

Software Engineering using Formal Methods

Formal Modeling with Linear Temporal Logic

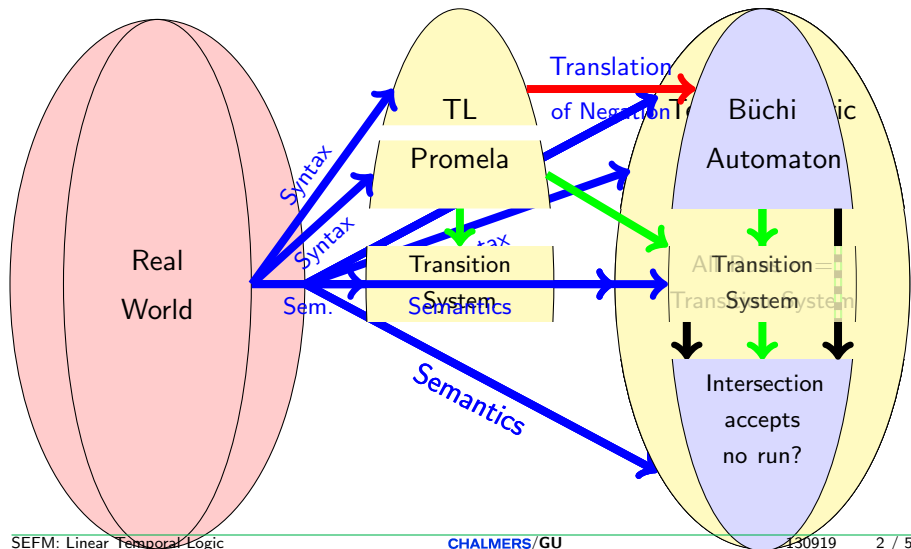
Wolfgang Ahrendt

19th September 2013

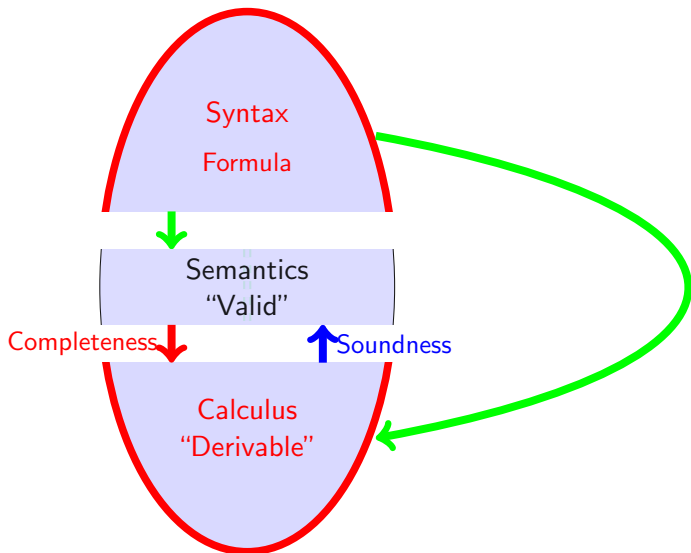
Recapitulation: Formalisation: Syntax, Semantics

Formalisation: Syntax, Semantics, Proving

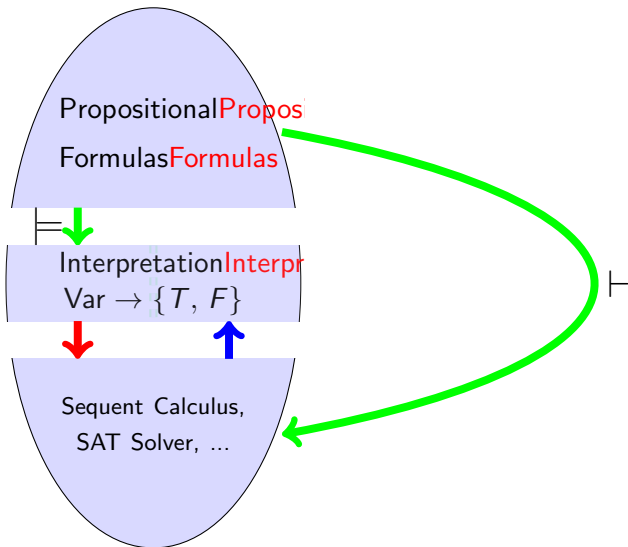
Formal Verification: Model Checking



The Big Picture: Syntax, Semantics, Calculus



Simplest Case: Propositional Logic—Syntax



Syntax of Propositional Logic

Signature

A set of **Propositional Variables** \mathcal{P} (with typical elements p, q, r, \dots)

Propositional Connectives

true, false, \wedge , \vee , \neg , \rightarrow , \leftrightarrow

Set of Propositional Formulas For_0

- ▶ Truth constants true, false and variables \mathcal{P} are formulas
- ▶ If ϕ and ψ are formulas then

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi, \quad \phi \leftrightarrow \psi$$

are also formulas

- ▶ There are no other formulas (inductive definition)

Remark on Concrete Syntax

	Text book	SPIN
Negation	\neg	!
Conjunction	\wedge	&&
Disjunction	\vee	
Implication	\rightarrow, \supset	\rightarrow
Equivalence	\leftrightarrow	\leftrightarrow

We use mostly the textbook notation
Except for tool-specific slides, input files

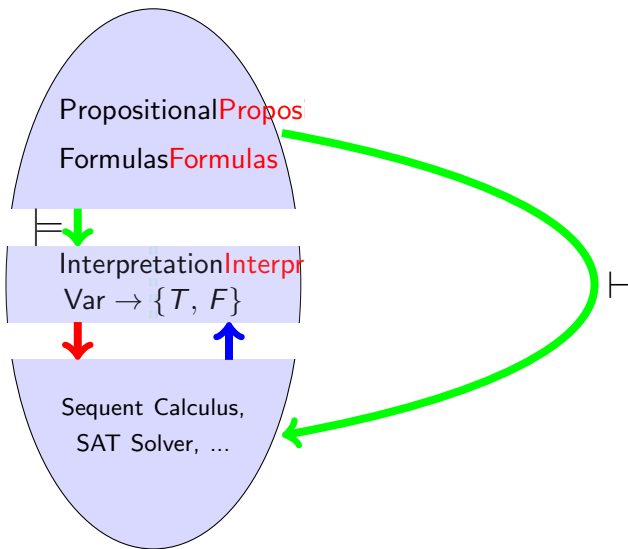
Propositional Logic Syntax: Examples

Let $\mathcal{P} = \{p, q, r\}$ be the set of propositional variables

Are the following character sequences also propositional formulas?

