Datastructures

Data Structures

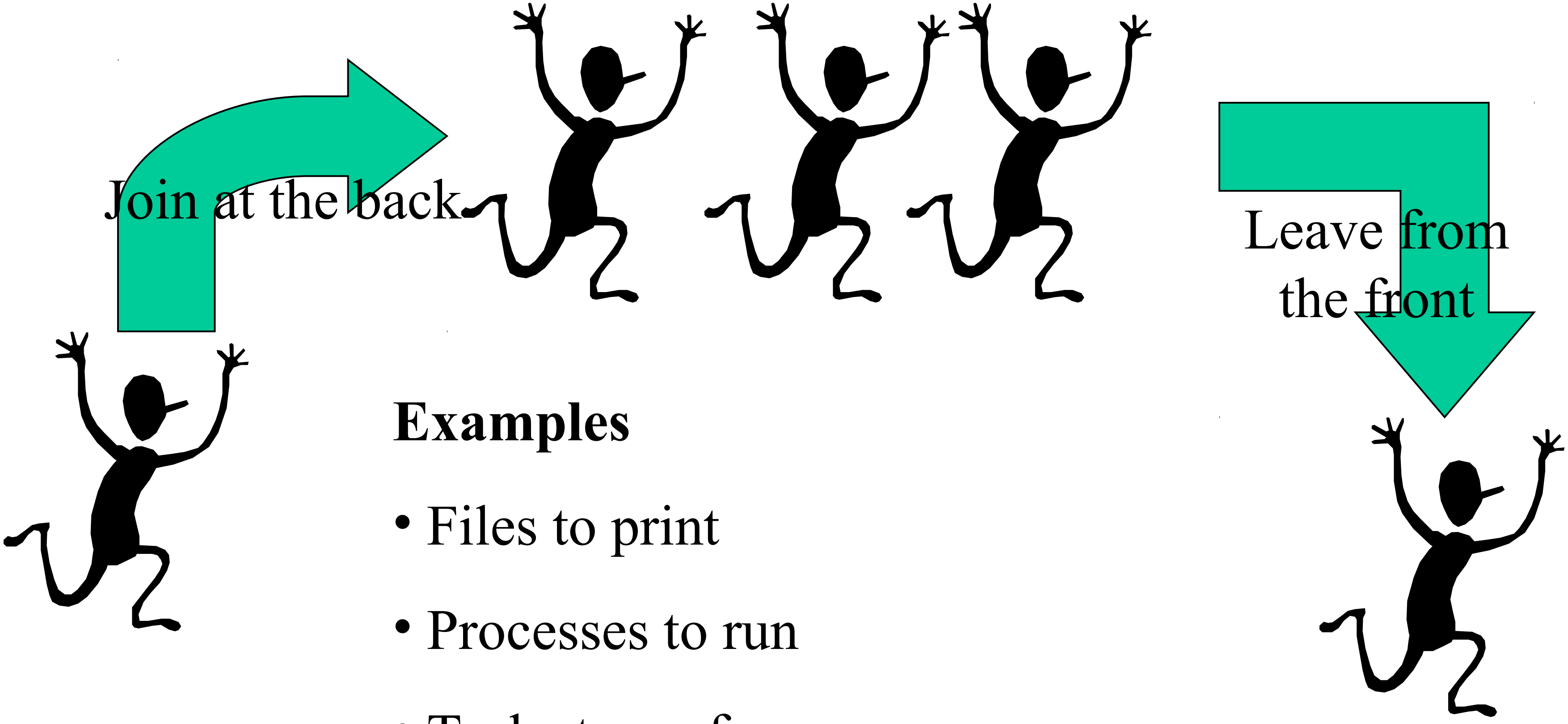
- Datatype
  - A model of something that we want to represent in our program
- Data structure
  - A particular way of *storing* data
  - How? Depending on what we want to do with the data
- Today: Two examples
  - Queues
  - Tables

Using QuickCheck to Develop Fast Queue Operations

What we’re going to do:

- Explain what a *queue* is, and give *slow* implementations of the queue operations, to act as a specification.
- Give a fast implementation of the queue.
- Formulate properties that say the fast implementation is ”correct”.
- Test them with QuickCheck.

