Q1) Let y_{ij} = number of passengers in class "i" travelling route "j"

where

$i=1$ - Super Flex $\qquad j=1$ - $S \to G$

$i=2$ - Flex $\qquad j=2$ - $G \to M$

$i=3$ - Economy $\qquad j=3$ - $S \to M$.

Let $A = \begin{bmatrix} 400 & 350 & 530 \\ 600 & 450 & 870 \\ 800 & 650 & 980 \end{bmatrix}$ "Ticket prices"

$B = \begin{bmatrix} 15 & 12 & 10 \\ 35 & 23 & 18 \\ 45 & 38 & 43 \end{bmatrix}$ "origin - dest./ fare combination upper bounds".

Number of passengers $S \to G$ = $\cancel{\sum_{i=1,2,3} \sum_{j}}$ $\underbrace{\sum_{i=1}^{3}}_{\text{all three fare classes}} \underbrace{\sum_{j=1,3}}_{S \to G,\ S \to M} y_{ij}$

Number of passengers $G \to M$ = $\sum_{i=1}^{3} \underbrace{\sum_{j=2,3}}_{G \to M,\ S \to M} y_{ij}$

L.P.:

$$\max_{\{y_{ij}\}} \sum_{i=1}^{3} \sum_{j=1}^{3} y_{ij}\, a_{ij}$$

s.t.

$$y_{ij} \le b_{ij} \quad \forall\, i \in 1 \cdots 3,\ j \in 1 \cdots 3$$

$$\sum_{i=1}^{3} \sum_{j=1,3} y_{ij} \le 80$$

$$\sum_{i=1}^{3} \sum_{j=2,3} y_{ij} \le 80$$

Q 2: **Weighted Vertex Cover:** $G = (V, E)$

$|V| = n$ (number of vertices)

$W = \{w_i : i \in V\}$ weights on vertices.

$|E| = m$ (number of edges).

$$\min_{y_1, \cdots y_n} \sum w_i y_i$$

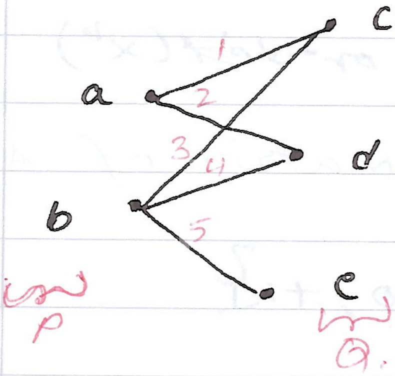
s.t. $\quad 0 \le y_i \le 1 \quad \forall\, i \in \{1 \cdots n\}$

$\quad y_i + y_j \geq 1 \quad \forall\, (i, j) \in E.$

$\boxed{\text{See MG. lemma 8.2.4 and 8.2.5}}$

Let A = incidence matrix $\in \mathbb{R}^{n \times m}$ s.t. $a_{ij} = \begin{cases} 1 & v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$

<u>example:</u>



; edges

$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline a & 1 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 1 & 1 \\ c & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 1 \end{array}$$

vertices

(a) T.P.T. A is unimodular $\Leftrightarrow$ Any square sub-matrix of A has determinant 0 or $+1$ or -1..
(X) has $\det(X) \in \{0, -1, +1\}$.

<u>Proof:</u> by induction on size "l" of $X^{l \times l}$.

<u>case 1</u> $l = 1$

$X = [0]$ or $[1] \Rightarrow \det(X) = 0$ or 1.

<u>case 2,</u> $l = 2$

$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or 15 other possibilities: $\Rightarrow \det(X) \in \{-1, 0, 1\}$

(can be enumerated)

general case: Assume all $(l-1)\times(l-1)$ submatrices of A have $\det(X') \in \{-1, 0, 1\}$.
(X')

Consider $X^{l\times l}$ submatrix of A. (of size "l")
then:

Note: each col of X can have at most 2 ones
[∵ each column corresponds to an edge]

case ① X has a column with all zeros.
$\det(X) = 0$.

case ② X has a column with single 1.
Let X'' be cofactor corresponding to the 1.

then $\det(X) = +\det(X'')$ or $-\det(X'')$

But: size $(l-1)\times(l-1)$ submatrix of A.
By assumption
$\det(X'') \in \{-1, 0, +1\}$.