- ▶ $\text{true} \rightarrow p$ ✓
- ▶ $(p(q \wedge r)) \vee p$ ✗
- ▶ $p \rightarrow (q \wedge)$ ✗
- ▶ $\text{false} \wedge (p \rightarrow (q \wedge r))$ ✓

Simplest Case: Propositional Logic—Syntax



Semantics of Propositional Logic

Interpretation \mathcal{I}

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
\mathcal{I}_1	F	F
\mathcal{I}_2	T	F
\vdots	\vdots	\vdots

Semantics of Propositional Logic

Interpretation \mathcal{I}

Assigns a truth value to each propositional variable

$$\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$$

Valuation Function

$val_{\mathcal{I}}$: Continuation of \mathcal{I} on For_0

$$val_{\mathcal{I}} : For_0 \rightarrow \{T, F\}$$

$$val_{\mathcal{I}}(\text{true}) = T$$

$$val_{\mathcal{I}}(\text{false}) = F$$

$$val_{\mathcal{I}}(p_i) = \mathcal{I}(p_i)$$

(cont'd next page)

Semantics of Propositional Logic (Cont'd)

Valuation function (Cont'd)

$$\text{val}_{\mathcal{I}}(\neg\phi) = \begin{cases} T & \text{if } \text{val}_{\mathcal{I}}(\phi) = F \\ F & \text{otherwise} \end{cases}$$

$$\text{val}_{\mathcal{I}}(\phi \wedge \psi) = \begin{cases} T & \text{if } \text{val}_{\mathcal{I}}(\phi) = T \text{ and } \text{val}_{\mathcal{I}}(\psi) = T \\ F & \text{otherwise} \end{cases}$$

$$\text{val}_{\mathcal{I}}(\phi \vee \psi) = \begin{cases} T & \text{if } \text{val}_{\mathcal{I}}(\phi) = T \text{ or } \text{val}_{\mathcal{I}}(\psi) = T \\ F & \text{otherwise} \end{cases}$$

$$\text{val}_{\mathcal{I}}(\phi \rightarrow \psi) = \begin{cases} T & \text{if } \text{val}_{\mathcal{I}}(\phi) = F \text{ or } \text{val}_{\mathcal{I}}(\psi) = T \\ F & \text{otherwise} \end{cases}$$

$$\text{val}_{\mathcal{I}}(\phi \leftrightarrow \psi) = \begin{cases} T & \text{if } \text{val}_{\mathcal{I}}(\phi) = \text{val}_{\mathcal{I}}(\psi) \\ F & \text{otherwise} \end{cases}$$

Valuation Examples

Example

Let $\mathcal{P} = \{p, q\}$

$$p \rightarrow (q \rightarrow p)$$

	p	q
\mathcal{I}_1	F	F
\mathcal{I}_2	T	F

...

How to evaluate $p \rightarrow (q \rightarrow p)$ in \mathcal{I}_2 ?

$$\text{val}_{\mathcal{I}_2}(p \rightarrow (q \rightarrow p)) = T \text{ iff } \text{val}_{\mathcal{I}_2}(p) = F \text{ or } \text{val}_{\mathcal{I}_2}(q \rightarrow p) = T$$

$$\text{val}_{\mathcal{I}_2}(p) = \mathcal{I}_2(p) = T$$

$$\text{val}_{\mathcal{I}_2}(q \rightarrow p) = T \text{ iff } \text{val}_{\mathcal{I}_2}(q) = F \text{ or } \text{val}_{\mathcal{I}_2}(p) = T$$

$$\text{val}_{\mathcal{I}_2}(q) = \mathcal{I}_2(q) = F$$

Semantic Notions of Propositional Logic

Let $\phi \in For_0$, $\Gamma \subseteq For_0$

Definition (Satisfying Interpretation, Consequence Relation)

\mathcal{I} satisfies ϕ (write: $\mathcal{I} \models \phi$) iff $val_{\mathcal{I}}(\phi) = T$

ϕ follows from Γ (write: $\Gamma \models \phi$) iff for all interpretations \mathcal{I} :

If $\mathcal{I} \models \psi$ for all $\psi \in \Gamma$ then also $\mathcal{I} \models \phi$

Definition (Satisfiability, Validity)

A formula is **satisfiable** if it is satisfied by **some** interpretation.

If **every** interpretation satisfies ϕ (write: $\models \phi$) then ϕ is called **valid**.

Semantics of Propositional Logic: Examples

Formula (same as before)

$$p \rightarrow (q \rightarrow p)$$

Is this formula valid?

$$\models p \rightarrow (q \rightarrow p) ?$$

Semantics of Propositional Logic: Examples

$$p \wedge ((\neg p) \vee q)$$

Satisfiable?



Satisfying Interpretation?

$$\mathcal{I}(p) = T, \mathcal{I}(q) = T$$

Other Satisfying Interpretations?



Therefore, also not valid!

$$p \wedge ((\neg p) \vee q) \models q \vee r$$

Does it hold? Yes. Why?

An Exercise in Formalisation

```
1 byte n;  
2 active proctype [2] P() {  
3   n = 0;  
4   n = n + 1  
5 }
```

Can we characterise the states of P propositionally?

Find a propositional formula ϕ_P which is true if and only if (iff) it describes a possible state of P.

$$\phi_P := \left(((PC0_3 \wedge \neg PC0_4 \wedge \neg PC0_5) \vee \dots) \wedge ((\neg PC0_5 \wedge \neg PC1_5) \implies (\neg N_5 \wedge \dots \wedge \neg N_7)) \wedge \dots \right)$$

An Exercise in Formalisation

```
1 byte n;  
2 active proctype [2] P() {  
3   n = 0;  
4   n = n + 1  
5 }
```

$\mathcal{P} : N_0, N_1, N_2, \dots, N_7$ 8-bit representation of byte
 $PC0_3, PC0_4, PC0_5, PC1_3, PC1_4, PC1_5$ next instruction pointer

Which interpretations do we need to “exclude”?

- ▶ The variable n is represented by eight bits, all values possible
- ▶ A process cannot be at two positions at the same time
- ▶ If neither process 0 nor process 1 are at position 5, then n is zero
- ▶ ...

$$\phi_P := \left(\begin{array}{l} ((PC0_3 \wedge \neg PC0_4 \wedge \neg PC0_5) \vee \dots) \wedge \\ ((\neg PC0_5 \wedge \neg PC1_5) \implies (\neg N_0 \wedge \dots \wedge \neg N_7)) \wedge \dots \end{array} \right)$$

Is Propositional Logic Enough?

Can design for a program P a formula Φ_P describing all reachable states

For a given property Ψ the consequence relation

$$\Phi_P \models \Psi$$

holds when Ψ is true in any possible state reachable in any run of P

But How to Express Properties Involving State Changes?

