# Intersection Testing
# Chapter 16

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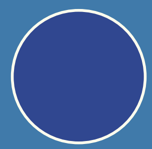
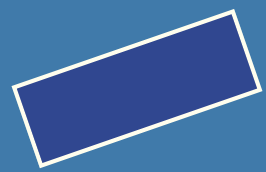
# What for?

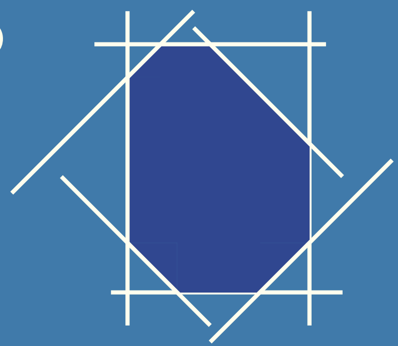
- A tool needed for the graphics people all the time…
- Very important components:
  - Need to make them fast!
- Finding if (and where) a ray hits an object
  - Picking
  - Ray tracing and global illumination
- For speed-up techniques
- Collision detection (treated in a later lecture)

# Example


Midtown Madness 3, DICE

# Some basic geometrical primitives

- Ray: 
- Sphere: 
- Box
  - Axis-aligned (AABB) 
  - Oriented (OBB) 
- *k*-DOP

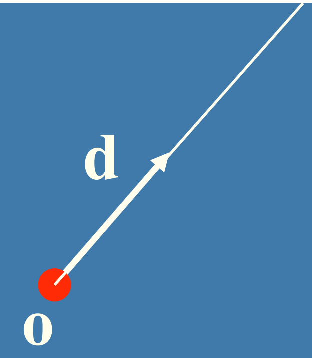


# Four different techniques

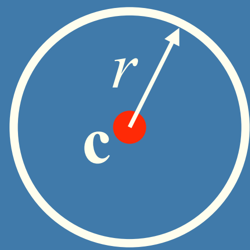
- Analytical
- Geometrical
- Separating axis theorem (SAT)
- Dynamic tests

- Given these, one can derive many tests quite easily
  - However, often tricks are needed to make them fast

# Analytical: Ray/sphere test




- Sphere center: $\mathbf{c}$, and radius $r$
- Ray: $\mathbf{r}(t)=\mathbf{o}+t\mathbf{d}$
- Sphere formula: $\|\mathbf{p}-\mathbf{c}\|=r$
- Replace $\mathbf{p}$ by $\mathbf{r}(t)$, and square it:




$$(\mathbf{r}(t)-\mathbf{c})\cdot(\mathbf{r}(t)-\mathbf{c})-r^2=0$$

$$(\mathbf{o}+t\mathbf{d}-\mathbf{c})\cdot(\mathbf{o}+t\mathbf{d}-\mathbf{c})-r^2=0$$

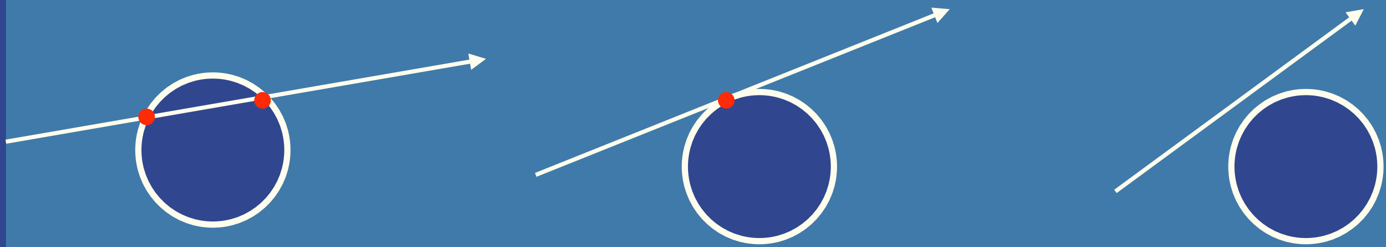
$$(t\mathbf{d}+(\mathbf{o}-\mathbf{c}))\cdot(t\mathbf{d}+(\mathbf{o}-\mathbf{c}))-r^2=0$$

$$(\mathbf{d}\cdot\mathbf{d})t^2+2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t+(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c})-r^2=0$$

$$t^2+2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t+(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c})-r^2=0 \qquad \|\mathbf{d}\|=1$$

# Analytical, continued

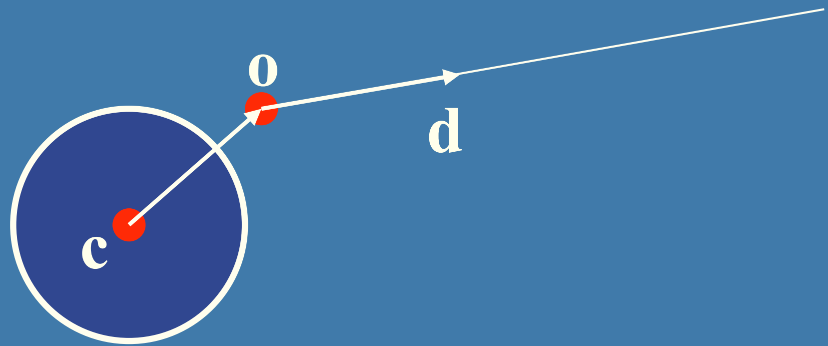
$$t^2 + 2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t + (\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) - r^2 = 0$$



- Be a little smart…

$$(\mathbf{o}-\mathbf{c})\cdot\mathbf{d} > 0 \ ?$$

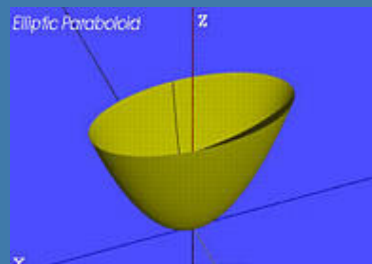
$$(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) - r^2 < 0 \ ?$$