What is a Queue?



Join at the back

Leave from the front

Examples

- Files to print
- Processes to run
- Tasks to perform

What is a Queue?

A *queue* contains a sequence of values. We can add elements at the back, and remove elements from the front.

We'll implement the following operations:

```
empty    :: Q a                 -- an empty queue
add      :: a -> Q a -> Q a     -- add an element at the back
remove   :: Q a -> Q a          -- remove an element from the front
front     :: Q a -> a           -- inspect the front element
isEmpty :: Q a -> Bool          -- check if the queue is empty
```

new
type

First Try

```
data Q a = Q [a] deriving (Eq, Show)

empty                = Q []
add x (Q xs)         = Q (xs++[x])
remove (Q (x:xs))    = Q xs
front (Q (x:xs))     = x
isEmpty (Q xs)       = null xs
```

“Obiously”
correct

Works, but slow

```
add x (Q xs) = Q (xs++[x])
```

```
[]     ++ ys = ys
(x:xs) ++ ys = x : (xs++ys)
```

As many recursive calls as there are elements in xs

Add 1, add 2, add 3, add 4, add 5…

Time is the *square* of the number of additions

Abstract data types

- Useful to separate the queue *interface* from the *implementation*
- Interface:

```
empty    :: Q a
add      :: a -> Q a -> Q a
remove   :: Q a -> Q a
front    :: Q a -> a
isEmpty :: Q a -> Bool
```

- Implementation:

```
data Q a = …
empty = ...
```

- Put the implementation in a *module*
- Allows programmers to switch implementation simply by changing imports