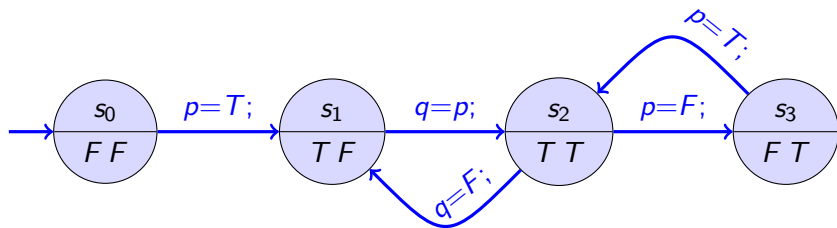
In any run of a program P

- ▶ n will become greater than 0 eventually?
- ▶ n changes its value infinitely often

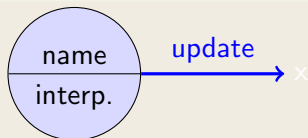
etc.

⇒ Need a more expressive logic: (Linear) Temporal Logic

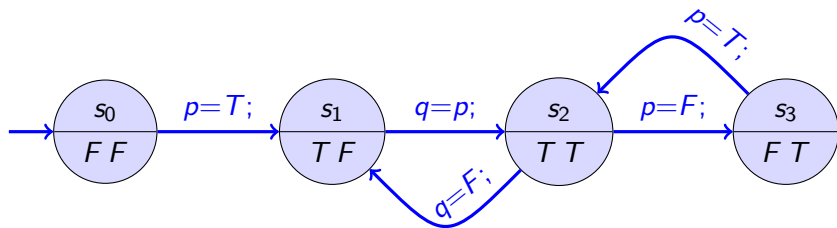
Transition systems (aka Kripke Structures)



Notation



Transition systems (aka Kripke Structures)

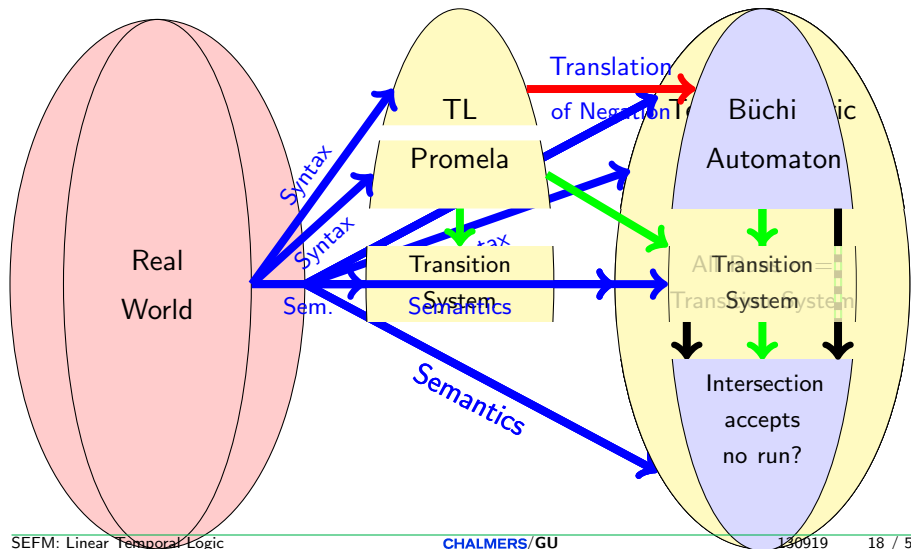


- ▶ Each state s_i has its own propositional interpretation I_i
 - ▶ Convention: list values of variables in ascending lexicographic order
- ▶ Computations, or **runs**, are *infinite* paths through states
 - ▶ Intuitively 'finite' runs modelled by looping on last state
- ▶ How to express (for example) that p changes its value infinitely often in each run?

Recapitulation: Formalisation

Formalisation: Syntax, Semantics

Formal Verification: Model Checking



(Linear) Temporal Logic—Syntax

An extension of propositional logic that allows to specify **properties of all runs**

Syntax

Based on propositional signature and syntax

Extension with three connectives:

Always If ϕ is a formula then so is $\Box\phi$

Eventually If ϕ is a formula then so is $\Diamond\phi$

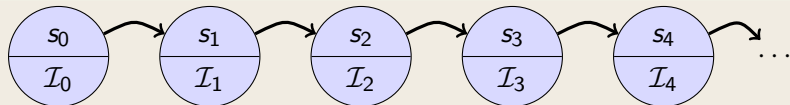
Until If ϕ and ψ are formulas then so is $\phi\mathcal{U}\psi$

Concrete Syntax

	text book	SPIN
Always	\Box	$[]$
Eventually	\Diamond	$\langle \rangle$
Until	\mathcal{U}	\mathcal{U}

Temporal Logic—Semantics

A run σ is an infinite chain of states



\mathcal{I}_j propositional interpretation of variables in j -th state

Write more compactly $s_0 s_1 s_2 s_3 \dots$

If $\sigma = s_0 s_1 \dots$, then $\sigma|_i$ denotes the **suffix** $s_i s_{i+1} \dots$ of σ .

Temporal Logic—Semantics (Cont'd)

Valuation of temporal formula relative to **run**: infinite sequence of states

Definition (Validity Relation)

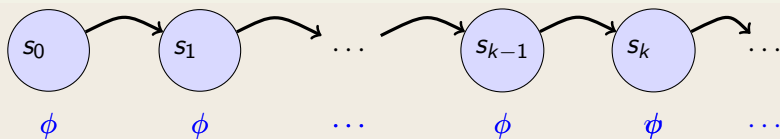
Validity of temporal formula depends on runs $\sigma = s_0 s_1 \dots$

$\sigma \models p$	iff	$\mathcal{I}_0(p) = T$, for $p \in \mathcal{P}$.
$\sigma \models \neg\phi$	iff	not $\sigma \models \phi$ (write $\sigma \not\models \phi$)
$\sigma \models \phi \wedge \psi$	iff	$\sigma \models \phi$ and $\sigma \models \psi$
$\sigma \models \phi \vee \psi$	iff	$\sigma \models \phi$ or $\sigma \models \psi$
$\sigma \models \phi \rightarrow \psi$	iff	$\sigma \not\models \phi$ or $\sigma \models \psi$

Temporal connectives?

