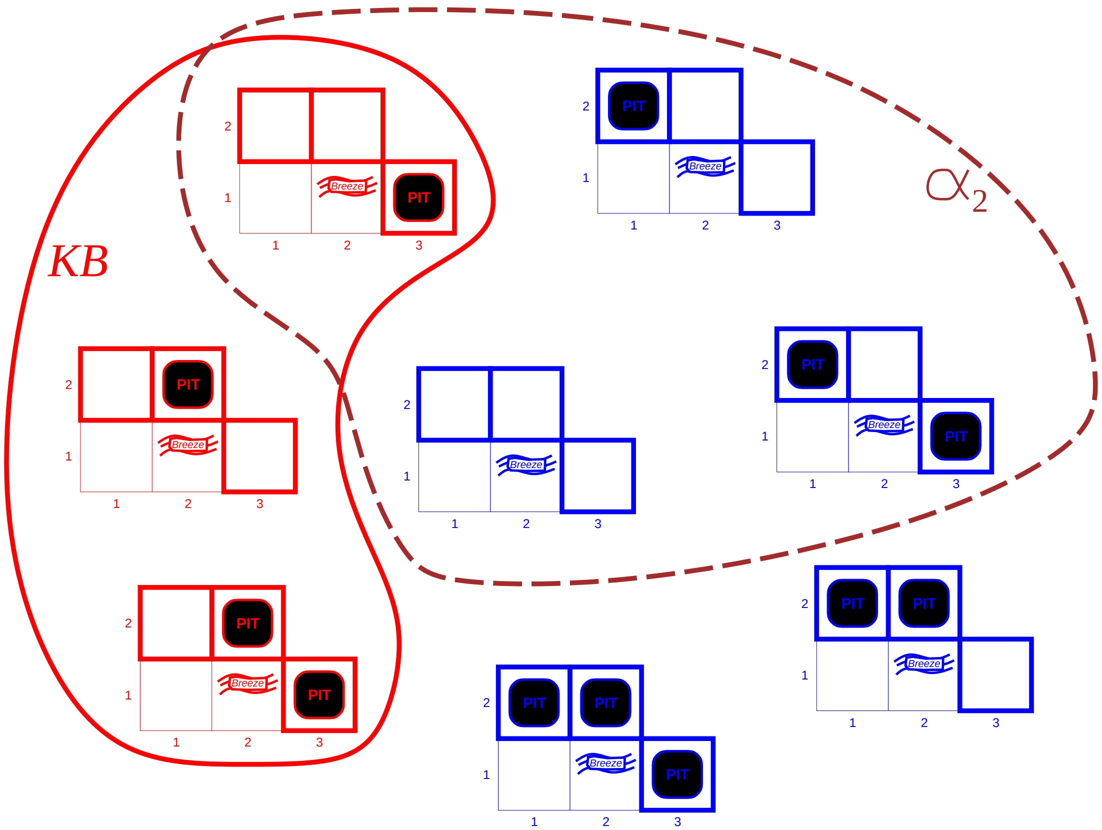
# Wumpus models




$KB$ = wumpus-world rules + observations

$\alpha_1$ = "[1,2] is safe", $KB \models \alpha_1$, proved by model checking

# Wumpus models




$KB$ = wumpus-world rules + observations

$\alpha_2$ = "[2,2] is safe", $KB \not\models \alpha_2$

# Inference

$KB \vdash_i \alpha$ = sentence $\alpha$ can be derived from $KB$ by procedure $i$

Consequences of $KB$ are a haystack; $\alpha$ is a needle.
Entailment = needle in haystack; inference = finding it

Soundness: $i$ is sound if
whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$

Completeness: $i$ is complete if
whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$

Preview: we will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure.

That is, the procedure will answer any question whose answer follows from what is known by the $KB$.

# Propositional logic: Syntax

Propositional logic is the simplest logic—illustrates basic ideas

The proposition symbols $P_1$, $P_2$ etc are sentences

If $S$ is a sentence, $\neg S$ is a sentence (negation)

If $S_1$ and $S_2$ are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)

If $S_1$ and $S_2$ are sentences, $S_1 \vee S_2$ is a sentence (disjunction)

If $S_1$ and $S_2$ are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)

If $S_1$ and $S_2$ are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

# Propositional logic: Semantics

Each model specifies true/false for each proposition symbol

E.g. $\begin{array}{ccc} P_{1,2} & P_{2,2} & P_{3,1} \\ true & true & false \end{array}$

(With these symbols, 8 possible models, can be enumerated automatically.)

Rules for evaluating truth with respect to a model $m$:

| | | | | | |
|---|---|---|---|---|---|
| $\neg S$ | is true iff | $S$ | is false | | |
| $S_1 \wedge S_2$ | is true iff | $S_1$ | is true **and** | $S_2$ | is true |
| $S_1 \vee S_2$ | is true iff | $S_1$ | is true **or** | $S_2$ | is true |
| $S_1 \Rightarrow S_2$ | is true iff | $S_1$ | is false **or** | $S_2$ | is true |
| i.e., | is false iff | $S_1$ | is true **and** | $S_2$ | is false |
| $S_1 \Leftrightarrow S_2$ | is true iff | $S_1 \Rightarrow S_2$ | is true **and** | $S_2 \Rightarrow S_1$ | is true |

Simple recursive process evaluates an arbitrary sentence, e.g.,
$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = true \wedge (false \vee true) = true \wedge true = true$

# Truth tables for connectives

| $P$ | $Q$ | $\neg P$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|---|
| *false* | *false* | *true* | *false* | *false* | *true* | *true* |
| *false* | *true* | *true* | *false* | *true* | *true* | *false* |
| *true* | *false* | *false* | *false* | *true* | *false* | *false* |
| *true* | *true* | *false* | *true* | *true* | *true* | *true* |

# Wumpus world sentences

Let $P_{i,j}$ be true if there is a pit in $[i, j]$.
Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.

$$\neg P_{1,1}$$
$$\neg B_{1,1}$$
$$B_{2,1}$$

How do we encode "pits cause breezes in adjacent squares"?

$$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$$
$$B_{2,1} \Leftrightarrow (P_{1,1} \lor P_{2,2} \lor P_{3,1})$$

In other words, "a square is breezy **if and only if** there is an adjacent pit"

# Truth tables for inference

| $B_{1,1}$ | $B_{2,1}$ | $P_{1,1}$ | $P_{1,2}$ | $P_{2,1}$ | $P_{2,2}$ | $P_{3,1}$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $KB$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *false* | *false* | *false* | *false* | *false* | *false* | *false* | *true* | *true* | *true* | *true* | *false* | *false* |
| *false* | *false* | *false* | *false* | *false* | *false* | *true* | *true* | *true* | *false* | *true* | *false* | *false* |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| *false* | *true* | *false* | *false* | *false* | *false* | *false* | *true* | *true* | *false* | *true* | *true* | *false* |
| *false* | *true* | *false* | *false* | *false* | *false* | *true* | *true* | *true* | *true* | *true* | *true* | *true* |
| *false* | *true* | *false* | *false* | *false* | *true* | *false* | *true* | *true* | *true* | *true* | *true* | *true* |
| *false* | *true* | *false* | *false* | *false* | *true* | *true* | *true* | *true* | *true* | *true* | *true* | *true* |
| *false* | *true* | *false* | *false* | *true* | *false* | *false* | *true* | *false* | *false* | *true* | *true* | *false* |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| *true* | *true* | *true* | *true* | *true* | *true* | *true* | *false* | *true* | *true* | *false* | *true* | *false* |

Enumerate rows (different assignments to symbols),
if KB is true in row, check that $\alpha$ is too