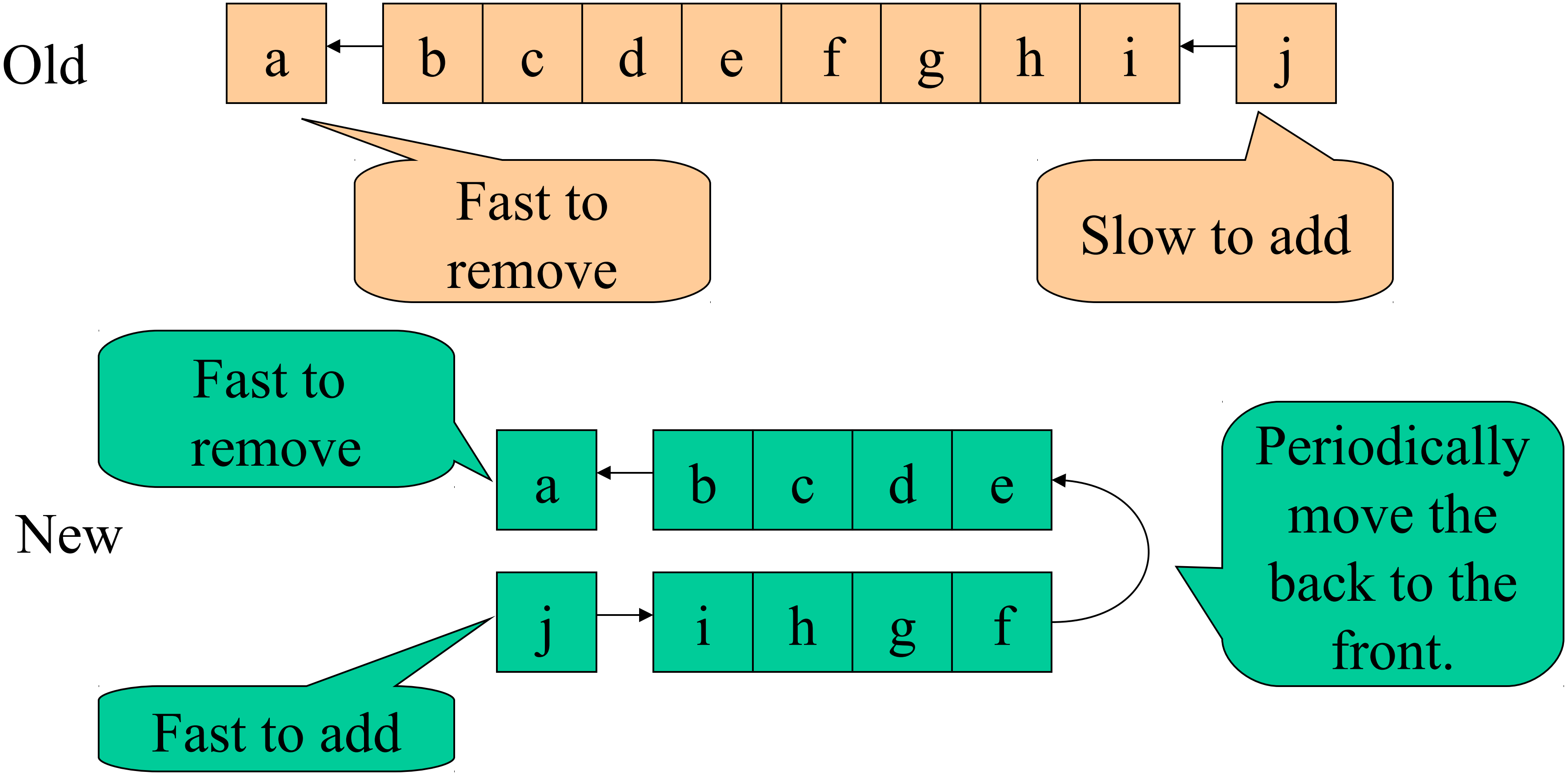
SlowQueue Module

```
module SlowQueue where

data Q a = Q [a] deriving (Eq, Show)

empty                = Q []
add x (Q xs)         = Q (xs++[x])
remove (Q (x:xs))    = Q xs
front (Q (x:xs))     = x
isEmpty (Q xs)       = null xs
```

New Idea: Store the Front and Back Separately



Old

Fast to remove

Slow to add

New

Fast to remove

Fast to add

Periodically move the back to the front.

Fast Datatype

```
data Q a = Q [a] [a]
  deriving (Eq, Show)
```

The front and the back part of the queue.

Fast Operations

```
empty                      = Q [] []
add x (Q front back)       = Q front (x:back)
remove (Q (x:front) back) = fixQ front back
front (Q (x:front) back)   = x
isEmpty (Q front back)     = nul    ont && null back
```

Flip the queue when we serve the last person in the front

Smart Constructor

```
fixQ []     back = Q (reverse back) []
fixQ front back = Q front back
```

This takes *one function call per element* in the back – each element is inserted into the back (one call), flipped (one call), and removed from the front (one call)

How can we test the fast functions?

- By using the original implementation as a *reference*
- The behaviour should be ”the same”
  - Check results
- First version is an *abstract model* that is ”obviously correct”

Comparing the Implementations

- They operate on different *types* of queues
- To compare, must convert between them
  - Can we convert a slow Q to a Q?
    - Where should we split the front from the back???
  - Can we convert a Q to a slow Q?

```
contents (Q front back) = Q (front++reverse back)
```

  - Retrieve the simple ”model” contents from the implementation

Accessing modules

```
import qualified SlowQueue as Slow

contents :: Q Int -> Slow.Q Int
contents (Q front back) =
                    Slow.Q (front ++ reverse back)
```

