Discrete Optimization Take Home Exam [1]

March 16	Take Home, Due March 17, 1600

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Max points:	60	
Grade criteria:	Chalmers	5:48, 4:36, 3:24
	GU	VG:48, G:28
	Doktorander	G:36
Helping material:	Course book, material on course page.	

- You are required to **work alone**.
- Recommended: First look through all questions and make sure that you understand them properly. In case of doubt, do not hesitate to ask.
- Answer concisely and to the point. (English if you can and Swedish if you must!)
- Code strictly forbidden! Motivated pseudocode or plain but clear English/Swedish description is fine.
- This exam counts for 50 % of the final grade, the other 50 % is from the weekly exercises.

Lycka till!

[1] 2011 LP3, TDA206/DIT370.

Problem 1 Helping Göteborg Steel [10] Göteborg Steel must decide how to allocate next week's time on a rolling mill which is a machine that takes unfinished slabs of steel as input and can produce one of two semi–finished products: bands and coils. The mill's two products come off the the rolling line at different rates:

Bands 200 tons/hr.

Coils 140 tons/hr.

They also have diifferent profits:

Bands SEK 250/ton

Coils SEK 300/ton

Based on the currently booked orders, the folliwng upper bounds are placed on the amount of each product to produce:

Bands 6000 tonnes

Coils 4000 tonnes

Given there are 40 hours of production time available this week, the problem is to decide how many tons of bands and how many tons of coils should be produced to yield the greatest profit. Formulate the problem as a linear problem and solve it graphically.

Problem 2 Facility Location - A student's take [10] Consider the facility location problem: we have m customers and n possible locations for facilities: facility i costs c_i to open and connecting customer j to facility i costs $d_{i,j}$. We need to open some facilities and connect each customer to an open facility to minimize the total cost (opening plus connecting). The ILP we formulated in class was:

$$\min \sum_{i=1}^{n} c_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{i,j} x_{i,j}$$

subject to

$$\sum_{i=1}^{n} x_{i,j} \geq 1, \quad j = 1 \ldots m$$

$$x_{i,j} \leq y_i, \quad i = 1 \ldots n, j = 1 \ldots m \tag{1}$$

$$x_{i,j}, y_i \in \{0, 1\}, \quad i = 1 \ldots n, j = 1 \ldots m$$

In a previous year, a student suggested a slightly different formulation: he replaced the constraint (1) by

$$\sum_{j=1}^{m} x_{i,j} \leq m y_i, \quad i = 1 \ldots n. \tag{2}$$

(Everything else is the same.) In this problem, you will analyse the difference between the two versions.

(a) Show first that the two ILP formulations are equivalent: both are an exact formulation of the original problem (the student was right!).

(b) Show however that the LP relaxations are *not* equivalent (the student was wrong, too bad!): the LP relaxation of the version (1) is *tighter* than the version (2) i.e. show that any solution to the LP relaxation of (1) is also a solution to the LP relaxation of version (2), but give an example to show the converse is not true.

Consider the following example: there are n customers and $\sqrt{n}$ facilities (assume n is a perfect square so $\sqrt{n}$ is an integer as well). The opening cost of each facility is $\sqrt{n}$. The customers are grouped into $\sqrt{n}$ groups with group i being $\{(i-1)\sqrt{n}+1, \ldots, i\sqrt{n}\}$. The connection cost of each customer in group i to facility i is $\frac{1}{\sqrt{n}}$ whereas to every other facility is is huge $M >> n$.

(c) What is the optimum solution to this example and what is the cost of this solution?

(d) What is the optimum value of the LP versions (1) and (2) for this example respectively? Just give the two optimal LP solutions and a brief justification if not a full proof.

(e) We stated in class that one can round the solution to version (1) to get a constant factor approximation to the original problem. Kalle claims he has a magic rounding rule to do something exactly similar with version (2) and get a constant factor approximation. Using (c) and (d) show why he is wrong.

Problem 3 Max Cut with equal size [10] Recall the MAXCUT problem: we have a graph $G = (V, E)$ with non-negative weights $(w_{u,v}, (u, v) \in E)$ an we want to partition the vertices into two parts A and B so as to maximize the sum $\sum_{u \in A, v \in B} w_{u,v}$ of the weights of the edges crossing the cut. Recall the integer program we wrote:

$$\max \sum_{(u,v) \in E} w_{u,v}(1 - y_u y_v)/2,$$

with

$$y_u \in \{-1, +1\}, u \in V$$

Recall then that we passed to the semi-definite programmng (SDP) relaxation:

$$\max \sum_{(u,v) \in E} w_{u,v}(1 - X_{u,v})/2,$$

subject to

$$X_{u,u} = 1, u \in V$$

and

$$X \succeq 0.$$

In this problem we want to find a partition of maximum weight where the two sides have *exactly equal* sizes i.e. $|A| = |B|$ (assume the graph has an even number of vertices).