- Such tests are called ”rejection tests”
- Other shapes: $p_x^2 + p_y^2 = r^2$

$$(p_x/a)^2 + (p_y/b)^2 + (p_z/c)^2 = 1$$

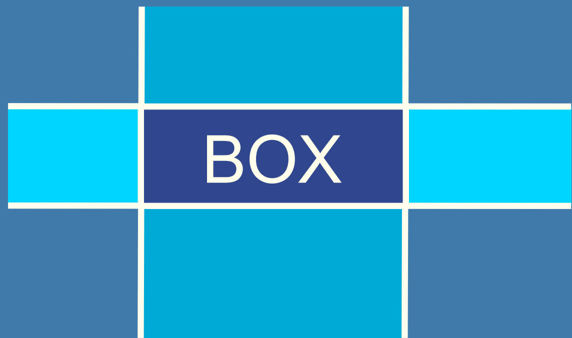
$$(p_x/a)^2 + (p_y/b)^2 - p_z = 0$$



# Geometrical: Ray/Box Intersection

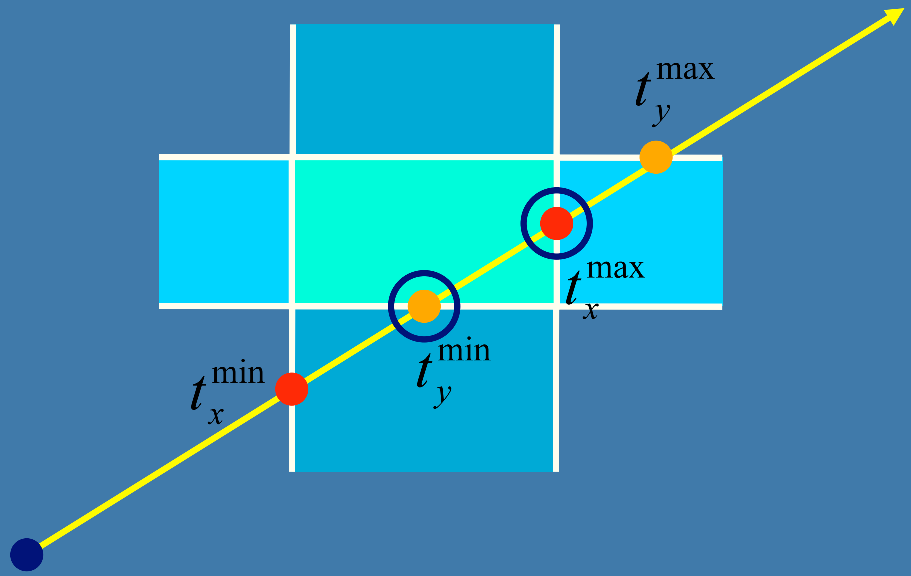
- Boxes and spheres often used as bounding volumes
- A slab is the volume between two parallell planes:

- A box is the logical intersection of three slabs (2 in 2D):



# Geometrical: Ray/Box Intersection (2)

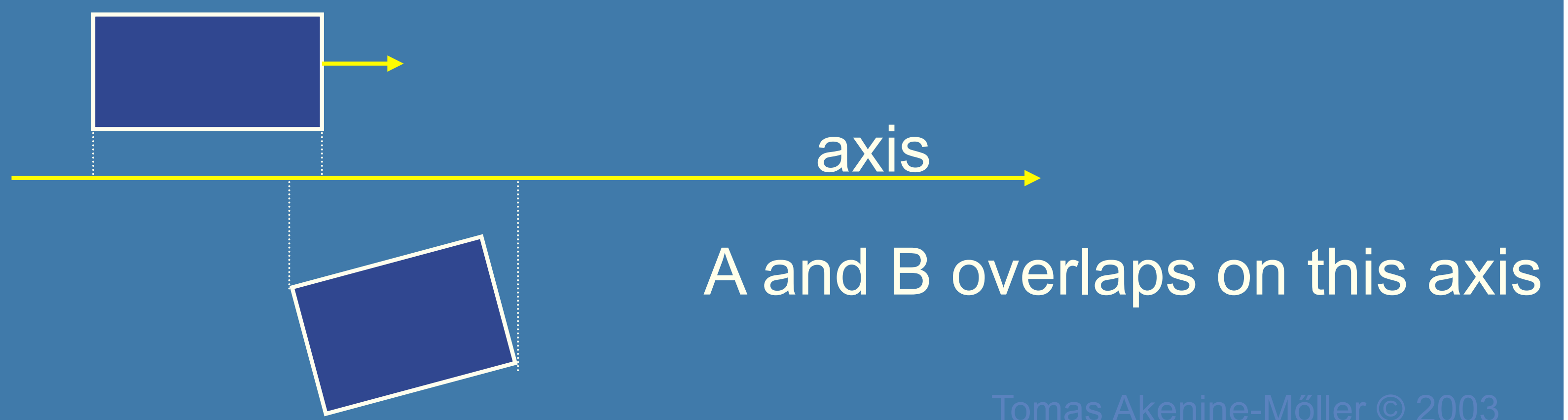
- Intersect the 2 planes of each slab with the ray




- Keep max of $t^{min}$ and min of $t^{max}$
- If $t^{min} < t^{max}$ then we got an intersection
- Special case when ray parallell to slab

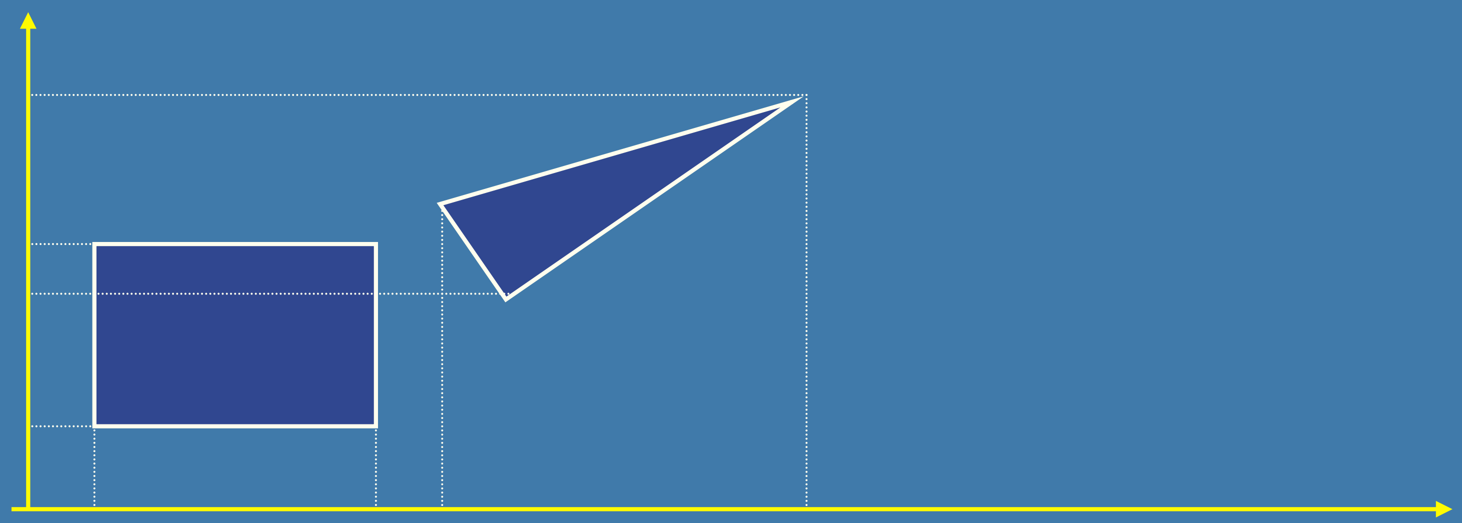
# Separating Axis Theorem (SAT) Page 563 in book