Temporal Logic—Semantics (Cont'd)

Run σ



Definition (Validity Relation for Temporal Connectives)

Given a run $\sigma = s_0 s_1 \dots$

$\sigma \models \Box\phi$ iff $\sigma|_k \models \phi$ for all $k \geq 0$

$\sigma \models \Diamond\phi$ iff $\sigma|_k \models \phi$ for some $k \geq 0$

$\sigma \models \phi\mathcal{U}\psi$ iff $\sigma|_k \models \psi$ for some $k \geq 0$, and $\sigma|_j \models \phi$ for all $0 \leq j < k$
(if $k = 0$ then ϕ needs never hold)

Safety and Liveness Properties

Safety Properties

- ▶ Always-formulas called **safety properties**:
“something bad never happens”
- ▶ Let `mutex` (“mutual exclusion”) be a variable that is true when two processes do not access a critical resource at the same time
- ▶ $\square \text{mutex}$ expresses that simultaneous access never happens

Liveness Properties

- ▶ Eventually-formulas called **liveness properties**:
“something good happens eventually”
- ▶ Let `s` be variable that is true when a process delivers a service
- ▶ $\diamond s$ expresses that service is eventually provided

What does this mean? Infinitely Often

$$\sigma \models \Box \Diamond \phi$$

“During run σ the formula ϕ becomes true infinitely often”

Validity of Temporal Logic

Definition (Validity)

ϕ is **valid**, write $\models \phi$, iff ϕ is valid in **all** runs $\sigma = s_0 s_1 \dots$.

Recall that each run $s_0 s_1 \dots$ essentially is an infinite sequence of interpretations $\mathcal{I}_0 \mathcal{I}_1 \dots$

Representation of Runs

Can represent a set of runs as a sequence of propositional formulas:

- ▶ $\phi_0 \phi_1, \dots$ represents all runs $s_0 s_1 \dots$ such that $s_i \models \phi_i$ for $i \geq 0$

Semantics of Temporal Logic: Examples

$$\diamond \Box \phi$$

Valid?

No, there is a run where it is not valid:

$$(\neg \phi \neg \phi \neg \phi \dots)$$

Valid in some run?

Yes, for example: $(\neg \phi \phi \phi \dots)$

$$\Box \phi \rightarrow \phi$$

$$(\neg \Box \phi) \leftrightarrow (\diamond \neg \phi)$$

$$\diamond \phi \leftrightarrow (\text{true } \mathcal{U} \phi)$$

All are valid! (proof is exercise)

- ▶ \Box is reflexive
- ▶ \Box and \diamond are dual connectives
- ▶ \Box and \diamond can be expressed with only using \mathcal{U}

Transition Systems: Formal Definition

Definition (Transition System)

A **transition system** $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$ is composed of a set of **states** S , a set $\emptyset \neq Ini \subseteq S$ of **initial states**, a **transition relation** $\delta \subseteq S \times S$, and a **labeling** \mathcal{I} of each state $s \in S$ with a propositional interpretation \mathcal{I}_s .

Definition (Run of Transition System)

A **run** of \mathcal{T} is a sequence of states $\sigma = s_0 s_1 \cdots$ such that $s_0 \in Ini$ and for all i is $s_i \in S$ as well as $(s_i, s_{i+1}) \in \delta$.

Temporal Logic—Semantics (Cont'd)

Extension of validity of temporal formulas to **transition systems**:

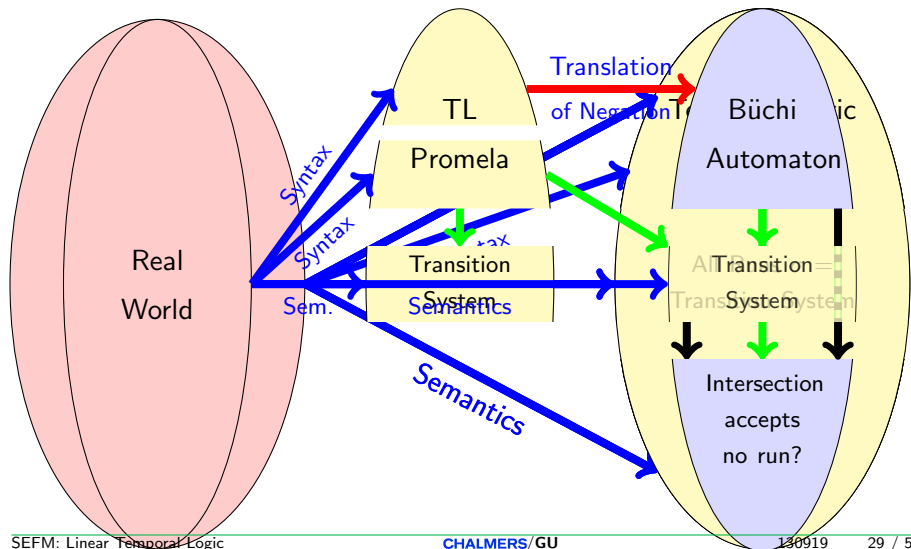
Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$, a temporal formula ϕ is **valid in \mathcal{T}** (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs σ of \mathcal{T} .

Recapitulation: Formalisation: Syntax, Semantics

Formalisation: Syntax, Semantics, Proving

Formal Verification: Model Checking



Given a finite alphabet (vocabulary) Σ

An ω -word $w \in \Sigma^{*\omega}$ is a **n** infinite sequence

$$w = a_0 \cdots a_n \cdots$$

with $a_i \in \Sigma, i \in \{0, \dots, n\} \mathbb{N}$

$\mathcal{L}^\omega \subseteq \Sigma^{*\omega}$ is called a **n** ω -language

Büchi Automaton

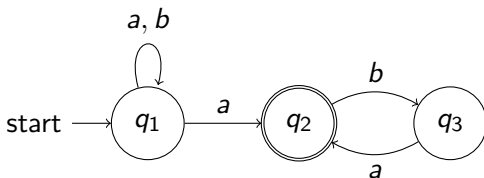
Definition (Büchi Automaton)

A (non-deterministic) **Büchi automaton** over an alphabet Σ consists of a

- ▶ finite, non-empty set of **locations** Q
- ▶ a non-empty set of **initial/start** locations $I \subseteq Q$
- ▶ a set of **accepting** locations $F = \{F_1, \dots, F_n\} \subseteq Q$
- ▶ a transition relation $\delta \subseteq Q \times \Sigma \times Q$

Example

$\Sigma = \{a, b\}$, $Q = \{q_1, q_2, q_3\}$, $I = \{q_1\}$, $F = \{q_2\}$



Büchi Automaton—Executions and Accepted Words

Definition (Execution)

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton over alphabet Σ .

An **execution** of \mathcal{B} is a pair (w, v) , with

▶ $w = a_0 \cdots a_k \cdots \in \Sigma^\omega$

▶ $v = q_0 \cdots q_k \cdots \in Q^\omega$

where $q_0 \in I$, and $(q_i, a_i, q_{i+1}) \in \delta$, for all $i \in \mathbb{N}$

Definition (Accepted Word)

A Büchi automaton \mathcal{B} **accepts** a word $w \in \Sigma^\omega$, if there exists an execution (w, v) of \mathcal{B} where **some accepting location** $f \in F$ appears **infinitely** often in v

Büchi Automaton—Language

Let $\mathcal{B} = (Q, I, F, \delta)$ be a Büchi automaton, then

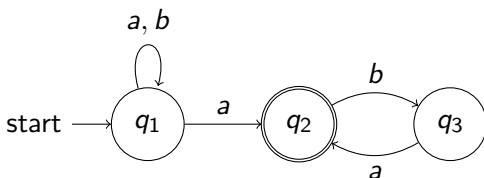
$$\mathcal{L}^\omega(\mathcal{B}) = \{w \in \Sigma^\omega \mid w \in \Sigma^\omega \text{ is an accepted word of } \mathcal{B}\}$$

denotes the ω -language **recognised** by \mathcal{B} .