Qualified name

The Properties

The behaviour is the same, except for type conversion

```
prop_empty =
    contents empty == Slow.empty
prop_add x q =
    contents (add x q) == Slow.add x (contents q)
prop_remove q =
    contents (remove q) == Slow.remove (contents q)
prop_front q =
    front q == Slow.front (contents q)
prop_isEmpty q =
    isEmpty q == Slow.isEmpty (contents q)
```

Generating Qs

```
instance Arbitrary a => Arbitrary (Q a) where
  arbitrary = do front <- arbitrary
                 back <- arbitrary
                 return (Q front back)
```

A Bug!

```
Queues> quickCheck prop_remove
*** Failed! Exception: 'Queue.hs:22:0-42: Non-exhaustive patterns in function remove' (after 1 test):
Q [] []
```

Preconditions

- A condition that *must hold* before a function is called

```
prop_remove q =
  not (isEmpty q) ==>
    contents (remove q) == remove (contents q)
prop_front q =
  not (isEmpty q) ==>
    front q == front (contents q)
```

- Useful to be precise about these

Another Bug!

```
Queues> quickCheck prop_remove
*** Failed! Exception: 'Queue.hs:22:0-42:
   Non-exhaustive patterns in function
   remove' (after 2 tests):
Q [] [-1,0]
```

But this ought not to happen!

An Invariant

- Q values ought *never* to have an empty front, and a non-empty back!
- Formulate an *invariant*

```
invariant (Q front back) =
    not (null front && not (null back))
```

Testing the Invariant

```
prop_invariant :: Q Int -> Bool
prop_invariant q = invariant q
```

- Of course, it fails…

```
Queues> quickCheck prop_invariant
Falsifiable, after 4 tests:
Q [] [-1]
```

Fixing the Generator

```
instance Arbitrary a => Arbitrary (Q a) where
  arbitrary = do front <- arbitrary
                 back <- arbitrary
                 return (Q front
                     (if null front then [] else back))
```

- Now prop_invariant passes the tests

Testing the Invariant

- We’ve *written down* the invariant
- We’ve made sure that we only generate valid Qs as *test data*
- We must ensure that the *queue functions* only build valid Q values!
    - It is at this stage that the invariant is most useful

Invariant Properties

```
prop_empty_inv =
    invariant empty
prop_add_inv x q =
    invariant (add x q)
prop_remove_inv q =
    not (isEmpty q) ==>
        invariant (remove q)
```

A Bug in the Q operations!

```
Queues> quickCheck prop_add_inv
Falsifiable, after 2 tests:
0
Q [] []

Queues> add 0 (Q [] [])
Q [] [0]
```

The invariant is False!

Fixing add

```
add x (Q front back) = fixQ front (x:back)
```

- We must flip the queue when *the first element is inserted* into an empty queue
- Previous bugs were in our understanding (our properties) – this one is in our implementation code

Summary

- Data structures *store data*
- Obeying an *invariant*
- ... that functions and operations
  - can make use of (to search faster)
  - have to respect (to not break the invariant)
- Writing down and testing invariants and properties is a good way of finding errors

Another Datastructure: Tables

A *table* holds a collection of *keys* and associated *values*.

For example, a phone book is a table whose keys are names, and whose values are telephone numbers.

Problem: Given a table and a key, find the associated value.

John Hughes	1001
Mary Sheeran	1013
Koen Claessen	5424
Hans Svensson	1079

Table Lookup Using Lists

Since a table may contain any kind of keys and values, define a parameterised type:

E.g. [(”x”,1), (”y”,2)] ::
Table String Int

type Table k v = [(k, v)]

lookup ”y” …
→ Just 2

lookup :: Eq k => k -> Table k v -> Maybe v

lookup ”z” ...
→ Nothing

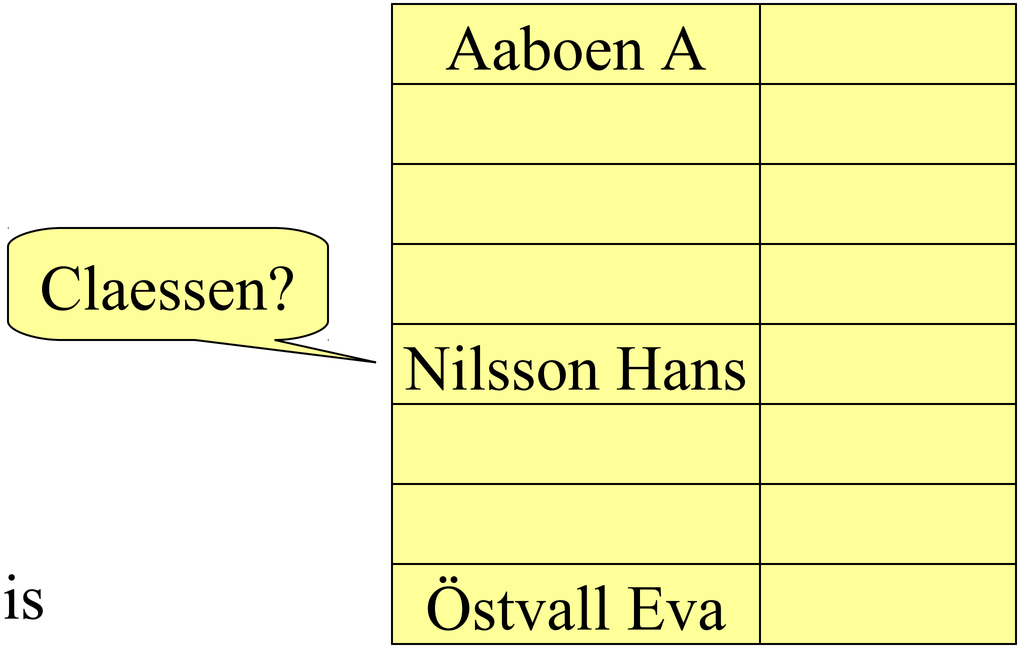
Finding Keys Fast

Finding keys by searching from the beginning is slow!

A better method:

look somewhere in the middle, and then look backwards or forwards depending on what you find.

(This assumes the table is sorted).



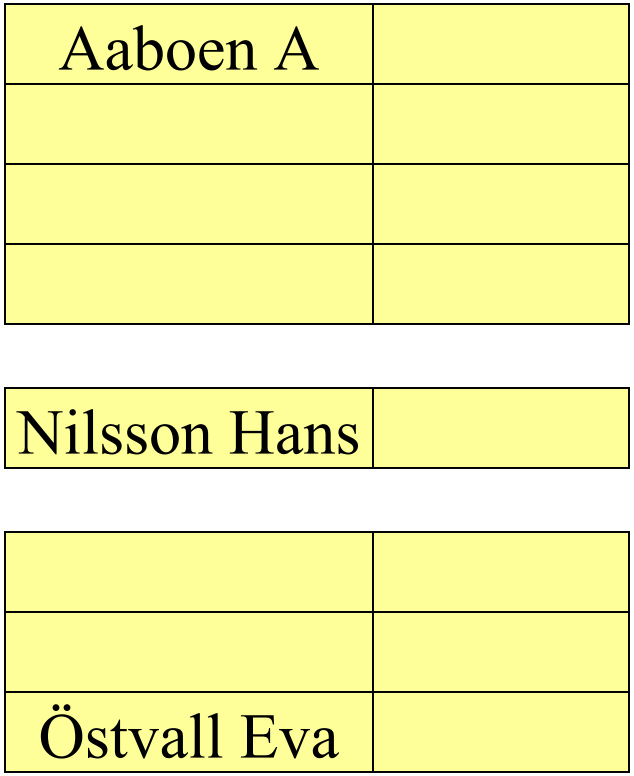
Claessen?