- Two convex polyhedrons, A and B, are disjoint if any of the following axes separate the objects:
  - An axis orthogonal to a face of A
  - An axis orthogonal to a face of B
  - An axis formed from the cross product of one edge from each of A and B



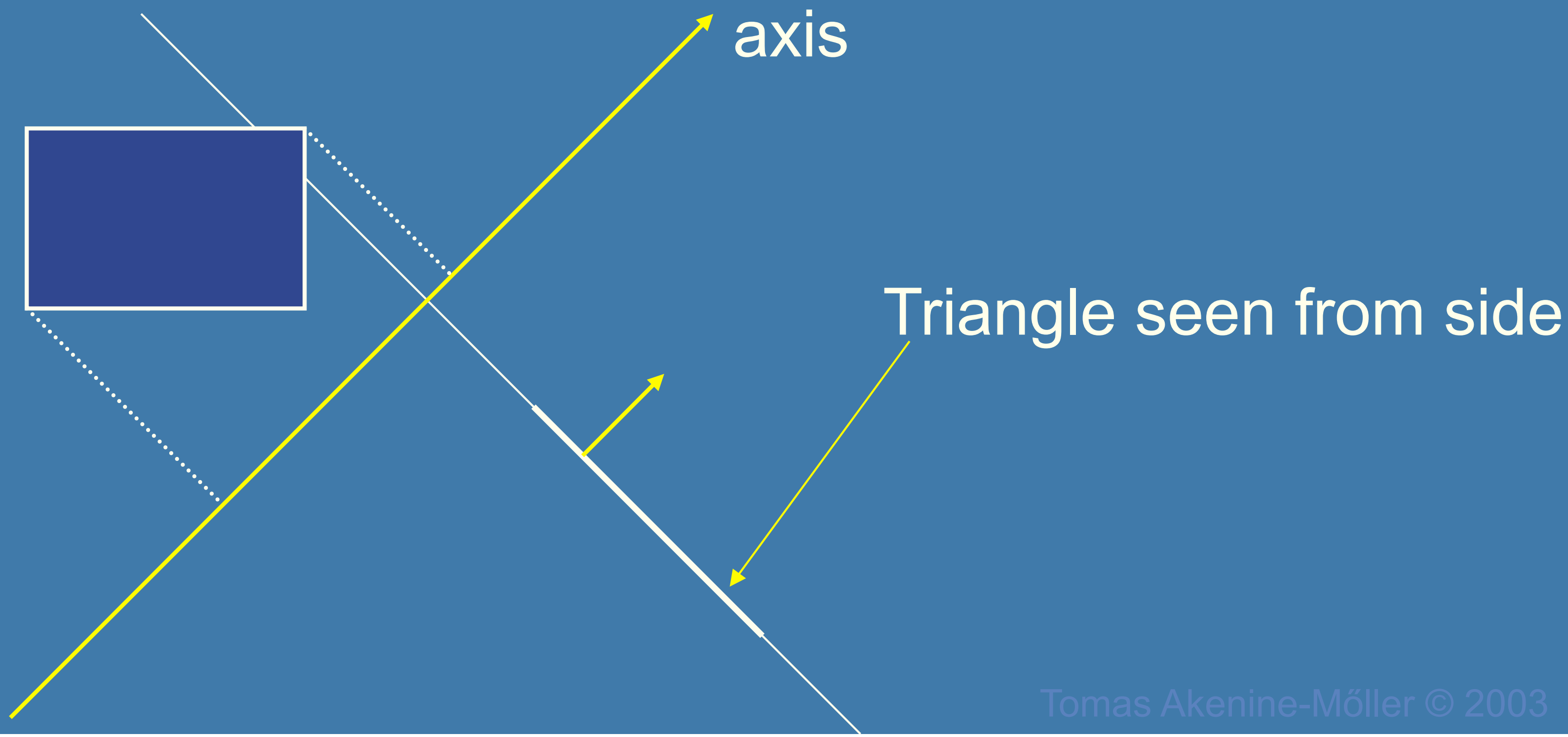
# SAT example: Triangle/Box

- E.g an axis-aligned box and a triangle
- **1)** test the axes that are orthogonal to the faces of the box
- That is, x,y, and z



# Triangle/Box with SAT (2)

- Assume that they overlapped on x,y,z
- Must continue testing
- **2)** Axis orthogonal to face of triangle



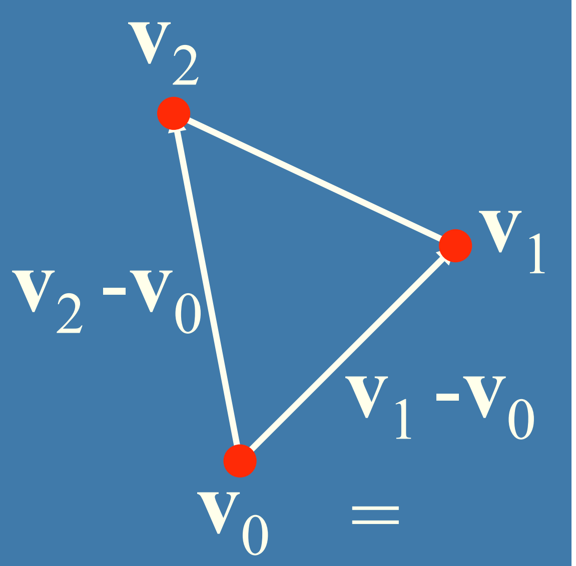
# Triangle/Box with SAT (3)

- If still no separating axis has been found…
- 3) Test axis: $\mathbf{t}=\mathbf{e}_{\text{box}} \times \mathbf{e}_{\text{triangle}}$
- Example:
  - x-axis from box: $\mathbf{e}_{\text{box}}=(1,0,0)$
  - $\mathbf{e}_{\text{triangle}}=\mathbf{v}_1-\mathbf{v}_0$
- Test all such combinations
- If there is at least one separating axis, then the objects do not collide
- Else they do overlap

# Rules of Thumb for Intersection Testing

- Acceptance and rejection test
  - Try them early on to make a fast exit
- Postpone expensive calculations if possible
- Use dimension reduction
  - E.g. 3 one-dimensional tests instead of one complex 3D test, or 2D instead of 3D
- Share computations between objects if possible
- Timing!!!

# Another analytical example: Ray/Triangle in detail




- Ray: $\mathbf{r}(t)=\mathbf{o}+t\mathbf{d}$
- Triangle vertices: $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$
- A point in the triangle:
- $\mathbf{t}(u,v)=\mathbf{v}_0+u(\mathbf{v}_1 - \mathbf{v}_0) + v(\mathbf{v}_2 - \mathbf{v}_0)=$ $(1-u-v)\mathbf{v}_0+u\mathbf{v}_1+v\mathbf{v}_2 \quad [u,v>=0,\ u+v<=1]$
- Set $\mathbf{t}(u,v)=\mathbf{r}(t)$, and solve!