An ω -language for which an accepting Büchi automaton exists is called **ω -regular** language.

Example, ω -Regular Expression

Which language is accepted by the following Büchi automaton?



Solution: $(a + b)^*(ab)^\omega$ [NB: $(ab)^\omega = a(ba)^\omega$]

ω -regular expressions like standard regular expression

ab a then b

$a + b$ a or b

a^* arbitrarily, but **finitely** often a

new: a^ω **infinitely** often a

Decidability, Closure Properties

Many properties for regular finite automata hold also for Büchi automata

Theorem (Decidability)

It is decidable whether the accepted language $\mathcal{L}^\omega(\mathcal{B})$ of a Büchi automaton \mathcal{B} is empty.

Theorem (Closure properties)

The set of ω -regular languages is closed with respect to intersection, union and complement:

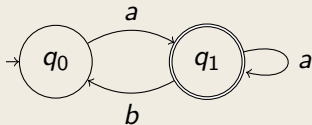
- ▶ *if $\mathcal{L}_1, \mathcal{L}_2$ are ω -regular then $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are ω -regular*
- ▶ *\mathcal{L} is ω -regular then $\Sigma^\omega \setminus \mathcal{L}$ is ω -regular*

But in contrast to regular finite automata

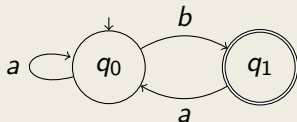
Non-deterministic Büchi automata are strictly more expressive than deterministic ones

Büchi Automata—More Examples

Language: $a(a + ba)^\omega$



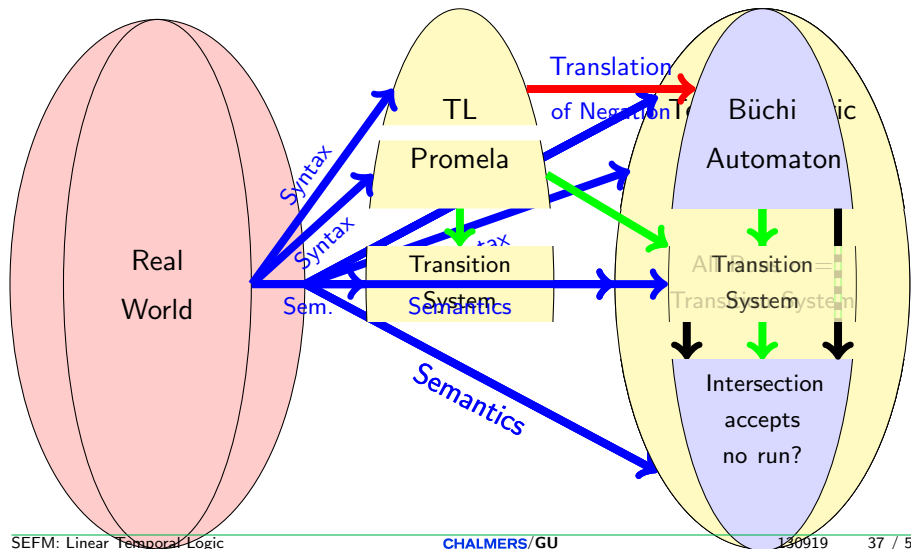
Language: $(a^*ba)^\omega$



Recapitulation: Formalisation: Syntax, Semantics

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Formal Verification: Model Checking



Linear Temporal Logic and Büchi Automata

LTL and Büchi Automata are connected

Recall

Definition (Validity Relation)

Given a transition system $\mathcal{T} = (S, Ini, \delta, \mathcal{I})$, a temporal formula ϕ is **valid in \mathcal{T}** (write $\mathcal{T} \models \phi$) iff $\sigma \models \phi$ for all runs σ of \mathcal{T} .

A run of the transition system is an infinite sequence of interpretations I

Intended Connection

Given an LTL formula ϕ :

Construct a Büchi automaton accepting exactly those runs (infinite sequences of interpretations) that satisfy ϕ

Encoding an LTL Formula as a Büchi Automaton

\mathcal{P} set of propositional variables, e.g., $\mathcal{P} = \{r, s\}$

Alphabet Σ of Büchi automaton

A state transition of Büchi automaton must represent an interpretation

Let Σ (i.e., the alphabet of the automata) be set of all interpretations over \mathcal{P} , i.e., $\Sigma = 2^{\mathcal{P}}$

Example

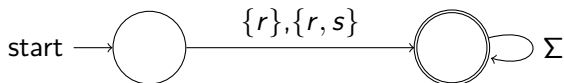
$$\Sigma = \{\emptyset, \{r\}, \{s\}, \{r, s\}\}$$

$$I_{\emptyset}(r) = F, I_{\emptyset}(s) = F, I_{\{r\}}(r) = T, I_{\{r\}}(s) = F, \dots$$

Büchi Automaton for LTL Formula By Example

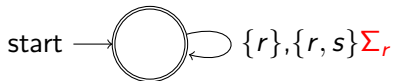
Example (Büchi automaton for formula r over $\mathcal{P} = \{r, s\}$)

A Büchi automaton \mathcal{B} accepting exactly those runs σ satisfying r



In the first state s_0 (of σ) at least r must hold, the rest is arbitrary

Example (Büchi automaton for formula $\Box r$ over $\mathcal{P} = \{r, s\}$)

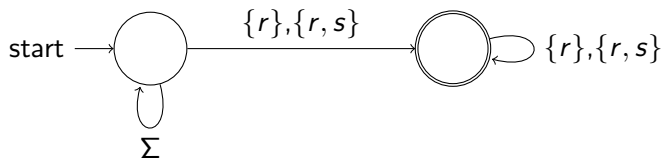


$$\Sigma_r := \{I \mid I \in \Sigma, r \in I\}$$

In **all** states s (of σ) at least r must hold

Büchi Automaton for LTL Formula By Example

Example (Büchi automaton for formula $\diamond\Box r$ over $\mathcal{P} = \{r, s\}$)