# Inference by enumeration

Depth-first enumeration of all models is sound and complete

```
function TT-ENTAILS?(KB, α) returns true or false
   inputs: KB, the knowledge base, a sentence in propositional logic
           α, the query, a sentence in propositional logic

   symbols ← a list of the proposition symbols in KB and α
   return TT-CHECK-ALL(KB, α, symbols, [ ])
```

---

```
function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
   if EMPTY?(symbols) then
      if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
      else return true
   else do
      P ← FIRST(symbols); rest ← REST(symbols)
      return TT-CHECK-ALL(KB, α, rest, EXTEND(P, true, model)) and
                 TT-CHECK-ALL(KB, α, rest, EXTEND(P, false, model))
```

$O(2^n)$ for $n$ symbols; problem is **co-NP-complete**

# Logical equivalence

Two sentences are logically equivalent iff they are true in the same models:

$\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$

| | | | |
|---|---|---|---|
| $(\alpha \land \beta)$ | $\equiv$ | $(\beta \land \alpha)$ | commutativity of $\land$ |
| $(\alpha \lor \beta)$ | $\equiv$ | $(\beta \lor \alpha)$ | commutativity of $\lor$ |
| $((\alpha \land \beta) \land \gamma)$ | $\equiv$ | $(\alpha \land (\beta \land \gamma))$ | associativity of $\land$ |
| $((\alpha \lor \beta) \lor \gamma)$ | $\equiv$ | $(\alpha \lor (\beta \lor \gamma))$ | associativity of $\lor$ |
| $\neg(\neg\alpha)$ | $\equiv$ | $\alpha$ | double-negation elimination |
| $(\alpha \Rightarrow \beta)$ | $\equiv$ | $(\neg\beta \Rightarrow \neg\alpha)$ | contraposition |
| $(\alpha \Rightarrow \beta)$ | $\equiv$ | $(\neg\alpha \lor \beta)$ | implication elimination |
| $(\alpha \Leftrightarrow \beta)$ | $\equiv$ | $((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$ | biconditional elimination |
| $\neg(\alpha \land \beta)$ | $\equiv$ | $(\neg\alpha \lor \neg\beta)$ | De Morgan |
| $\neg(\alpha \lor \beta)$ | $\equiv$ | $(\neg\alpha \land \neg\beta)$ | De Morgan |
| $(\alpha \land (\beta \lor \gamma))$ | $\equiv$ | $((\alpha \land \beta) \lor (\alpha \land \gamma))$ | distributivity of $\land$ over $\lor$ |
| $(\alpha \lor (\beta \land \gamma))$ | $\equiv$ | $((\alpha \lor \beta) \land (\alpha \lor \gamma))$ | distributivity of $\lor$ over $\land$ |

# Validity and satisfiability

A sentence is valid if it is true in **all** models,

e.g., $True$, $A \lor \neg A$, $A \Rightarrow A$, $(A \land (A \Rightarrow B)) \Rightarrow B$

Validity is connected to inference via the Deduction Theorem:

$KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid

A sentence is satisfiable if it is true in **some** model

e.g., $A \lor B$, $C$

A sentence is unsatisfiable if it is true in **no** models

e.g., $A \land \neg A$

Satisfiability is connected to inference via the following:

$KB \models \alpha$ iff $(KB \land \neg\alpha)$ is unsatisfiable

i.e., prove $\alpha$ by *reductio ad absurdum*

# Proof methods

Proof methods divide into (roughly) two kinds:

Application of inference rules

– Legitimate (sound) generation of new sentences from old

– Proof = a sequence of inference rule applications
  can use inference rules as operators in a standard search algorithm

– Typically require translation of sentences into a normal form

Model checking

– Truth table enumeration (always exponential in $n$)

– Improved backtracking, e.g., Davis–Putnam–Logemann–Loveland

– Heuristic search in model space (sound but incomplete)
  e.g., min-conflicts-like hill-climbing algorithms

# Forward and backward chaining

Horn Form (restricted)

KB = **conjunction** of **Horn clauses**

Horn clause =

– proposition symbol; or

– (conjunction of symbols) $\Rightarrow$ symbol

Example KB: $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

Modus Ponens (for Horn Form): complete for Horn KBs

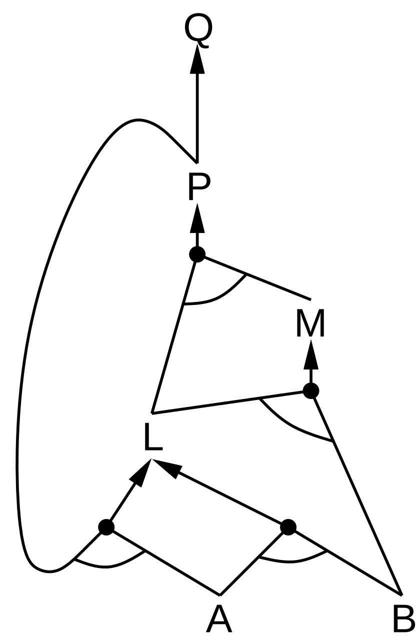
$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

Can be used with forward chaining or backward chaining.
These algorithms are very natural and run in **linear** time

# Forward chaining

Idea: fire any rule whose premises are satisfied in the $KB$,
add its conclusion to the $KB$, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

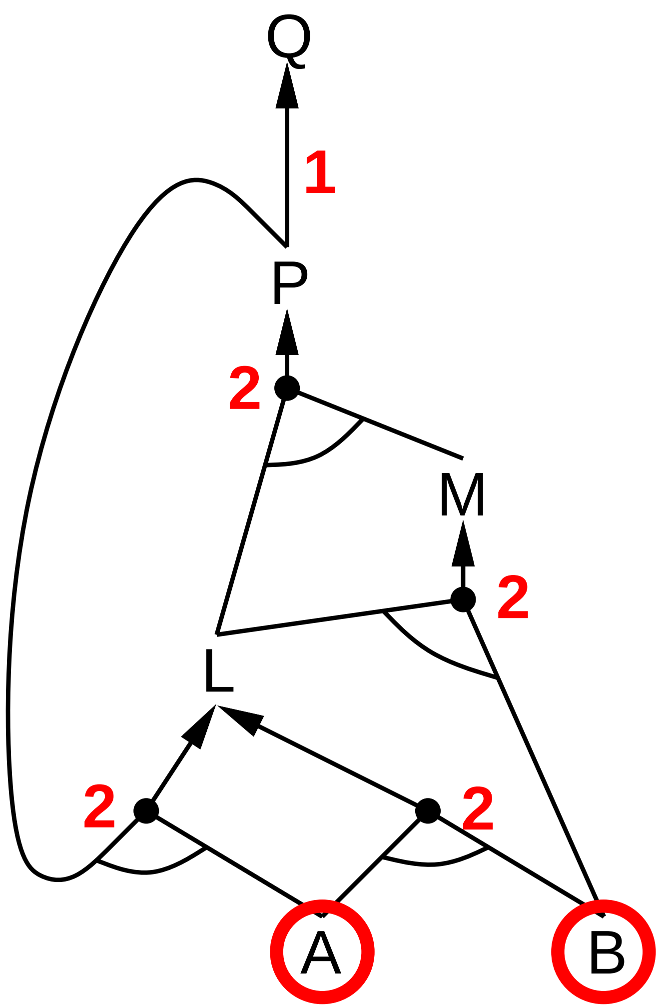


# Forward chaining algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  inputs: KB, the knowledge base, a set of propositional Horn clauses
          q, the query, a proposition symbol
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known in KB

