

Finite Automata and Formal Languages

TMV026/DIT321 – LP4 2012

Formal Proofs, Alphabets and Words

Week 1

In these exercises, book sections and pages refer to those in the third edition of the course book.

Let \mathbb{N} is the set of all positive integers $\{0, 1, 2, \dots\}$ (see page 22 of the text book: “Integers as recursively defined concepts”).

1. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined by recursion as

$$f(0) = 0 \quad f(n + 1) = f(n) + n$$

What are the values of $f(2)$ and $f(3)$?

Use mathematical induction to show that for all $n \in \mathbb{N}$ we have

$$2f(n) = n^2 - n$$

2. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.
3. Let us define by recursion the following function:

$$0! = 1 \quad (n + 1)! = (n + 1) \times n!$$

Show that $n! \geq 2^n$ for $n \geq 4$ by analogy with the proof of example 1.17, page 21 of the text book.

4. Let us define by recursion the following two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$

$$\begin{aligned} f(0) &= 0 & g(0) &= 1 \\ f(n + 1) &= g(n) & g(n + 1) &= f(n) \end{aligned}$$

What are the values of $g(2)$ and $f(4)$? Show by mathematical induction that for all $n \in \mathbb{N}$ we have

$$f(n) + g(n) = 1 \quad f(n)g(n) = 0$$

Show by *mutual* induction that $f(n) = 0$ iff $g(n) = 1$ iff n is even and that $f(n) = 1$ iff $g(n) = 0$ iff n is odd, in analogy to the proof in pages 26–28 in the text book.

5. Let us define the Fibonacci function:

$$f(0) = 0 \quad f(1) = 1 \quad f(n+2) = f(n+1) + f(n)$$

We then define $s(0) = 0$, $s(n+1) = s(n) + f(n+1)$.

Prove by induction that we have

$$\forall n. s(n) = f(n+2) - 1.$$

Now we define

$$l(0) = 2, \quad l(1) = 1, \quad l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have $l(n+1) = f(n) + f(n+2)$.

6. If $\Sigma = \{a, b, c\}$, what are Σ^1 , Σ^2 and Σ^0 ?

7. If $\Sigma = \{0, 1\}$, find a counterexample to the following alleged theorem: $\forall x, y \in \Sigma^*$ we have

$$x^2y = xyx$$

(cf. section 1.3.4)

8. Let $\Sigma = \{0, 1\}$. We define $\phi : \Sigma^* \rightarrow \Sigma^*$ by recursion as follows

$$\phi(\epsilon) = \epsilon \quad \phi(w0) = \phi(w)1 \quad \phi(w1) = \phi(w)0$$

What are $\phi(1011)$ and $\phi(1101)$?

Show by induction on $|w|$ that

$$|\phi(w)| = |w|.$$

9. Let $\Sigma = \{0, 1\}$. We define the reverse function on Σ^* by the equations

$$\text{rev}(\epsilon) = \epsilon \quad \text{rev}(ax) = \text{rev}(x)a$$

What are $\text{rev}(010)$ and $\text{rev}(10)$?

Show by induction on y that we have

$$\text{rev}(yx) = \text{rev}(x)\text{rev}(y).$$

Show by induction on $n \in \mathbb{N}$ that we have

$$\text{rev}(x^n) = (\text{rev}(x))^n.$$

10. Given a finite alphabet Σ , when can we have $x^2 = y^3$ with $x, y \in \Sigma^*$?