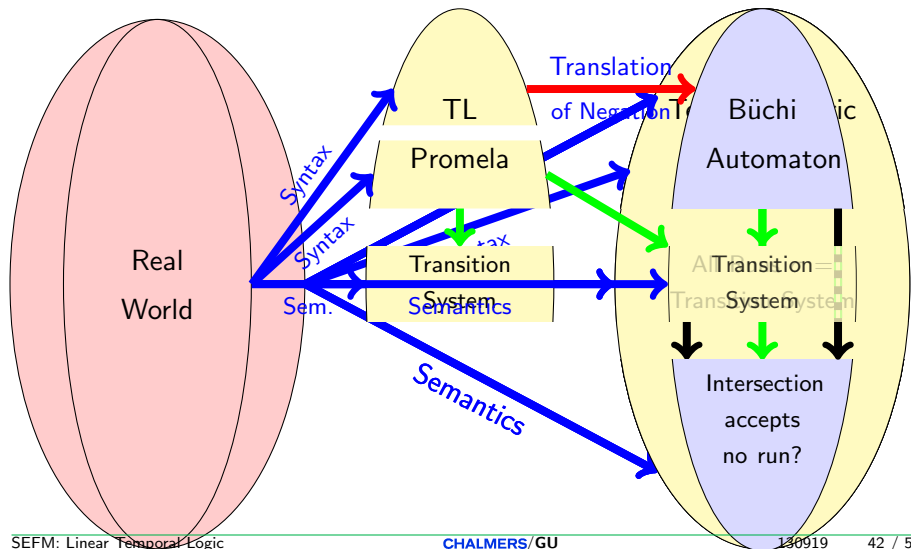


Recapitulation: Formalisation

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Formal Verification: Model Checking



Model Checking

Check whether a formula is valid in all runs of a transition system

Given a transition system \mathcal{T} (e.g., derived from a PROMELA program)

Verification task: is the LTL formula ϕ satisfied in all runs of \mathcal{T} , i.e.,

$$\mathcal{T} \models \phi \quad ?$$

Temporal model checking with SPIN: Topic of next lecture

Today: Basic principle behind SPIN model checking

$$\mathcal{T} \models \phi \quad ?$$

1. Represent transition system \mathcal{T} as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ such that $\mathcal{B}_{\mathcal{T}}$ accepts exactly those words corresponding to runs through \mathcal{T}
2. Construct Büchi automaton $\mathcal{B}_{\neg\phi}$ for **negation** of formula ϕ
3. If

$$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) = \emptyset$$

then ϕ holds.

If

$$\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi}) \neq \emptyset$$

then each element of the set is a counterexample for ϕ .

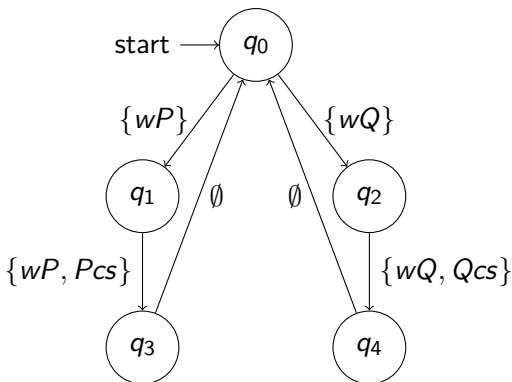
To check $\mathcal{L}^{\omega}(\mathcal{B}_{\mathcal{T}}) \cap \mathcal{L}^{\omega}(\mathcal{B}_{\neg\phi})$ construct intersection automaton and search for cycle through accepting state

Representing a Model as a Büchi Automaton

First Step: Represent transition system \mathcal{T} as Büchi automaton $\mathcal{B}_{\mathcal{T}}$ accepting exactly those words representing a run of \mathcal{T}

Example

```
active proctype P () {  
do  
  :: atomic {  
    !wQ; wP = true  
  };  
  Pcs = true;  
  atomic {  
    Pcs = false;  
    wP = false  
  }  
od }
```



First location skipped and second made atomic just to keep automaton small; similar code for process Q

Büchi Automaton $B_{\neg\phi}$ for $\neg\phi$

Second Step:

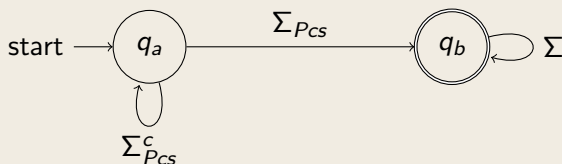
Construct Büchi Automaton corresponding to negated LTL formula

$\mathcal{T} \models \phi$ holds iff there is **no** accepting run of \mathcal{T} for $\neg\phi$

Simplify $\neg\phi = \neg\Box\neg Pcs = \Diamond Pcs$

Büchi Automaton $B_{\neg\phi}$

$$\mathcal{P} = \{wP, wQ, Pcs, Qcs\}, \Sigma = 2^{\mathcal{P}}$$



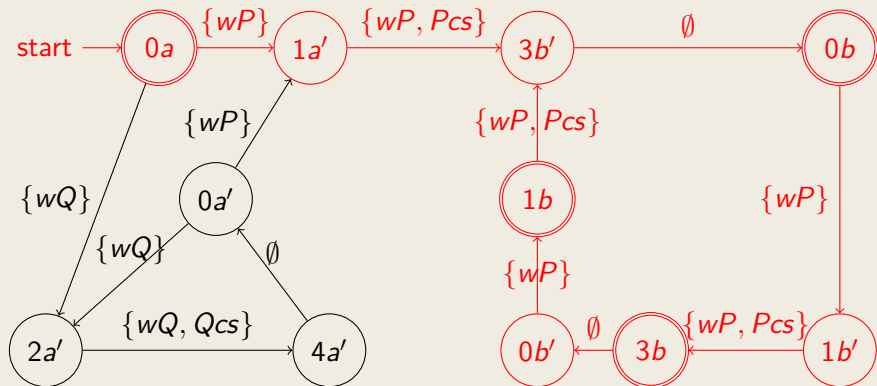
$$\Sigma_{Pcs} = \{I \mid I \in \Sigma, Pcs \in I\}, \quad \Sigma_{Pcs}^c = \Sigma - \Sigma_{Pcs}$$

Checking for Emptiness of Intersection Automaton

Third Step: $\mathcal{L}^\omega(\mathcal{B}_T) \cap \mathcal{L}^\omega(\mathcal{B}_{\neg\phi}) = \neq \emptyset$?

Counterexample Construction of intersection automaton: Appendix

Intersection Automaton



Literature for this Lecture

Ben-Ari Section 5.2.1
(only syntax of LTL)

Baier and Katoen Principles of Model Checking, May 2008, The MIT Press, ISBN: 0-262-02649-X

Appendix I:

Intersection Automaton

—

Construction

Construction of Intersection Automaton

Given: two Büchi automata $\mathcal{B}_i = (Q_i, \delta_i, l_i, F_i)$, $i = 1, 2$

Wanted: a Büchi automaton

$$\mathcal{B}_{1 \cap 2} = (Q_{1 \cap 2}, \delta_{1 \cap 2}, l_{1 \cap 2}, F_{1 \cap 2})$$

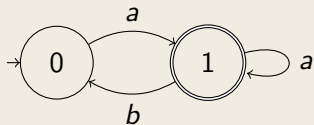
accepting a word w iff w is accepted by \mathcal{B}_1 **and** \mathcal{B}_2

Maybe just the product automaton as for regular automata?