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{v}_1-\mathbf{v}_0 & \mathbf{v}_2-\mathbf{v}_0 \\ | & | & | \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{o}-\mathbf{v}_0 \\ | \end{pmatrix}$$

# Ray/Triangle (2)

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{v}_1-\mathbf{v}_0 & \mathbf{v}_2-\mathbf{v}_0 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{o}-\mathbf{v}_0 \\ | \end{pmatrix}$$

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_0 \qquad \mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_0 \qquad \mathbf{s} = \mathbf{o} - \mathbf{v}_0$$

- Solve with Cramer's rule:

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{e}_1 & \mathbf{e}_2 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{s} \\ | \end{pmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

$$A\,x = b$$

$$x = \frac{\begin{vmatrix} \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} & \begin{matrix} b \\ e \\ h \end{matrix} & \begin{matrix} c \\ f \\ i \end{matrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \begin{matrix} a \\ d \\ g \end{matrix} & \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} & \begin{matrix} c \\ f \\ i \end{matrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad \text{and } z = \frac{\begin{vmatrix} \begin{matrix} a \\ d \\ g \end{matrix} & \begin{matrix} b \\ e \\ h \end{matrix} & \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{\det(-\mathbf{d},\mathbf{e}_1,\mathbf{e}_2)}\begin{pmatrix} \det(\mathbf{s},\mathbf{e}_1,\mathbf{e}_2) \\ \det(-\mathbf{d},\mathbf{s},\mathbf{e}_2) \\ \det(-\mathbf{d},\mathbf{e}_1,\mathbf{s}) \end{pmatrix}$$

# Ray/Triangle (2)

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{v}_1-\mathbf{v}_0 & \mathbf{v}_2-\mathbf{v}_0 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{o}-\mathbf{v}_0 \\ | \end{pmatrix}$$

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_0 \qquad \mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_0 \qquad \mathbf{s} = \mathbf{o} - \mathbf{v}_0$$

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{e}_1 & \mathbf{e}_2 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{s} \\ | \end{pmatrix}$$

- Solve with Cramer's rule:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{\det(-\mathbf{d}, \mathbf{e}_1, \mathbf{e}_2)}\begin{pmatrix} \det(\mathbf{s}, \mathbf{e}_1, \mathbf{e}_2) \\ \det(-\mathbf{d}, \mathbf{s}, \mathbf{e}_2) \\ \det(-\mathbf{d}, \mathbf{e}_1, \mathbf{s}) \end{pmatrix}$$

$$\text{Use this fact:} \quad \det(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}$$

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{(\mathbf{d}\times\mathbf{e}_2)\cdot\mathbf{e}_1}\begin{pmatrix} (\mathbf{s}\times\mathbf{e}_1)\cdot\mathbf{e}_2 \\ (\mathbf{d}\times\mathbf{e}_2)\cdot\mathbf{s} \\ (\mathbf{s}\times\mathbf{e}_1)\cdot\mathbf{d} \end{pmatrix}$$

- Share factors to speed up computations

# Ray/Triangle (3) Implementation

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{(\mathbf{d}\times\mathbf{e}_2)\cdot\mathbf{e}_1}\begin{pmatrix} (\mathbf{s}\times\mathbf{e}_1)\cdot\mathbf{e}_2 \\ (\mathbf{d}\times\mathbf{e}_2)\cdot\mathbf{s} \\ (\mathbf{s}\times\mathbf{e}_1)\cdot\mathbf{d} \end{pmatrix}$$

- Be smart!
  - Compute as little as possible. Then test
- Examples: $\mathbf{p} = \mathbf{d}\times\mathbf{e}_2$, $a = \mathbf{p}\cdot\mathbf{e}_1$, $f = 1/a$
- Compute $u = f(\mathbf{p}\cdot\mathbf{s})$
- Then test valid bounds
- `if (u<0 or u>1) exit;`

$$\text{Plane}: \quad \pi : \mathbf{n} \cdot \mathbf{p} + d = 0$$

# Point/Plane

- Insert a point $\mathbf{x}$ into plane equation:

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d = ?$$

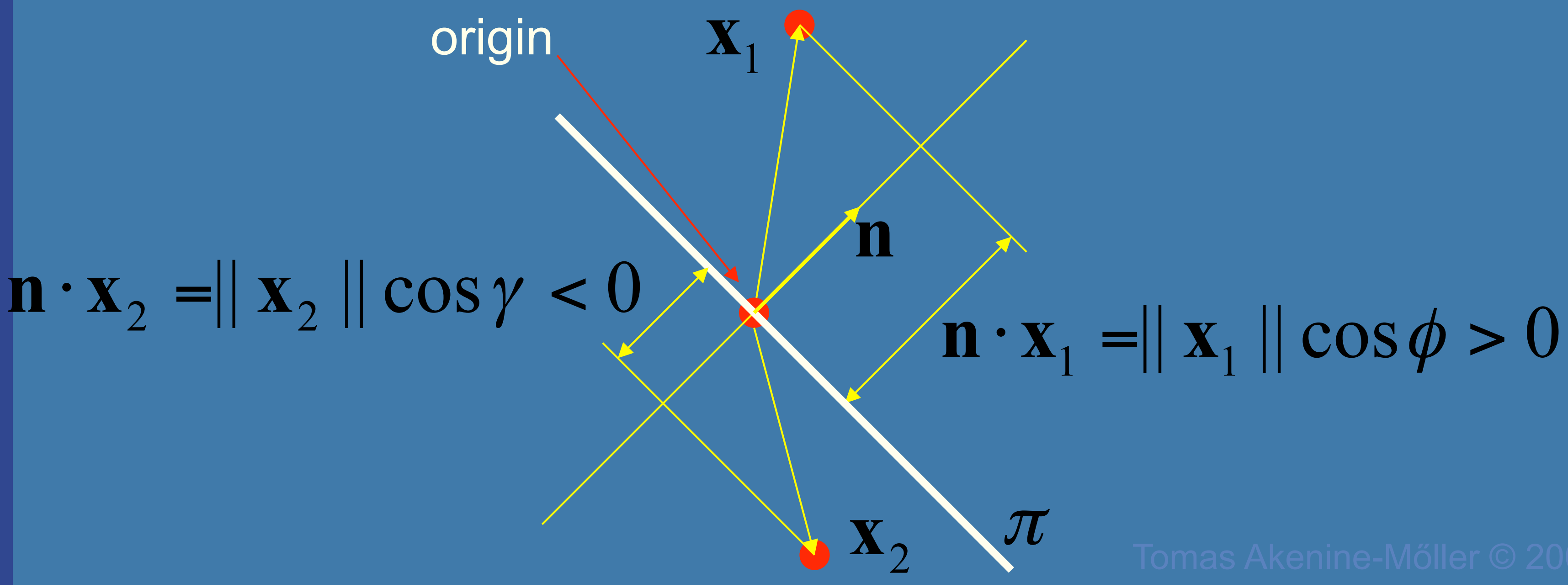
$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d = 0 \qquad \text{for } \mathbf{x}\text{'s on the plane}$$