Aaboen A	
Nilsson Hans	
Östvall Eva	

Representing Tables

We must be able to break up a table fast, into:

- A smaller table of entries before the middle one,
- the middle entry,
- a table of entries after it.



```
data Table k v =
      Join (Table k v) k v (Table k v)
```

Quiz

What’s wrong with this (recursive) type?

```
data Table k v = Join (Table k v) k v (Table k v)
```

Quiz

What’s wrong with this (recursive) type? No base case!

```
data Table k v = Join (Table k v) k v (Table k v)
               | Empty
```

Add a base case.

Looking Up a Key

To look up a key in a table:

- If the table is empty, then the key is not found.
- Compare the key with the key of the middle element.
- If they are equal, return the associated value.
- If the key is less than the key in the middle, look in the first half of the table.
- If the key is greater than the key in the middle, look in the second half of the table.

Quiz

Define

```
lookupT :: Ord k => k -> Table k v -> Maybe v
```

Recall

```
data Table k v = Join (Table k v) k v (Table k v)
               |  Empty
```

Quiz

Define

```
        lookupT :: Ord k => k -> Table k v -> Maybe v


lookupT key Empty = Nothing

lookupT key (Join left k v right)

        | key == k        = Just v

        | key < k         = lookupT key left

        | key > k         = lookupT key right
```

Recursive type means a recursive function!

Inserting a New Key

We also need function to build tables. We define

```
insertT :: Ord k => k -> v -> Table k v -> Table k v
```

to insert a new key and value into a table.

We must be careful to insert the new entry in the right place, so that the keys remain in order.

Idea: Compare the new key against the middle one. Insert into the first or second half as appropriate.

Defining Insert

```
insertT key val Empty = Join Empty key val Empty
insertT key val (Join left k v right)
        | key <= k = Join (insertT key val left) k v right
        | key > k   = Join left k v (insertT key val right)
```

Many forget to join up the new right half with the old left half again.

Efficiency

On average, how many comparisons does it take to find a key in a table of 1000 entries, using a list and using the new method?

Using a list: 500

Using the new method: 10

Testing

- How should we test the Table operations?
  - By comparison with the list operations

```
prop_lookupT k t =
   lookupT k t == lookup k (contents t)
prop_insertT k v t =
   contents (insertT k v t) == insert (k,v) (contents t)
```

```
contents :: Table k v -> [(k,v)]
```

Generating Random Tables

- Recursive types need recursive generators

```
instance (Arbitrary k, Arbitrary v) =>
                Arbitrary (Table k v) where
```

We can generate arbitrary Tables...

...provided we can generate keys and values

Generating Random Tables

- Recursive types need recursive generators

```
instance (Arbitrary k, Arbitrary v) =>
    Arbitrary (Table k v) where
    arbitrary = oneof [ return Empty,
                        do k <- arbitrary
                           v <- arbitrary
                           left <- arbitrary
                           right <- arbitrary
                           return (Join left k v right) ]
```

Quiz:
What is wrong with
this generator?

Controlling the Size of Tables

- Generate tables with *at most n elements*

```
table s = frequency [(1, return Empty),
                     (s, do k <- arbitrary
                            v <- arbitrary
                            l <- table (s `div` 2)
                            r <- table (s `div` 2)
                            return (Join l k v r))]
```

```
instance (Arbitrary k, Arbitrary v) =>
                    Arbitrary (Table k v) where
  arbitrary = sized table
```

Testing Table Properties

```
prop_lookupT k t = lookupT k t == lookup k (contents t)
```

```
Main> quickCheck prop_lookupT
Falsifiable, after 10 tests:
0
Join Empty 2 (-2) (Join Empty 0 0 Empty)
Main> contents (Join Empty 2 (-2)  …)
[(2,-2),(0,0)]
```

What's wrong?