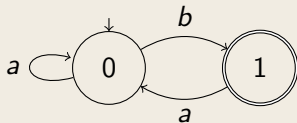
First Attempt: Product Automata for Intersection

$\Sigma = \{a, b\}$, $a(a + ba)^\omega \cap (a^*ba)^\omega = \emptyset$? No, e.g., $a(ba)^\omega$

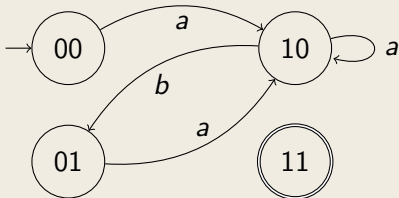
$a(a + ba)^\omega$:



$(a^*ba)^\omega$:

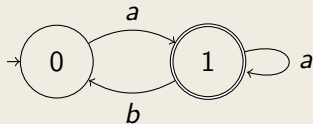


Product Automaton: **accepting location 11 never reached**

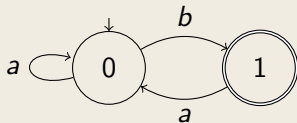


Explicit Construction of Intersection Automaton

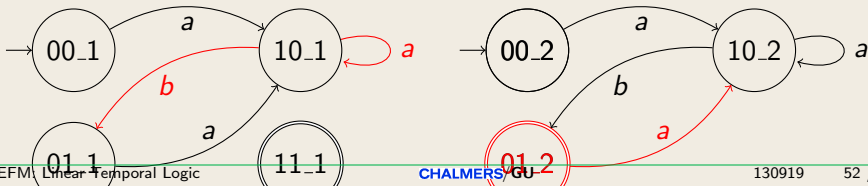
$a(a + ba)^\omega$:



$(a^*ba)^\omega$:

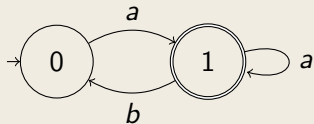


- (i) Product Automaton (ii) Reachable States (iii) Clone (iv) Initial States Restricted to First Copy (v) Final States Restricted to First Automaton of First Copy (vi) Ensure Acceptance in Both Copies $1 \rightarrow 2$ (vii) Ensure Acceptance in Both Copies $2 \rightarrow 1$ (viii) Transitions of Product Automaton

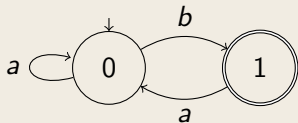


Explicit Construction of Intersection Automaton

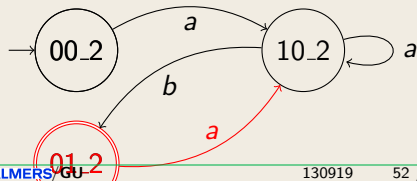
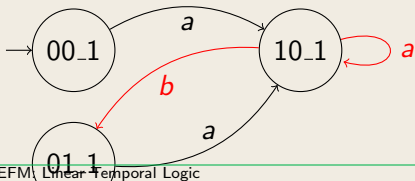
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$(a^*ba)^\omega$:

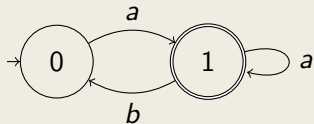


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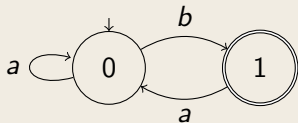


Explicit Construction of Intersection Automaton

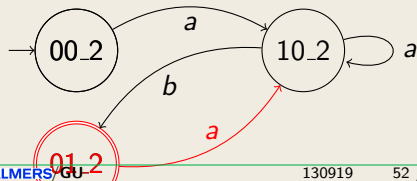
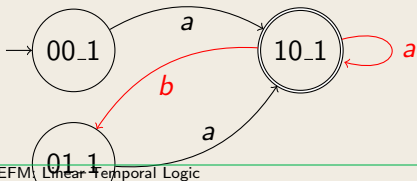
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$(a^*ba)^\omega$:

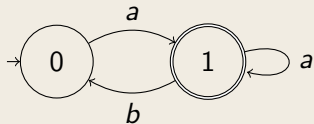


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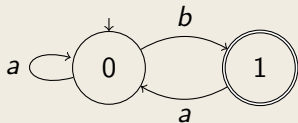


Explicit Construction of Intersection Automaton

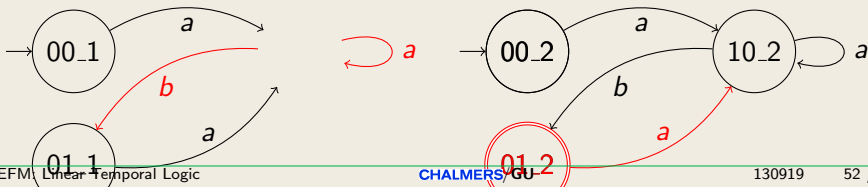
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$(a^*ba)^\omega$:

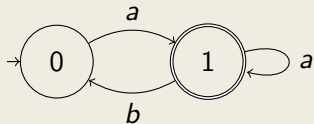


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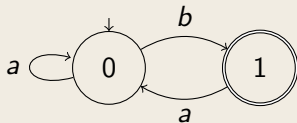


Explicit Construction of Intersection Automaton

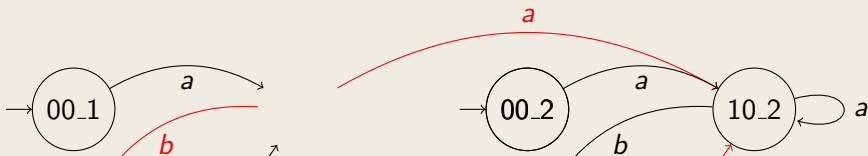
$a(a + ba)^\omega$:



$(a^*ba)^\omega$:

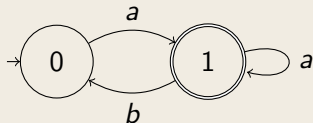


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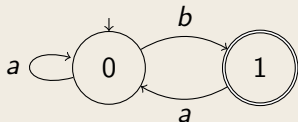