Negative half space

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d < 0 \qquad \text{for } \mathbf{x}\text{'s on one side of the plane}$$

Positive half space

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d > 0 \qquad \text{for } \mathbf{x}\text{'s on the other side}$$

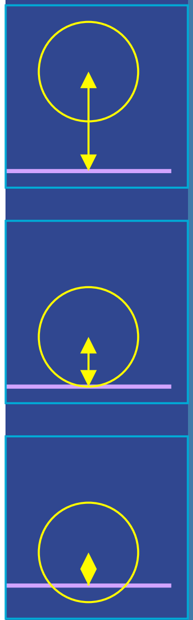


# Sphere/Plane
# AABB/Plane

Plane: $\pi : \mathbf{n} \cdot \mathbf{p} + d = 0$

Sphere: $\mathbf{c} \quad r$

Box: $\mathbf{b}^{\min} \quad \mathbf{b}^{\max}$



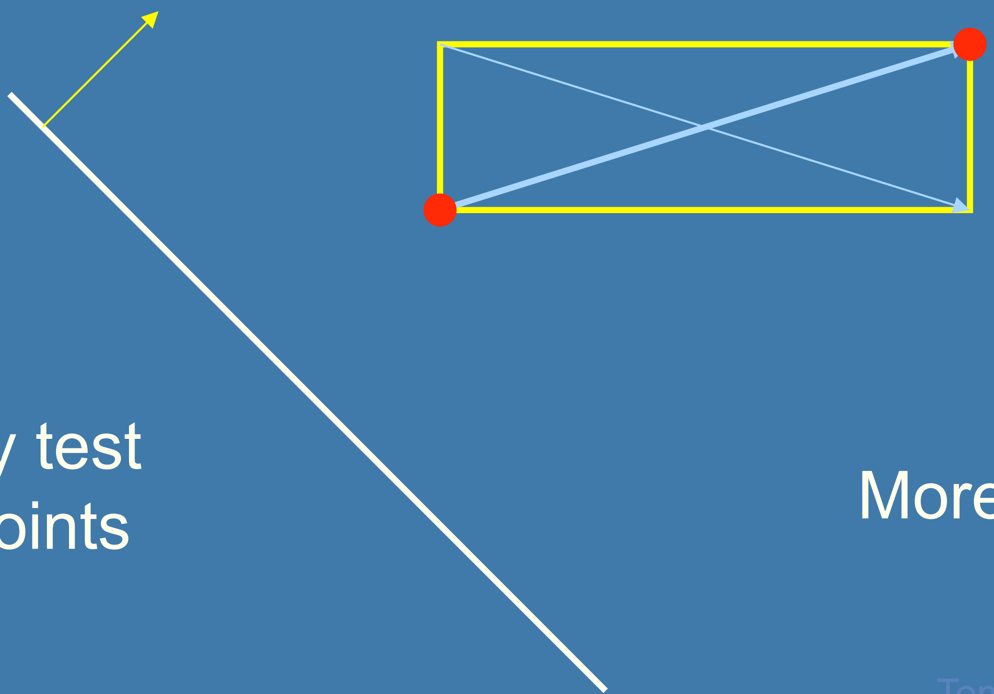
- Sphere: compute $f(\mathbf{c}) = \mathbf{n} \cdot \mathbf{c} + d$
- $f(\mathbf{c})$ is the signed distance ($\mathbf{n}$ normalized)
- $\text{abs}(f(\mathbf{c})) > r$ no collision
- $\text{abs}(f(\mathbf{c})) = r$ sphere touches the plane
- $\text{abs}(f(\mathbf{c})) < r$ sphere intersects plane

- Box: insert all 8 corners
- If all $f$'s have the same sign, then all points are on the same side, and no collision

# AABB/plane

$$\text{Plane}: \quad \pi : \mathbf{n} \cdot \mathbf{p} + d = 0$$
$$\text{Sphere}: \quad \mathbf{c} \qquad r$$
$$\text{Box}: \quad \mathbf{b}^{\min} \qquad \mathbf{b}^{\max}$$

- The smart way (shown in 2D)
- Find diagonal that is most closely aligned with plane normal