Tables must be Ordered!

```
prop_invTable :: Table Integer Integer -> Bool
prop_invTable tab = ordered ks
    where ks = [k | (k,v) <- contents tab]
```

- Tables should satisfy an important *invariant*.

```
Main> quickCheck prop_invTable
Falsifiable, after 4 tests:
Join Empty 3 3 (Join Empty 0 3 Empty)
```

How to Generate Ordered Tables?

- Generate a random list,
  - Take the *first* (key,value) to be at the root
  - Take all the *smaller* keys to go in the left subtree
  - Take all the *larger* keys to go in the right subtree

Converting a List to a Table

```
-- table kvs converts a list of key-value pairs into a Table
-- satisfying the ordering invariant
table :: Ord k => [(k,v)] -> Table k v
table []             = Empty
table ((k,v):kvs) = Join (table smaller) k v (table larger)
  where
   smaller = [(k',v') | (k',v') <- kvs, k' < k]
   larger   = [(k',v') | (k',v') <- kvs, k' > k]
```

Generating Ordered Tables

Keys must have an ordering

```
instance (Ord k, Arbitrary k, Arbitrary v) =>
                          Arbitrary (Table k v) where
   arbitrary = do kvs <- arbitrary
                  return (table kvs)
```

List of keys and values

Testing the Properties

- Now the invariant holds, but the properties don’t!

```
Main> quickCheck prop_invTable
OK, passed 100 tests.
Main> quickCheck prop_lookupT
Falsifiable, after 7 tests:
-1
Join (Join Empty (-1) (-2) Empty) (-1) (-1) Empty
```

More Testing

```
prop_insertT k v t =
   insert (k,v) (contents t)
   == contents (insertT k v t)
```

```
Main> quickCheck prop_insertT
Falsifiable, after 8 tests:
0
0
Join Empty 0 (-1) Empty
```

What's
wrong?

The Bug

```
insertT key val Empty = Join Empty key val Empty
insertT key val (Join left k v right) =
    | key <= k = Join (insertT key val left) k v right
    | key > k   = Join left k v (insertT key val right)
```

Inserts duplicate keys!

The Fix

```
insertT key val Empty = Join Empty key val Empty
insertT key val (Join left k v right) =
   | key < k   = Join (insertT key val left) k v right
   | key==k    = Join left k val right
   | key > k   = Join left k v (insertT key val right)
```

```
prop_invTable :: Table Integer Integer -> Bool
prop_invTable tab = ordered ks && ks == nub ks
   where ks = [k | (k,v) <- contents tab]
```

(and fix the table generator)

Testing Again

```
Main> quickCheck prop_insertT
Falsifiable, after 6 tests:
-2
2
Join Empty (-2) 1 Empty
```

Testing Again

```
Main> quickCheck prop_insertT
Falsifiable, after 6 tests:
-2
2
Join Empty (-2) 1 Empty

Main> insertT (-2) 2 (Join Empty (-2) 1 Empty)
Join Empty (-2) 2 Empty
```

Testing Again

```
Main> quickCheck prop_insertT
Falsifiable, after 6 tests:
-2
2
Join Empty (-2) 1 Empty

Main> insertT (-2) 2 (Join Empty (-2) 1 Empty)
Join Empty (-2) 2 Empty

Main> insert (-2,2) [(-2,1)]
[(-2,1),(-2,2)]
```

insert doesn't *remove* the old key-value pair when keys clash – the wrong model!

Fixing prop_insertT

- Ad hoc fix:

```
prop_insertT k v t =
  insert (k,v) [(k',v') | (k',v') <- contents t, k' /= k] ==
     contents (insertT k v t)
```

Data.Map

- The standard module Data.Map contains an advanced tree-based implementation of tables

Summary

- Recursive data-types can store data in different ways
- Clever choices of datatypes and algorithms can improve performance dramatically
- Careful thought about *invariants* is needed to get such algorithms right!
- Formulating properties and invariants, and testing them, reveals bugs early