  while agenda is not empty do
      p ← POP(agenda)
      unless inferred[p] do
          inferred[p] ← true
          for each Horn clause c in whose premise p appears do
              decrement count[c]
              if count[c] = 0 then do
                  if HEAD[c] = q then return true
                  PUSH(HEAD[c], agenda)
  return false
```

# Forward chaining example



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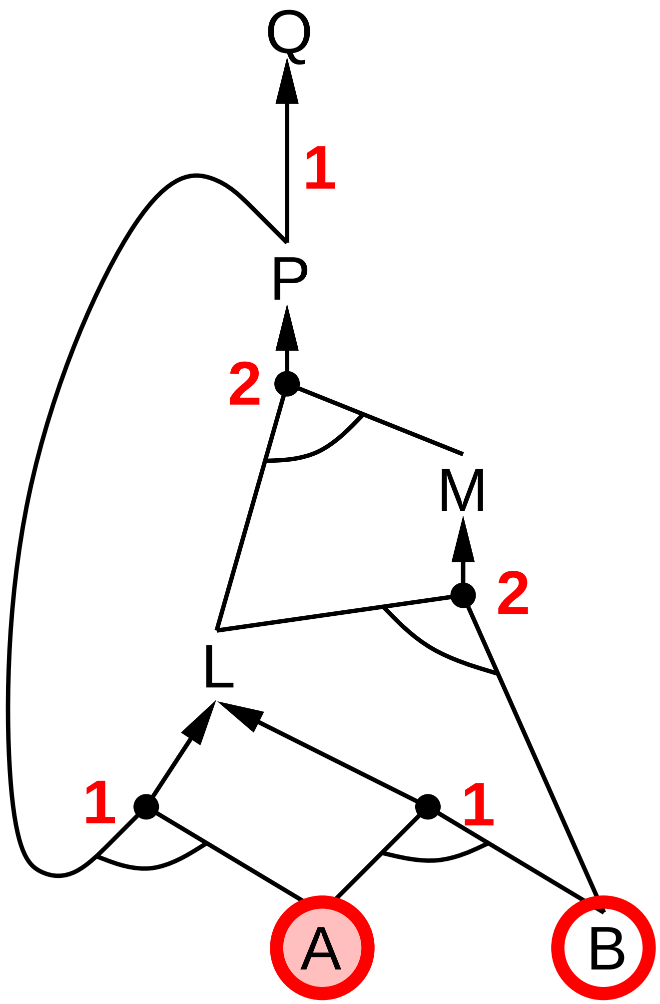

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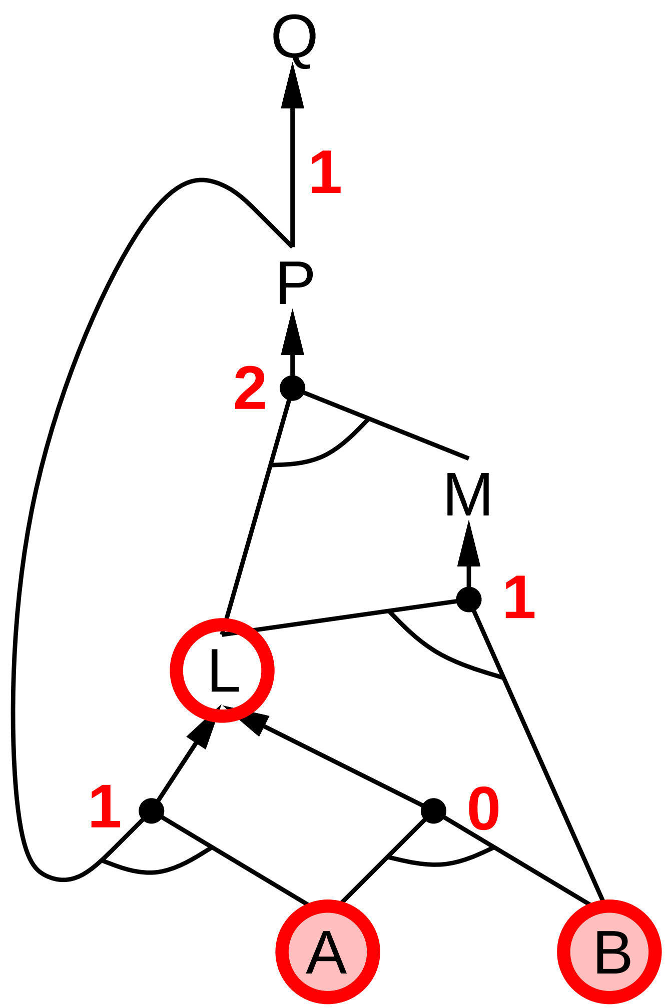

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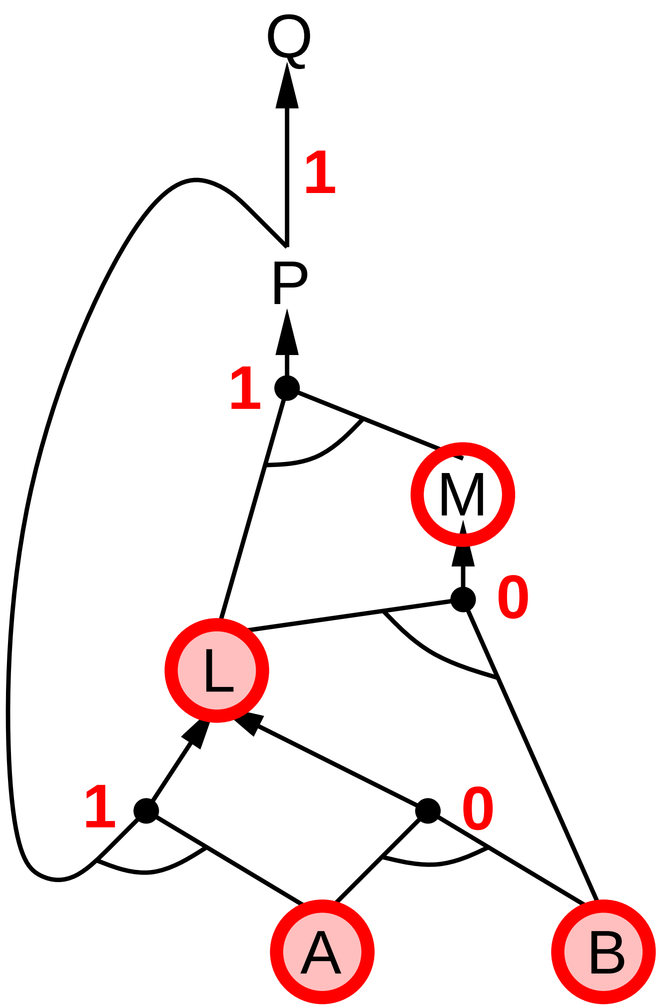

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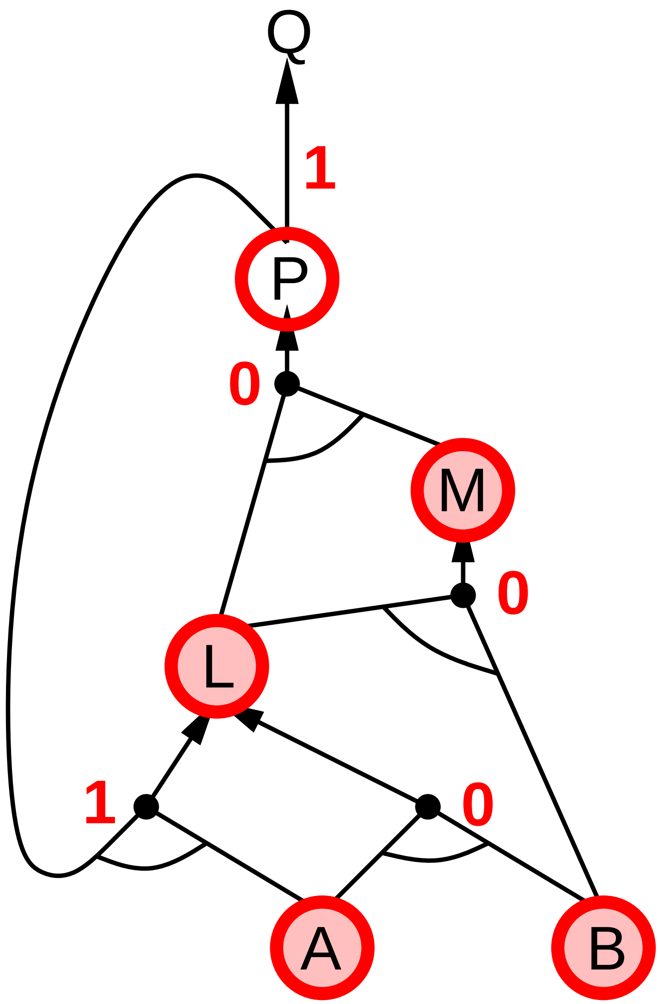

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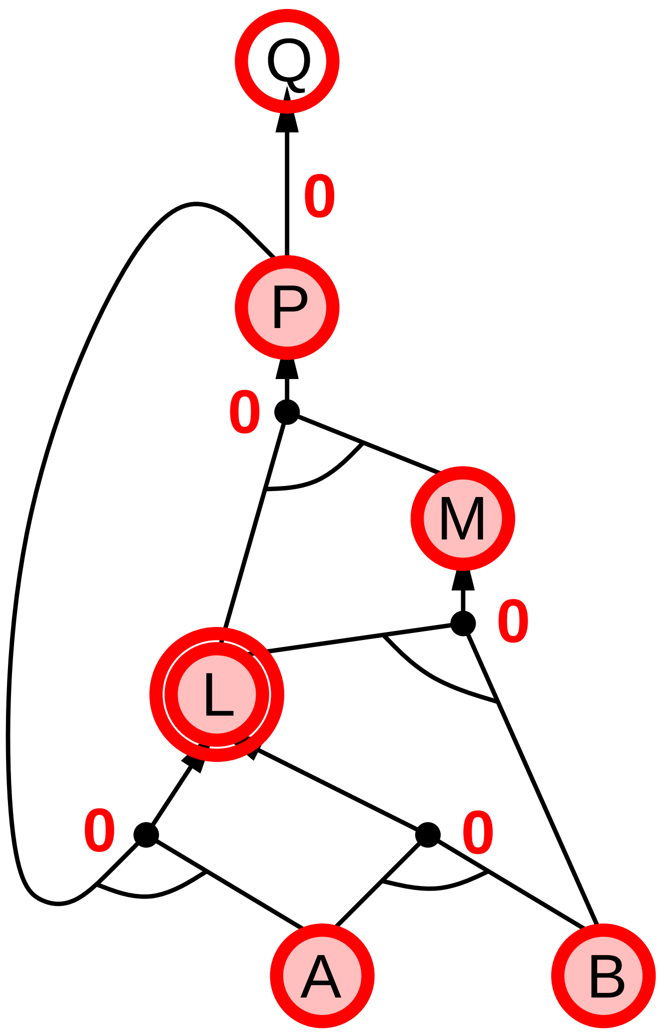

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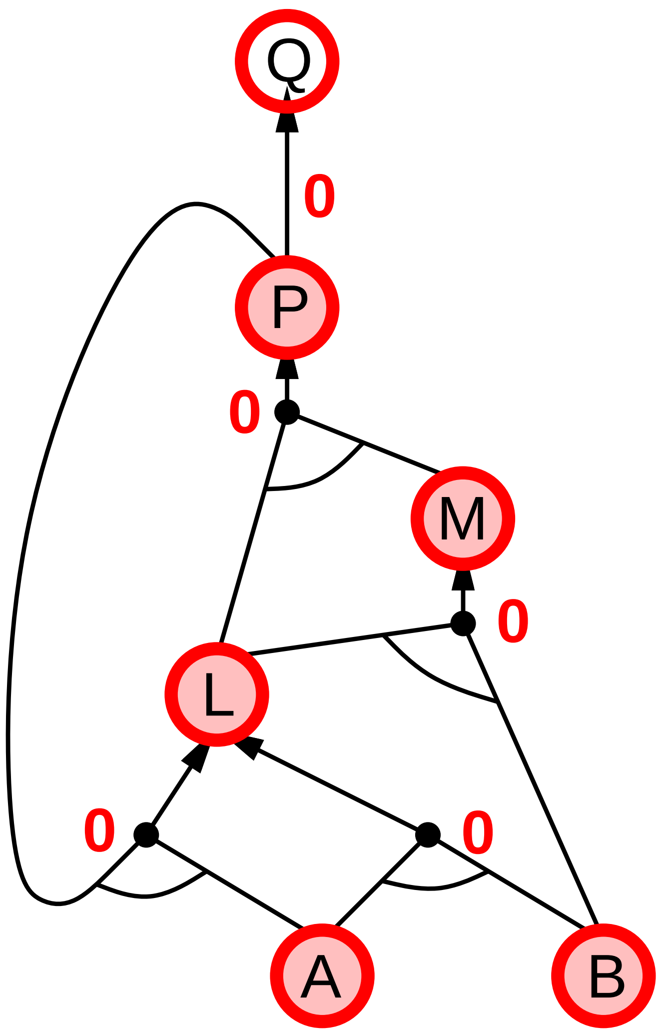