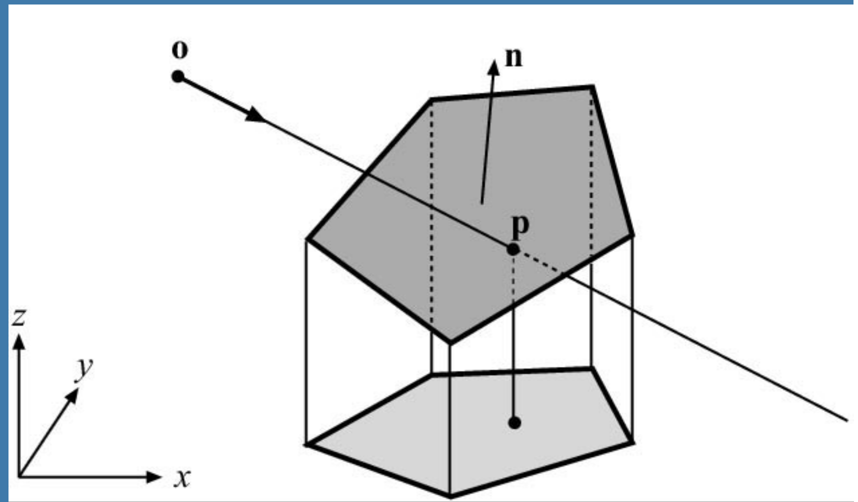


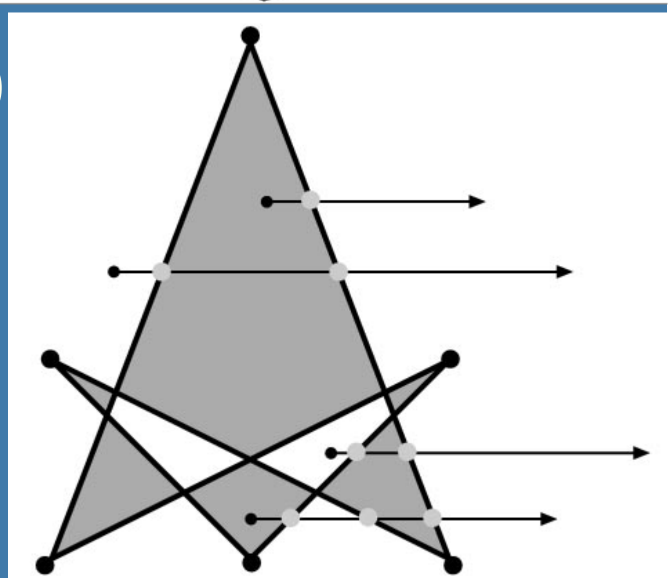
Need only test the red points

More details in book

# Ray/Polygon: very briefly

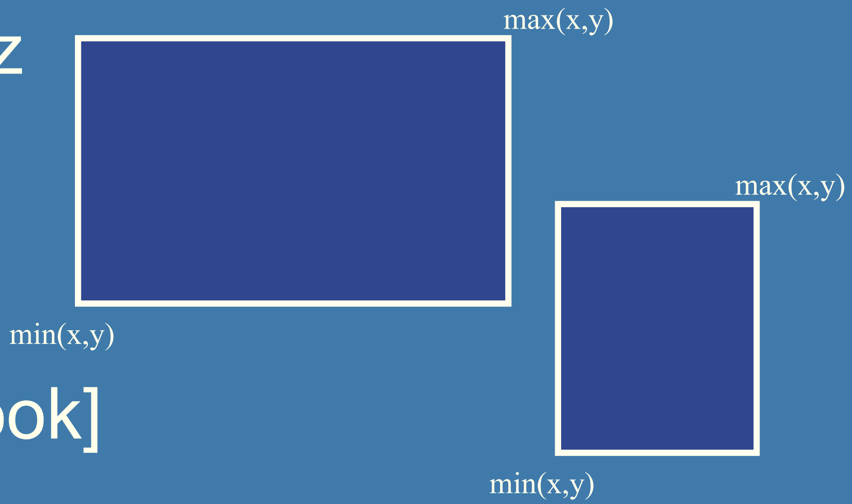
- Intersect ray with polygon plane
- Project from 3D to 2D
- How?
- Find $\max(|n_x|,|n_y|,|n_z|)$
- Skip that coordinate!
- Then, count crossing in 2D



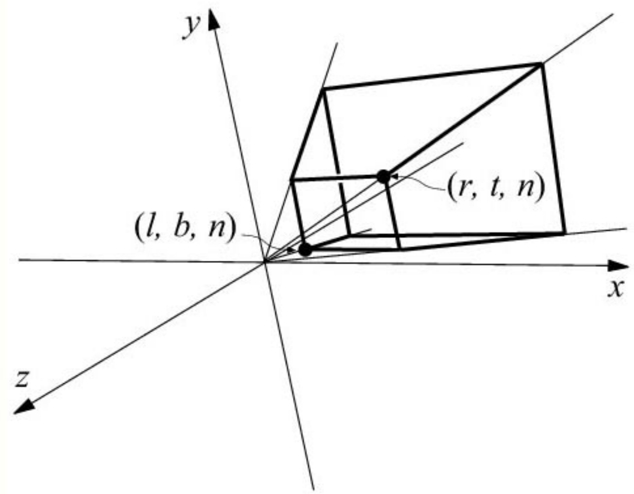
# Volume/Volume tests

- Used in collision detection
- Sphere/sphere
  - Compute squared distance between sphere centers, and compare to $(r_1+r_2)^2$
- Axis-Aligned Bounding Box/AABB
  - Test in 1D for x,y, and z



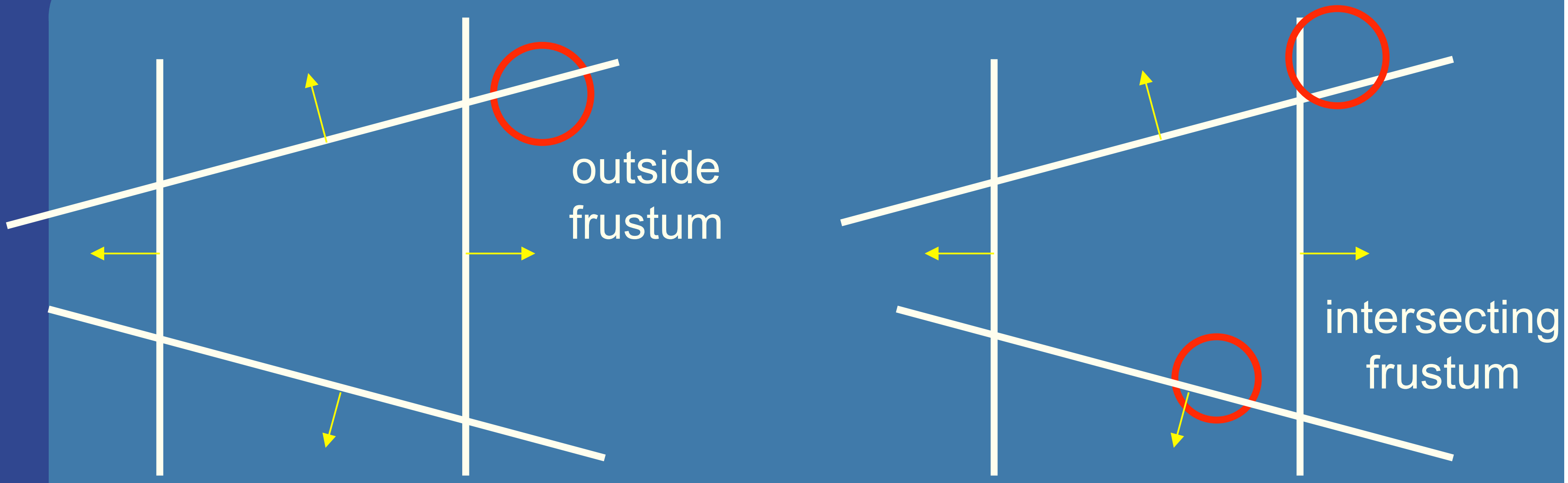

- Oriented Bounding boxes
  - Use SAT [details in book]

# View frustum testing




- View frustum is 6 planes:
- Near, far, right, left, top, bottom
- Create planes from projection matrix
  - Let all positive half spaces be outside frustum
  - Not dealt with here -- p. 773-774, 3rd ed.
- Sphere/frustum common approach:
  - Test sphere against 6 frustum planes
  - If in positive half space and $r$ distances from plane => no intersection
  - If intersecting plane or in negative half space, continue
  - If not outside after all six planes, then inside or intersecting
- Example follows…

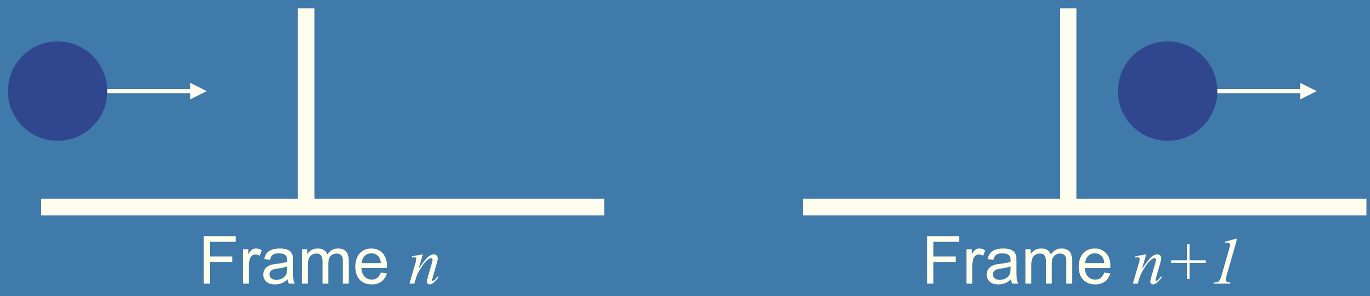
# View frustum testing example




- Not exact test, but not incorrect
  - A sphere that is reported to be inside, can be outside
  - Not vice versa
- Similarly for boxes