$\therefore \det(X) \in \{-1, 0, +1\}$.

case ③ All columns of X have two 1's.

eg: $X = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 1 \\ c & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 \end{array}$ [see previous example]

Each edge has one endpoint in "p"

and one end point in "Q".

<u>In Figure</u> ~~P = {a, 2}~~

$P = \{a, b\}$ ~~$Q = \{3, 4, 5\}$~~

$Q = \{c, d, e\}$.

let $\vec{x}_i$ denote a row of x corresponding to vertex 'i'.

then $\sum_{i \in P} \vec{x}_i = \vec{x}_a + \vec{x}_b = (1, 1 \quad 1 \; 1)$

$\sum_{i \in Q} \vec{x}_i = \vec{x}_c + \vec{x}_d = (1, 1 \quad 1 \; 1)$

$\therefore \quad \sum_{i \in P} \vec{x}_i - \sum_{i \in Q} \vec{x}_i = (0 \; 0 \; 0 \; 0)$

$\Rightarrow$ columns are linearly dependent.

$\Rightarrow \quad \det(X) = 0$.

(b) We can rewrite vertex cover as:

$$\min_{y} \; 1^T y$$
$$\text{s.t.} \quad A^T y \geq 1$$
$$0 \leq y_i \leq 1$$

$\boxed{A = \text{is unimodular.}}$

<u>Can rewrite as:</u>

$$\min_{\bar{y}} \; c^T \bar{y}$$
$$\text{s.t.} \quad \bar{A}^T \bar{y} \leq \bar{b}$$
$$\bar{y} \geq 0$$

this absorbs $A^T y \geq 1$, $y_i \leq 1$

~~where~~ ~~y~~ = then converting to standard form & using simplex to solve; one gets

$$y^* = \frac{1}{\det(A')} \text{adj}(A)\, b.$$

→ square submatrix of unimodular matrix.

$$\det(A') \in \{-1, 0, 1\}$$

if problem has solution $\det(A') \neq 0$.

then $\det(A') = +1$ or -1 ⇒

$$y^* = \pm \text{adj}(A)\, b$$

matrix multiplication all integers

∴ y^* ⇒ integer solution.

⇒ LP yields integral solution over bipartite graphs

Q 3 : **Vertex Cover again**

$G = (V, E)$ graph

$n = |V|$ number of vertices

$w = \{w_i : i \in V\}$ weight " "

$m = |E|$ number of edges.

L.P. $\min_{\{y_1 - y_n\}} \sum w_i y_i$

s.t. $y_i + y_j \geq 1 \quad \forall (i,j) \in E$

$y_i \geq 0 \quad \forall i \in V.$

Assumption: Let y^* be a basic feasible solution, $\left(\overset{\shortparallel}{y_1^* \cdots y_n^*}\right)$ having some $y_i^* > 1/2$ or $0 < y_i^* < 1/2$ we define

$$V_+ = \{ i \mid 1/2 < y_i^* < 1 \}$$

$$V_- = \{ i \mid 0 < y_i^* < 1/2 \}.$$

Note: $V_+ \cap V_- = \phi.$

Example let $y^* = (0 \quad 0.1 \quad 1/2 \quad 0.35 \quad 0.54 \quad 1)$

then $V_- = \{2, 4\}$

$V_+ = \{5\}$ $\quad V_- \cap V_+ = \phi.$

For $\epsilon > 0$, define

$$y' = \begin{cases} y_i^* + \epsilon & \text{if } i \in V_+ \\ y_i^* - \epsilon & \text{if } i \in V_- \\ y_i^* & \text{otherwise.} \end{cases} \quad ; \quad y'' = \begin{cases} y_i^* - \epsilon & \text{if } i \in V_+ \\ y_i^* + \epsilon & \text{if } i \in V_- \\ y_i^* & \text{otherwise} \end{cases}$$

Choose ϵ smallest which satisfies:

① $y_i^* - \epsilon \geq 0 \quad \forall\ i \in V_+ \cup V_-$

② $y_i^* + y_j^* - \epsilon \geq 1 \quad \forall\ i \in V_{+/-}$ and $j \notin V_+ \cup V_-$ or $j \in V_{+/-}$ and $i \notin V_+ \cup V_-$

$\forall (i,j)$ s.t. $y_i^* + y_j^* > 1$

if one of them lies in V_+ or V_-.