(a) Show that the new problem is also NP-hard by giving a reduction: that is, given an instance of the original MAXCUT problem, create an instance for the modified problem such that an equal–sized MAXCUT cut in the latter allows you to compute a MAXCUT in the original problem easily.

(b) Write a linear condition involving the variables $y_u, u \in V$ above to express this property that the two sides of the cut must have the same size.

(c) Using the identity $(\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i\neq j} a_i a_j$, rewrite this property in terms of the variables $X_{u,v}$ above to pass to a SDP relaxation.

(d) Recall the randomized rounding rule for the original problem: first factorize the matrix X obtained by solving the SDP relaxation as $X = UV$ using a Cholesky decomposition to get a column vector b_v corresponding to each vertex $v \in V$. Then pick a random vector r and set the sign of vertex u to be $+1$ if $r \cdot b_u \geq 0$ and -1 otherwise i.e. depending on which side of r the vector b_v falls on. Suggest how to modify this rule to produce a partition with exactly equal sizes. (HINT: One way is to think about what happens if you repeat this procedure several times and look at who ends up on the same side as a fixed vertex.)

Problem 4 Checking optimality [10] Use complementary slackness to check if the following claims are correct:

(a) The problem

$$\max 18x_1 - 7x_2 + 12x_3 + 5x_4 + 8x_6$$

subject to

$$\begin{aligned}
2x_1 - 6x_2 + 2x_3 + 7x_4 + 3x_5 + 8x_6 &\leq 1 \\
-3x_1 - x_2 + 4x_3 - 3x_4 + x_5 + 2x_6 &\leq -2 \\
8x_1 - 3x_2 + 5x_3 - 2x_4 + 2x_6 &\leq 4 \\
4x_1 + 8x_3 + 7x_4 - x_5 + 3x_6 &\leq 1 \\
5x_1 + 2x_2 - 3x_3 + 6x_4 - 2x_5 - x_6 &\leq 5 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

has optimal solution

$$x_1^* = 2, x_2^* = 4, x_3^* = 0, x_4^* = 0, x_5^* = 7, x_6^* = 0.$$

(b) The problem

$$\max 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5$$

subject to

$$\begin{aligned}
2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 &\leq 1 \\
x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 &\leq 1 \\
5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 &\leq 22 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

has optimal solution:

$$x_1^* = 0, x_2^* = 2, x_3^* = 0, x_4^* = 7, x_5^* = 0.$$

Problem 5 Checking Feasibility with Simplex [10] Show that the following LP is not feasible:

$$\max -3y_1 - 4y_2 - y_3$$

subject to

$$\begin{aligned} y_1 - y_2 + y_3 &\leq -2 \\ y_1 + y_2 - y_3 &\leq -3 \\ y_1 - y_2 - 2y_3 &\leq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(HINT: Write the dual and use the simplex algorithm.)

Problem 6 Vertex Cover in Hypergraphs [10] A *hypergraph* $H = (V, E)$ is a generalization of a graph given by a vertex set V (as usual) and a set E of *hyperedges*. A hyperedge $e \in E$ is just a subset of V (but unlike a usual edge, it can have more than 2 elements or less). For example, on the vertex set $\{1, 2, 3, 4, 5\}$, we can have hyperedges $\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3\}, \{5\}$. Each vertex $v \in V$ has a weight $w(v) \geq 0$ and each hyperedge $e \in E$ has a coverage *requirement* $r(e)$ which is a non-negative integer. The problem is to find a subset $U \subseteq V$ such that $|U \cap e| \geq r(e)$ for all $e \in E$ and the total weight $w(U) := \sum_{v \in U} w(v)$ is as small as possible. (Obviously we need $r(e) \leq |e|$ for it to be possible at all.)

(a) Formlulate the problem as an ILP and pass to the LP relaxation.

(b) Write the dual to the LP.

(c) Write the complementary slackness conditions for this primal–dual pair.

(d) Develop a primal–dual algorithm for the problem based on the continuous time view of raising dual variables until constraints become tight and using the complementary slackness conditions as a guide.

(e) By using the dual solution give an analysis of the performance of approximation quality of the algorithm in terms of some parameters of the underlying hypergraph.

(f) Give a discrete implementation of the algorithm and state the running time.