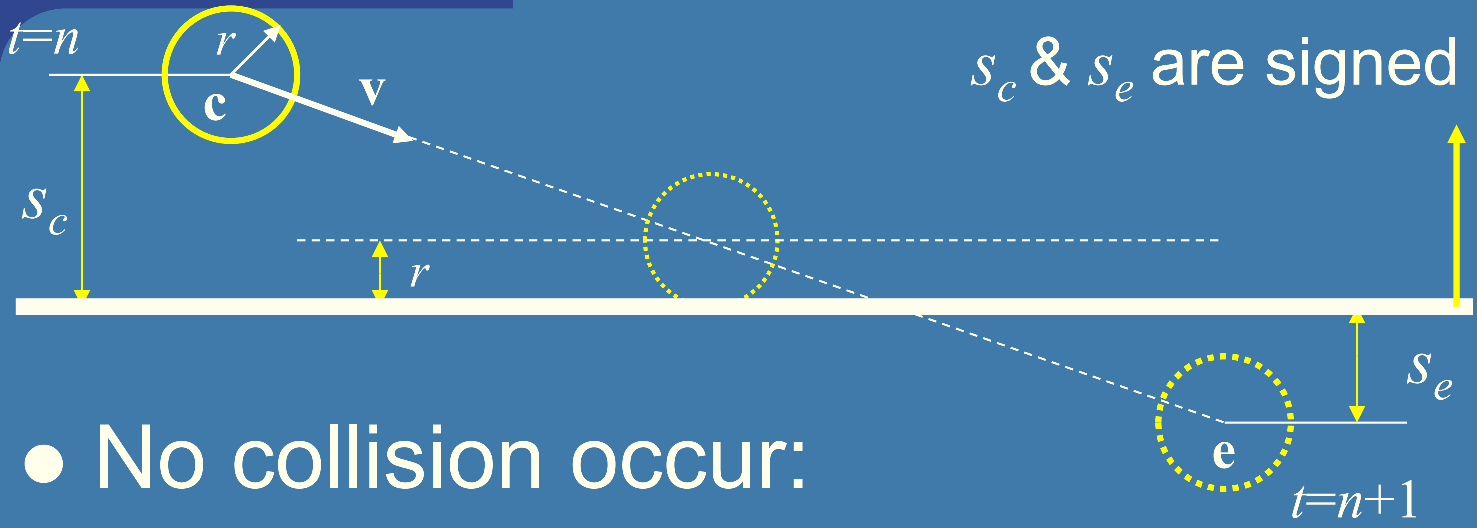
# Dynamic Intersection Testing
**[In book: 620-628]**

- Testing is often done every rendered frame, i.e., at discrete time intervals
- Therefore, you can get ”quantum effects”




- Dynamic testing deals with this
- Is more expensive
- Deals with a time interval: time between two frames

# Dynamic intersection testing Sphere/Plane

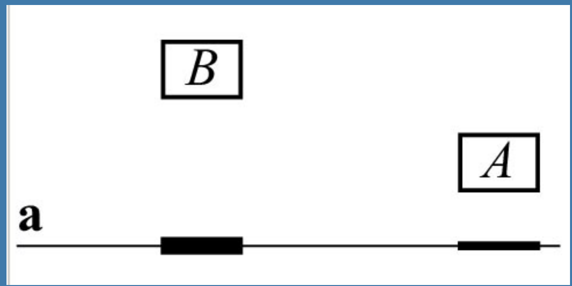


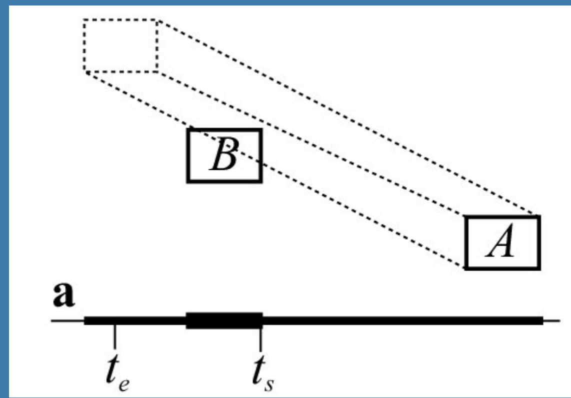

- No collision occur:
  - If they are on the same side of the plane ($s_c s_e > 0$)
  - Plus $|s_c| > r$ and $|s_e| > r$
- Otherwise, sphere can move $|s_c| - r$
- Time of collision: $t_{cd} = n + \frac{s_c - r}{s_c - s_e}$
- Response: reflect $\mathbf{v}$ around $\mathbf{n}$, and move $(1-t_{cd})\mathbf{r}$ ($\mathbf{r}$=refl vector)

**BONUS**

# Dynamic Separating Axis Theorem

- SAT: tests one axis at a time for overlap




- Same with DSAT, but:
  - Need to adjust the projection on the axis so that the interval moves on the axis as well
- Need to test same axes as with SAT
- Same criteria for overlap/disjoint:
  - If no overlap on axis => disjoint
  - If overlap on all axes => objects overlap

BONUS

# Dynamic Sweep-and-Prune

- http://graphics.idav.ucdavis.edu/~dcoming/papers/coming_staadt_vriphys05.pdf

# Exercises

- Create a function (by writing code on paper) that tests for intersection between:
  - two spheres
  - a ray and a sphere
  - view frustum and a sphere