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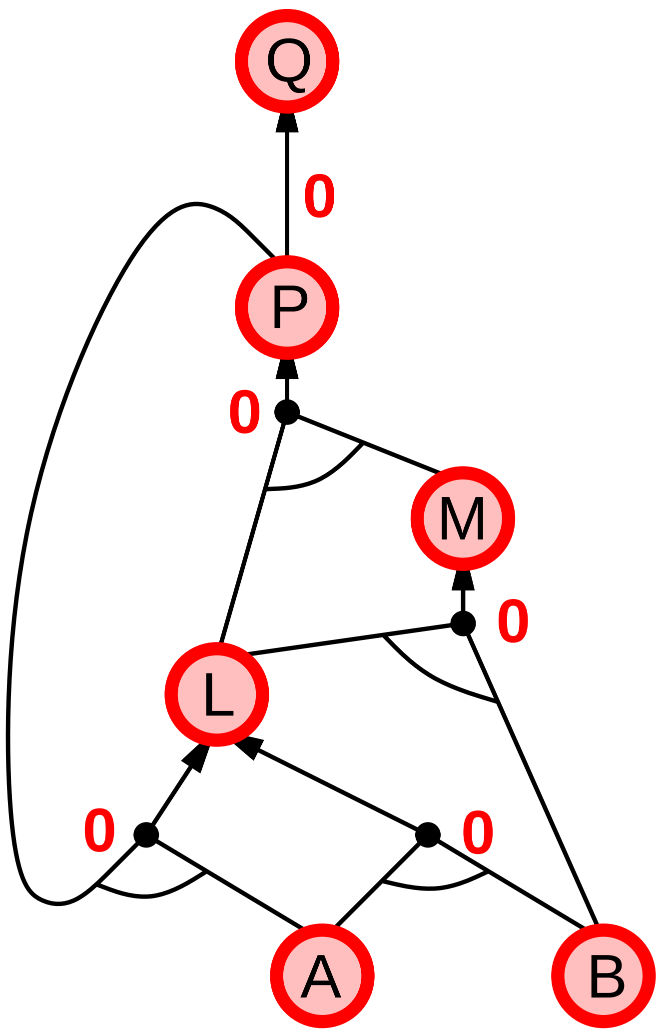


# Proof of completeness

FC derives every atomic sentence that is entailed by $KB$

1. FC reaches a fixed point where no new atomic sentences are derived
2. Consider the final state as a model $m$, assigning true/false to symbols
3. Every clause in the original $KB$ is true in $m$
   **Proof**: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in $m$
   Then $a_1 \wedge \ldots \wedge a_k$ is true in $m$ and $b$ is false in $m$
   Therefore the algorithm has not reached a fixed point!
4. Hence $m$ is a model of $KB$
5. If $KB \models q$, then $q$ is true in **every** model of $KB$, including $m$

General idea: construct any model of $KB$ by sound inference, check $\alpha$

# Backward chaining

Idea: work backwards from the query $q$:

to prove $q$ by BC,

check if $q$ is known already, or

prove by BC all premises of some rule concluding $q$

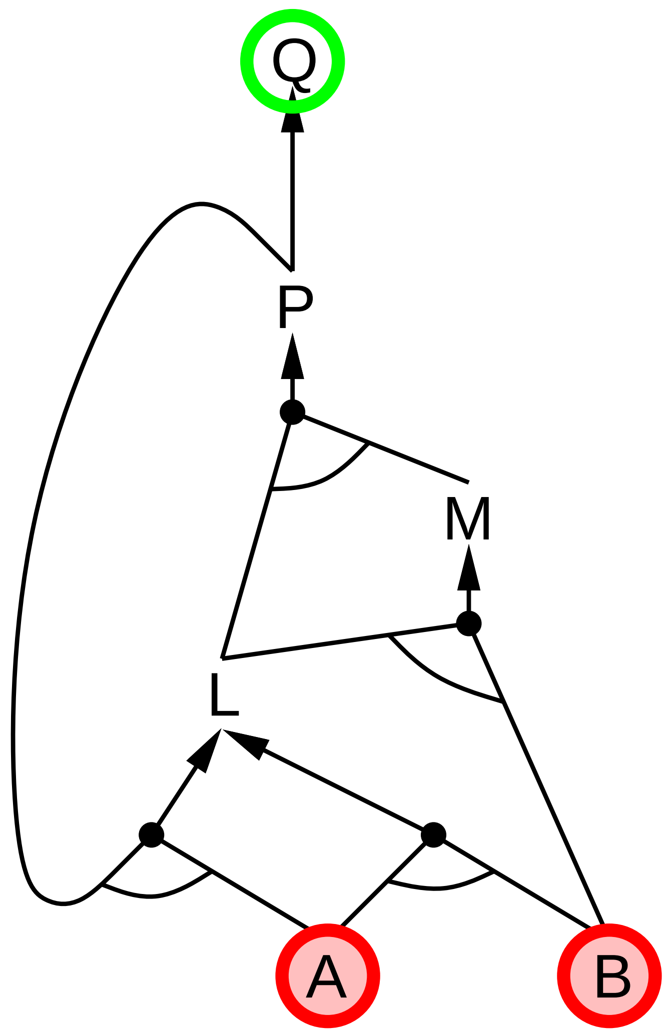
Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1) has already been proved true, or

2) has already failed

# Backward chaining example



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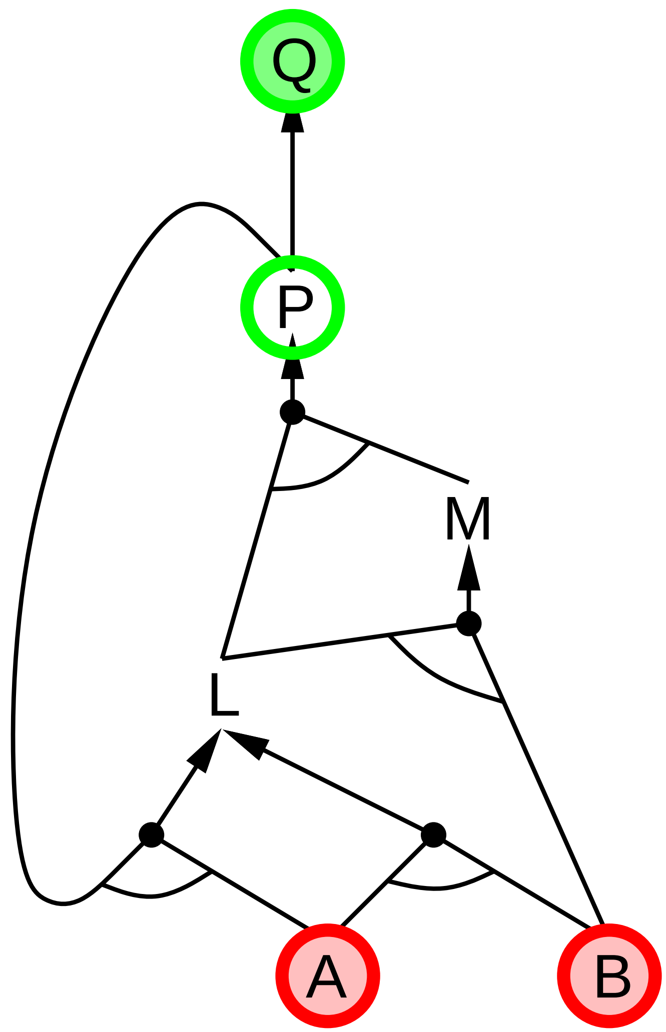

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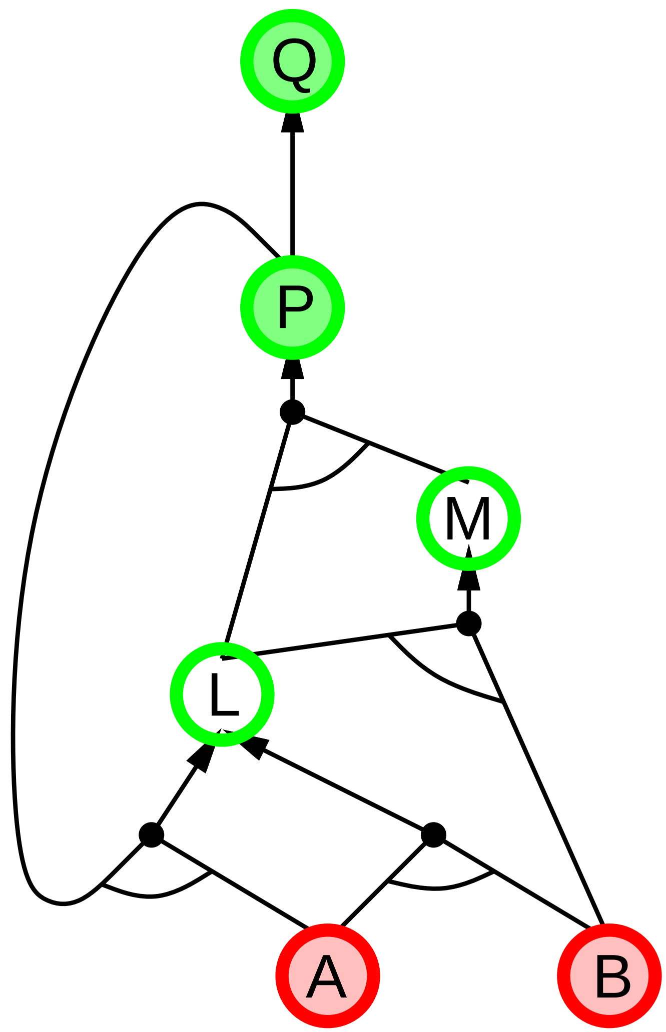
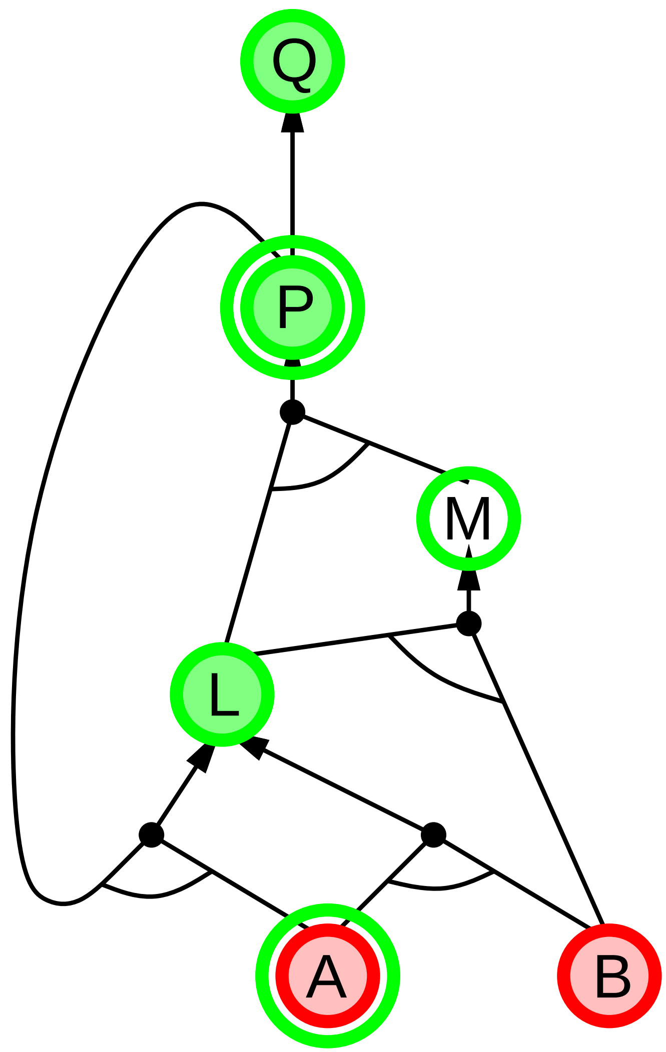
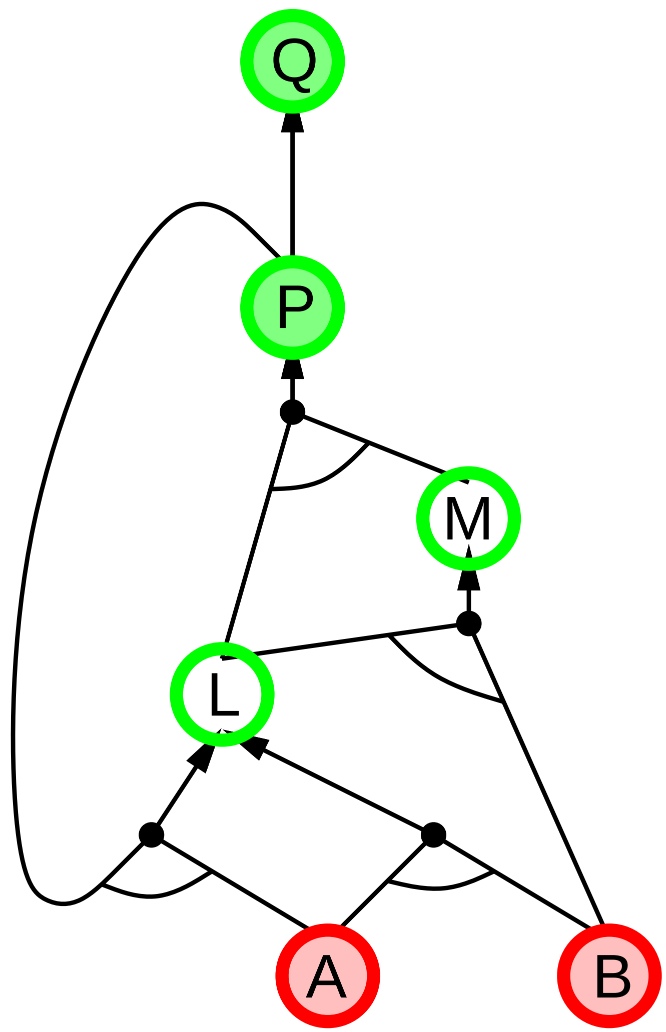


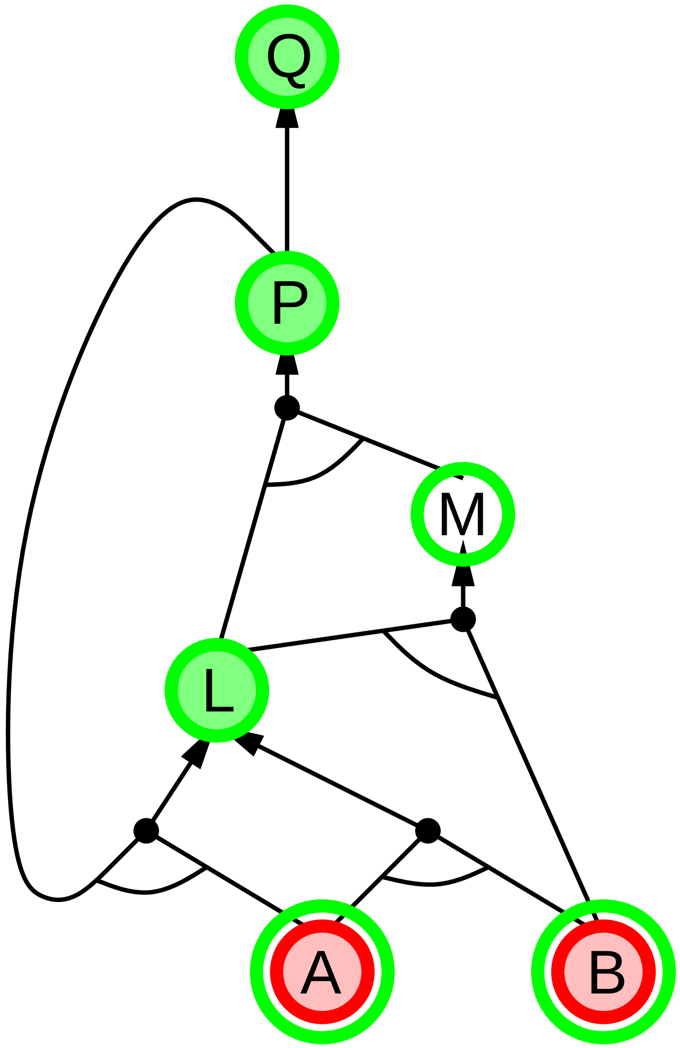
# Backward chaining example



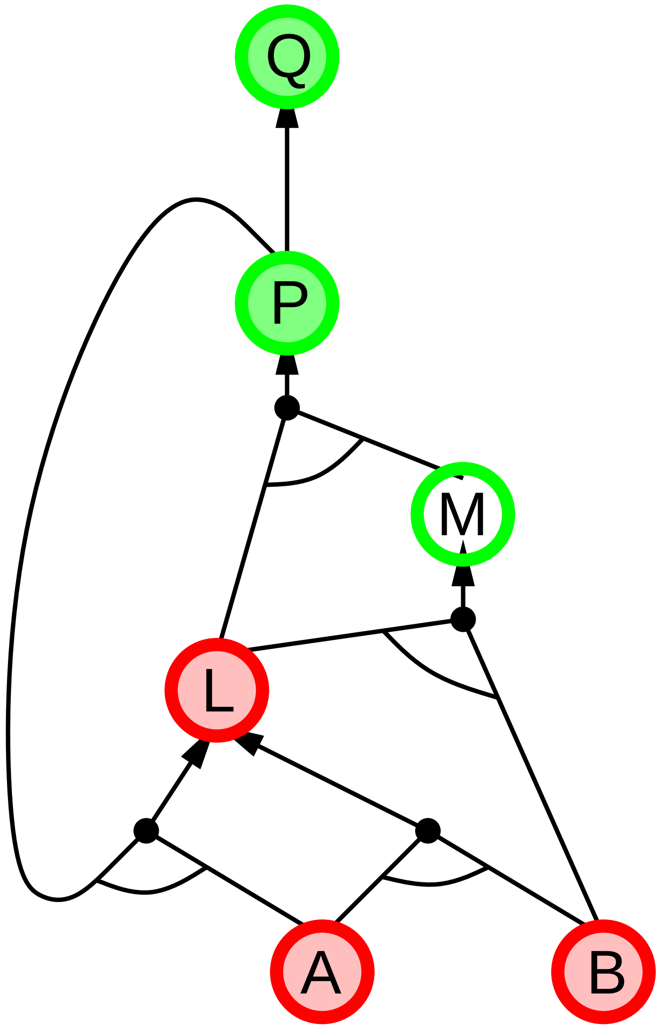
# Backward chaining example



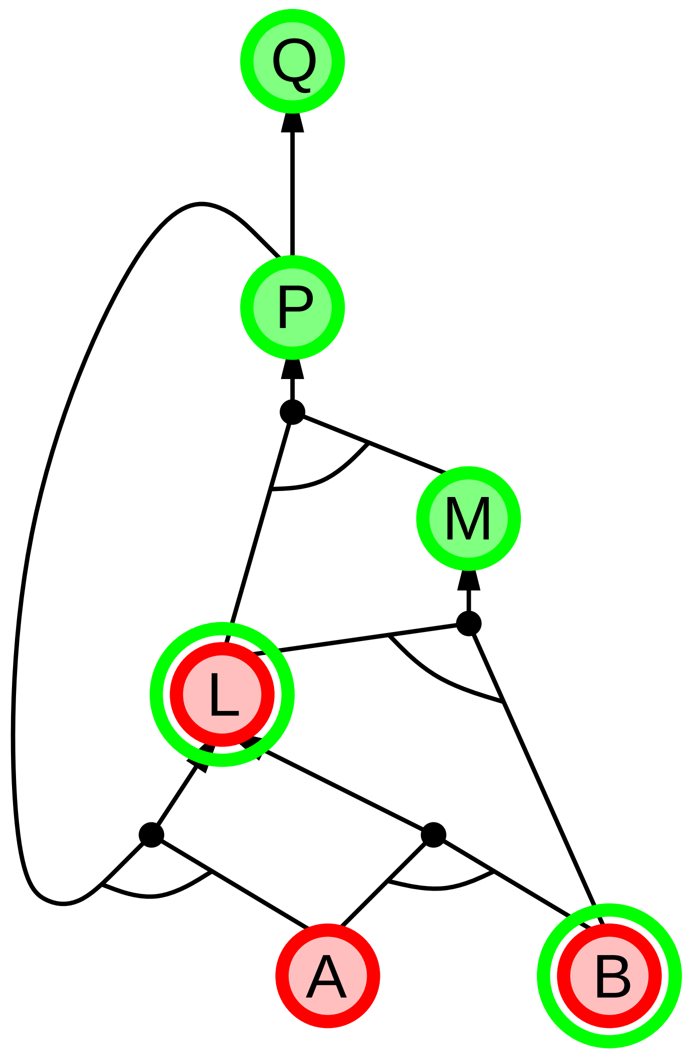
# Backward chaining example



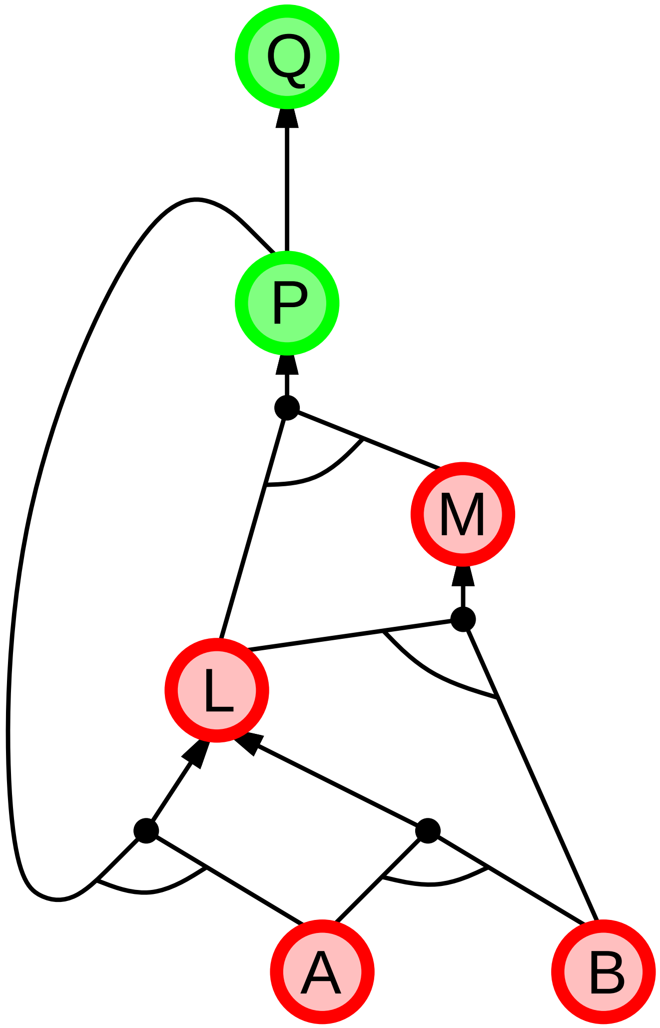
# Backward chaining example



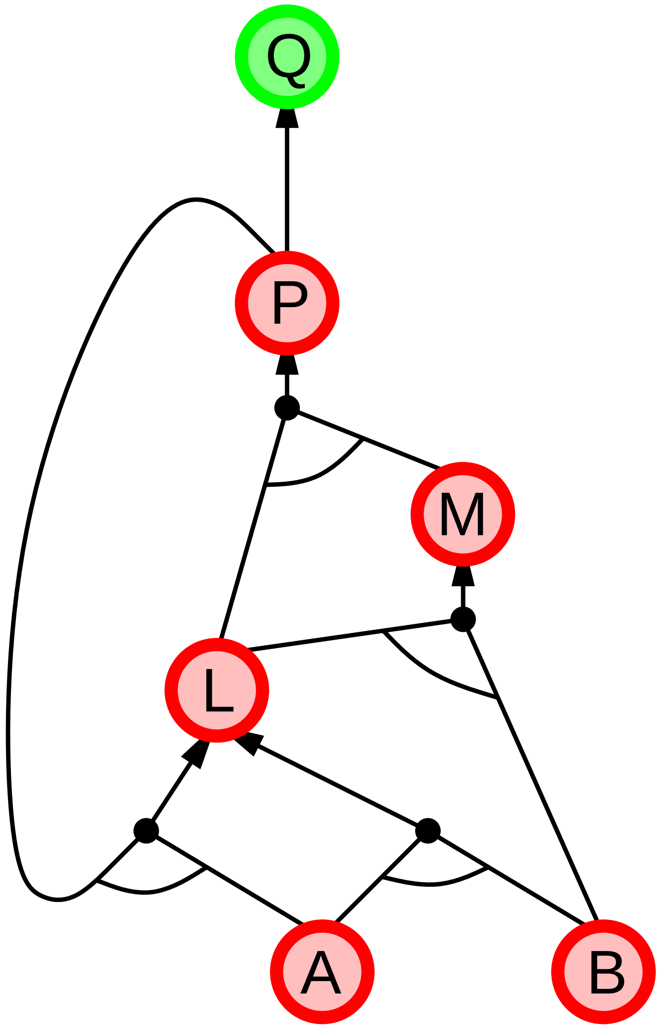
# Backward chaining example



# Backward chaining example



# Backward chaining example



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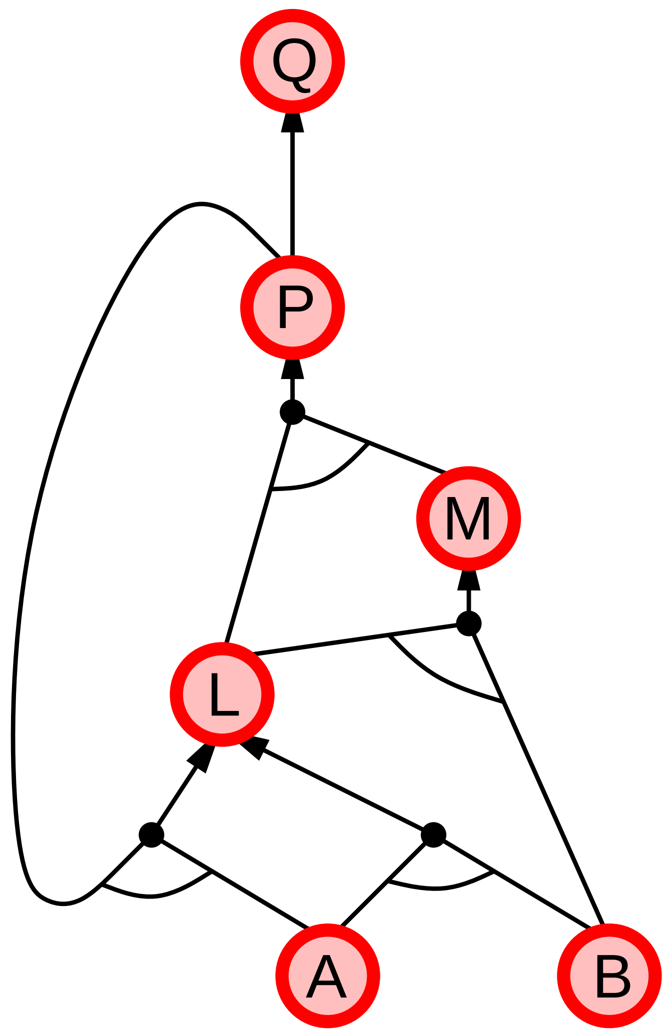


# Forward vs. backward chaining

FC is data-driven, cf. automatic, unconscious processing,