Explicit Construction of Intersection Automaton

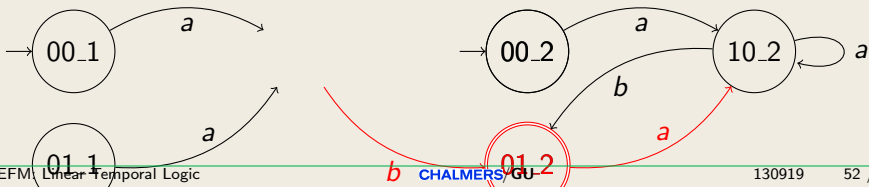
$a(a + ba)^\omega$:



$(a^*ba)^\omega$:

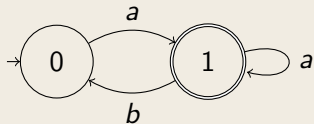


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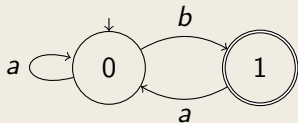


Explicit Construction of Intersection Automaton

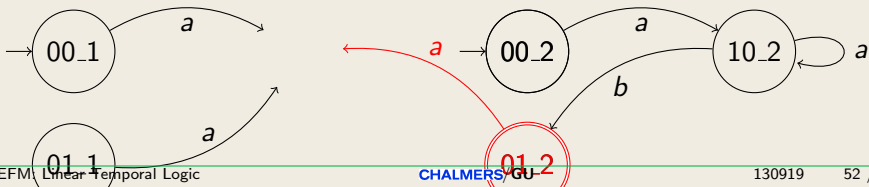
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Appendix II:

Construction of a Büchi
Automaton \mathcal{B}_ϕ
for an
LTL-Formula ϕ

The General Case: Generalised Büchi Automata

Generalize Büchi automata so that **sets** of interpretations accepted

A **generalised** Büchi automaton is defined as:

$$\mathcal{B}^g = (Q, \delta, I, \mathbb{F})$$

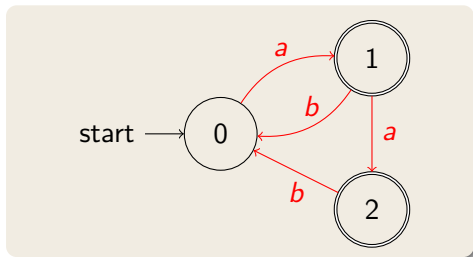
Q, δ, I as for standard Büchi automata

$\mathbb{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$, where $\mathcal{F}_i = \{q_{i1}, \dots, q_{im_i}\} \subseteq Q$

Definition (Acceptance for generalised Büchi automata)

A generalised Büchi automaton **accepts** an ω -word $w \in \Sigma^\omega$ iff for every $i \in \{1, \dots, n\}$ **at least one** $q_{ik} \in \mathcal{F}_i$ is visited infinitely often.

Normal vs. Generalised Büchi Automata: Example



\mathcal{B}^{normal} with $\mathcal{F} = \{1, 2\}$, $\mathcal{B}^{general}$ with $\mathbb{F} = \{\overbrace{\{1\}}^{\mathcal{F}_1}, \overbrace{\{2\}}^{\mathcal{F}_2}\}$

Which ω -word is accepted by which automaton?

ω -word	\mathcal{B}^{normal}	$\mathcal{B}^{general}$
$(ab)^\omega$	✓	✗
$(aab)^\omega$	✓	✓

Fischer-Ladner Closure

Fischer-Ladner closure of an LTL-formula ϕ

$$FL(\phi) = \{\varphi \mid \varphi \text{ is subformula or negated subformula of } \phi\}$$

($\neg\neg\varphi$ is identified with φ)

Example

$$FL(rUs) = \{r, \neg r, s, \neg s, rUs, \neg(rUs)\}$$

\mathcal{B}_ϕ -Construction: Locations

Assumption:

\mathcal{U} only temporal logic operator in LTL-formula (can express \Box, \Diamond with \mathcal{U})

Locations of \mathcal{B}_ϕ are $Q \subseteq 2^{FL(\phi)}$ where each $q \in Q$ satisfies:

- Consistent, Total**
- ▶ $\psi \in FL(\phi)$: exactly one of ψ and $\neg\psi$ in q
 - ▶ $\psi_1 \mathcal{U} \psi_2 \in (FL(\phi) \setminus q)$ then $\psi_2 \notin q$
- Downward Closed**
- ▶ $\psi_1 \wedge \psi_2 \in q$: $\psi_1 \in q$ and $\psi_2 \in q$
 - ▶ ... other propositional connectives similar
 - ▶ $\psi_1 \mathcal{U} \psi_2 \in q$ then $\psi_1 \in q$ or $\psi_2 \in q$

$$FL(rUs) = \{r, \neg r, s, \neg s, rUs, \neg(rUs)\}$$

	$\in Q$
$\{rUs, \neg r, s\}$	✓
$\{rUs, \neg r, \neg s\}$	✗
$\{\neg(rUs), r, s\}$	✗
$\{\neg(rUs), r, \neg s\}$	✓

\mathcal{B}_ϕ -Construction: Transitions

$$\underbrace{\{rUs, \neg r, s\}}_{q_1}, \underbrace{\{rUs, r, \neg s\}}_{q_2}, \underbrace{\{rUs, r, s\}}_{q_3}, \underbrace{\{\neg(rUs), r, \neg s\}}_{q_4}, \underbrace{\{\neg(rUs), \neg r, \neg s\}}_{q_5}$$

Transitions $(q, \alpha, q') \in \delta_\phi$:

$$\alpha = q \cap \mathcal{P}$$

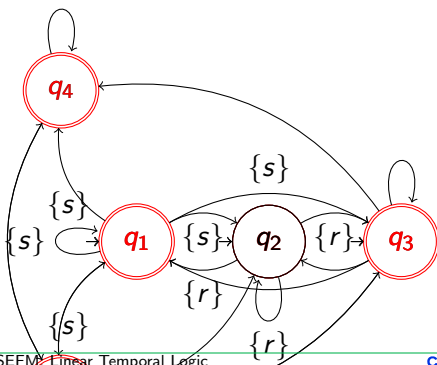
\mathcal{P} set of propositional variables
 outgoing edges of q_1 labeled $\{s\}$,
 of q_2 labeled $\{r\}$, etc.

1. If $\psi_1 U \psi_2 \in q$ and $\psi_2 \notin q$ then $\psi_1 U \psi_2 \in q'$
2. If $\psi_1 U \psi_2 \in (FL(\phi) \setminus q)$ and $\psi_1 \in q$ then $\psi_1 U \psi_2 \notin q'$

Initial locations

$$q \in I_\phi \text{ iff } \phi \in q$$

Accepting locations



Remarks on Generalized Büchi Automata

- ▶ Construction **always** gives exponential number of states in $|\phi|$
- ▶ Satisfiability checking of LTL is PSPACE-complete
- ▶ There exist (more complex) constructions that minimize number of required states
 - ▶ One of these is used in SPIN, which moreover computes the states lazily