# Scan Line Fill

Set active edges to AB and AC

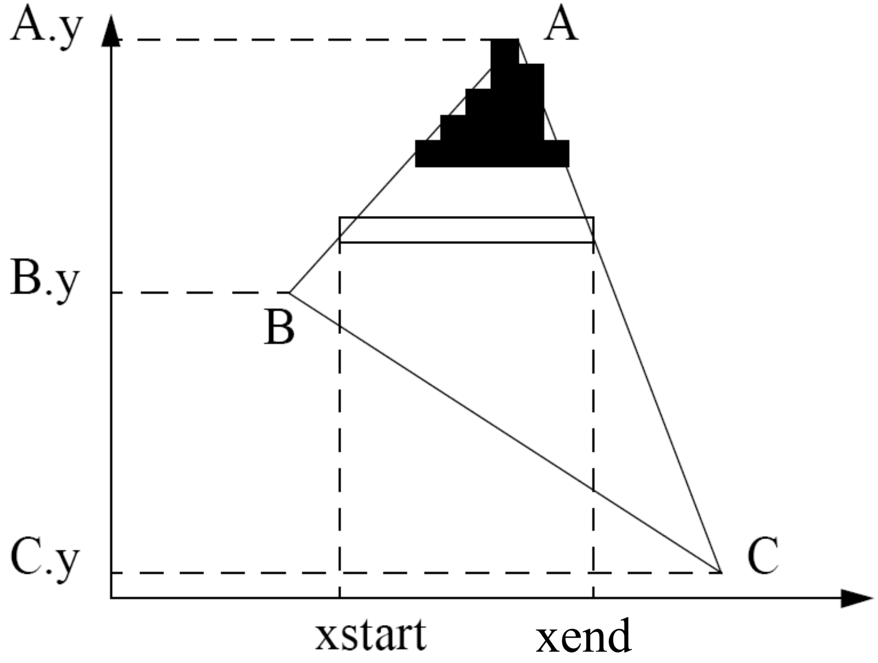
For y = A.y, A.y-1,...,C.y

If y=B.y → exchange AB with BC

Compute xstart and xend. Interpolate color, depth, texcoords etc for points (xstart,y) and (xend,y)

For x = xstart, xstart+1, ...,xend

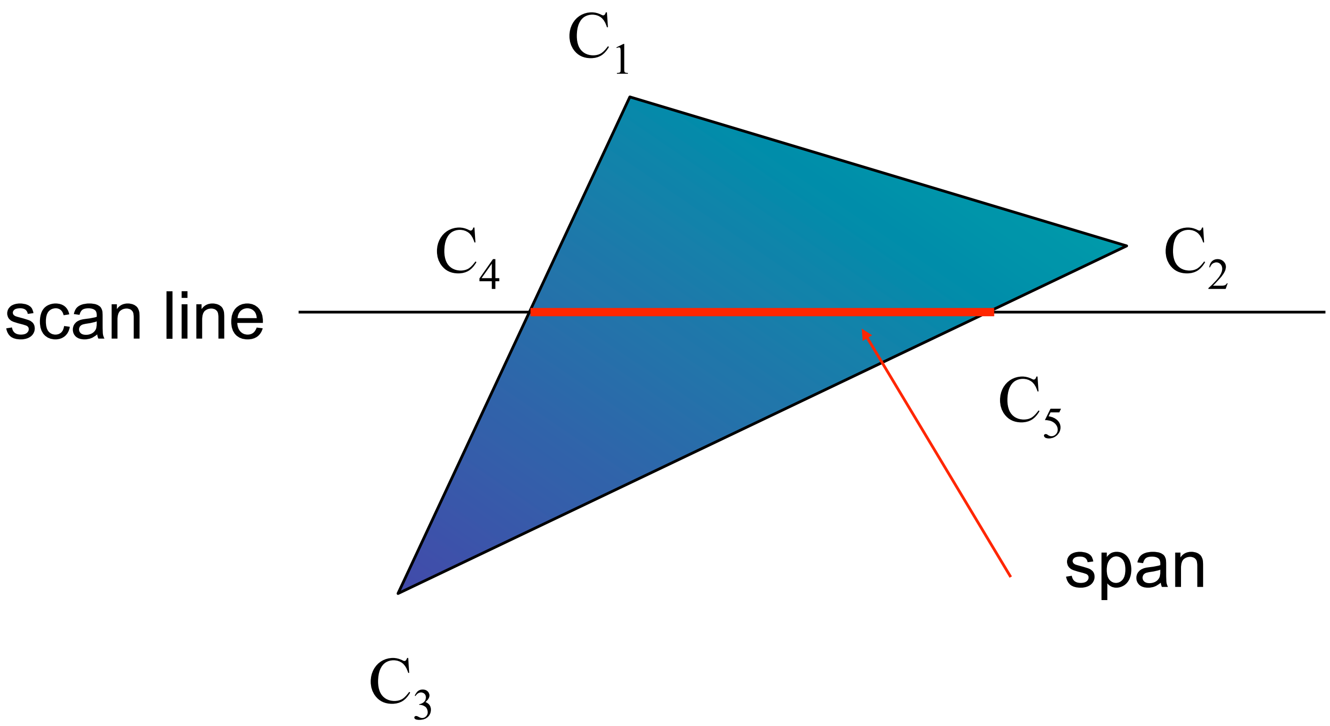
Compute color, depth etc for (x,y) using interpolation.



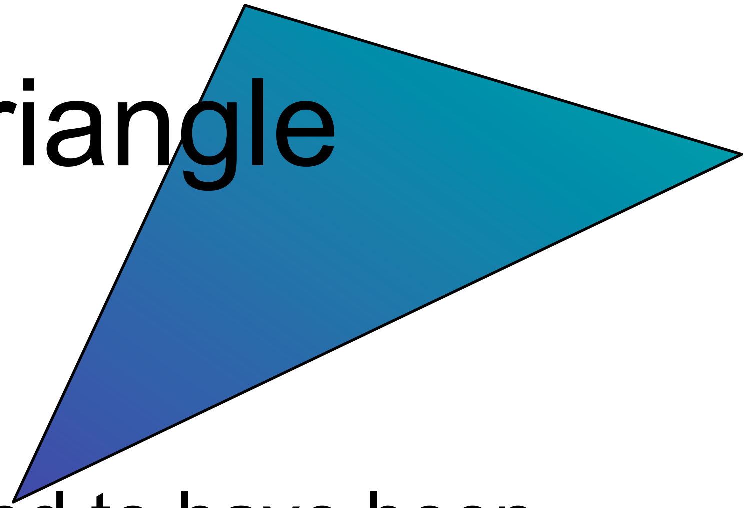

**This is the modern way to rasterize a triangle**

# Using Interpolation

$C_1$ $C_2$ $C_3$ specified by **`glColor`** or by vertex shading
$C_4$ determined by interpolating between $C_1$ and $C_2$
$C_5$ determined by interpolating between $C_2$ and $C_3$
interpolate between $C_4$ and $C_5$ along span



# Rasterizing a Triangle



- –Convex Polygons only
- –Nonconvex polygons assumed to have been tessellated
- –Shades (colors) have been computed for vertices (Gouraud shading)
- –Combine with z-buffer algorithm
  - March across scan lines interpolating shades
  - Incremental work small

# Flood Fill

- Fill can be done recursively if we know a seed point located inside (WHITE)
- Scan convert edges into buffer in edge/inside color (BLACK)

```
flood_fill(int x, int y) {
    if(read_pixel(x,y)= = WHITE) {
        write_pixel(x,y,BLACK);
        flood_fill(x-1, y);
        flood_fill(x+1, y);
        flood_fill(x, y+1);
        flood_fill(x, y-1);
}   }
```