e.g., object recognition, routine decisions

May do lots of work that is irrelevant to the goal

BC is goal-driven, appropriate for problem-solving,
e.g., Where are my keys? How do I get into a PhD program?

Complexity of BC can be **much less** than linear in size of KB

# Resolution

Conjunctive Normal Form (CNF—universal)

**conjunction** of $\underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$

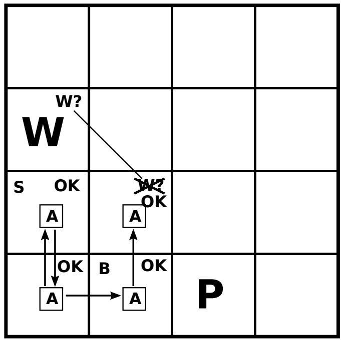
E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

Resolution inference rule (for CNF): complete for propositional logic

$$\frac{\ell_1 \lor \cdots \lor \ell_i \lor \cdots \lor \ell_k, \qquad m_1 \lor \cdots \lor m_j \lor \cdots \lor m_n}{\ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n}$$

where $\ell_i \equiv \neg m_j$ are complementary literals. E.g.,

$$\frac{W_{1,3} \lor W_{2,2}, \qquad \neg W_{2,2}}{W_{1,3}}$$




Resolution is sound and complete for propositional logic

# Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \ \Rightarrow \ \beta) \wedge (\beta \ \Rightarrow \ \alpha)$.

   $(B_{1,1} \ \Rightarrow \ (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \ \Rightarrow \ B_{1,1})$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and double-negation:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributivity law ($\vee$ over $\wedge$) and flatten:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

# Resolution algorithm

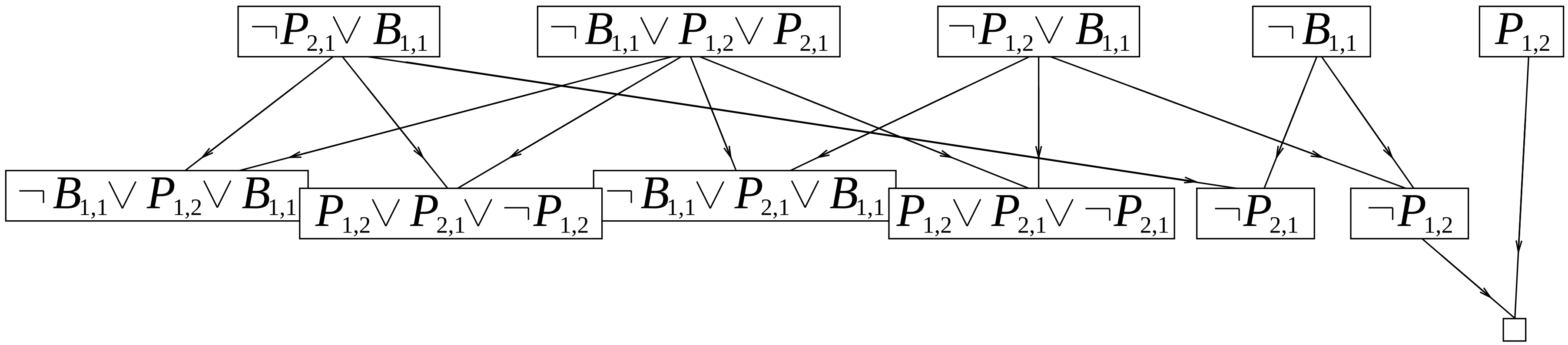
Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

**function** PL-RESOLUTION($KB, \alpha$) **returns** *true* or *false*
  **inputs**: $KB$, the knowledge base, a sentence in propositional logic
    $\alpha$, the query, a sentence in propositional logic

  *clauses* ← the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  *new* ← { }
  **loop do**
    **for each** $C_i$, $C_j$ **in** *clauses* **do**
      *resolvents* ← PL-RESOLVE($C_i, C_j$)
      **if** *resolvents* contains the empty clause **then return** *true*
      *new* ← *new* ∪ *resolvents*
    **if** *new* ⊆ *clauses* **then return** *false*
    *clauses* ← *clauses* ∪ *new*

# Resolution example

$KB = (B_{1,1} \;\Leftrightarrow\; (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1}$ $\alpha = \neg P_{1,2}$



# Summary

Logical agents apply inference to a knowledge base
to derive new information and make decisions

Basic concepts of logic:
- syntax: formal structure of sentences
- semantics: truth of sentences wrt models
- entailment: necessary truth of one sentence given another
- inference: deriving sentences from other sentences
- soundness: derivations produce only entailed sentences
- completeness: derivations can produce all entailed sentences

Wumpus world requires the ability to represent partial and negated information, reason by cases, etc.

Forward, backward chaining are linear-time, complete for Horn clauses
Resolution is complete for propositional logic

Propositional logic lacks expressive power