③ $y_i^* + y_j^* - 2\epsilon \geq 1$ if $i \in V_+ \cup V_-$ & $j \in V_+ \cup V_-$.

(Both integral)

<u>Then:</u> y' and y'' are feasible solution and $\boxed{y^* = \frac{1}{2}(y' + y'')}$ convex combination.

Consider $\forall (i,j)$ s.t.
$\hookrightarrow y_i^* + y_j^* = 1$

<u>then</u> ① $y_i^* = 1$ & $y_j^* = 0$ or vice versa

② $y_i^* = y_j^* = 1/2$

③ $y_i^* \in V_+$ & $y_j^* \in V_-$ or vice versa.

These are the only possible cases.

<u>And</u>:

$y_i^* + y_j^* = y_i' + y_j' = y_i'' + y_j'' = 1$.

$\Rightarrow$ Since convex combination, y^* is not a basic feasible solution if it has $y_i^* \notin \{0, 1/2, 1\}$

Factor-2 approximation

Any basic feasible solution y^*.
of vertex cover has $y_i^* \in \{0, \frac{1}{2}, 1\}$

Choose vertex if $y_i^* = \frac{1}{2}$ or 1

factor-2 approximation

Since

$$\sum w_i y_i^* \leq \sum w_i 1$$

possibly all $\frac{1}{2}$

Q.4. **Set Cover**

$U = \{1 \dots n\}$

$S_1, S_2, \dots S_m \subseteq U$

a) Let $G=(V,E)$ be a graph with $\Rightarrow W = \{w_v : v \in V\}$ set of weights on the vertices.

Let $|V| = n$

$|E| = m$

then define:

$N(v) = \{e : v \in e\}$

the set of edges incident to a vertex "v"

Example

set	elements	weight
S_2	1, 2, 4	10
S_1	6, 4	5
S_3	3, 4	3
S_4	3, 5	20
S_5	5	10

min weight set collection $\Rightarrow S_1, S_2, S_3, S_5$

cost $= 10 + 5 + 3 + 10 = 28$

Then: $\boxed{U = \{v_1 \dots v_n, e_1 \dots e_m\}}$ (vertices + edges)

Define the sets

$\boxed{S_i := \{v_i, N(v_i)\}}$

$\forall i \in \{1 \dots n\}$ [vertex + its incident edges]

with weights $\boxed{w_{S_i} = w_i}$ (weight on the vertex)

Then the set cover problem over these sets, is the same as original vertex cover problem.

(b) Let $x_i = \begin{cases} 1 & \text{if set } S_i \text{ is chosen} \\ 0 & \text{" " } S_i \text{ is } \underline{\text{not}} \text{ chosen.} \end{cases}$

Then, the total weight:

$$W = \sum w_i x_i \qquad \text{—— (1)}$$

The constraint matrix is given by

$$\underset{\mathbb{R}^{n \times m}}{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \\ a_{n1} & & a_{nm} \end{bmatrix}$$

where $a_{ij} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{otherwise.} \end{cases}$

In the previous example;

$$A = \begin{array}{c|ccccc} & S_2 & S_1 & S_3 & S_4 & S_5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 1 \\ 6 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Then, the ILP is given by

$$\min_{\{x_i\}} \sum w_i x_i$$
s.t.
$$Ax \geq 1$$
$$x_i \in \{0, 1\}$$

$\sum a_{ij} x_j \geq 1$ corresponds to the i^{th} point being in at least one set.

and its L.P. relaxation

$$\min \sum w_i x_i$$
s.t.
$$Ax \geq 1$$
$$0 \leq x_i \leq 1 \quad \forall i \in \{1 \ldots m\}$$

Rounding rule:

~~$x_i = 1$~~

$$\cancel{\boxed{\text{Chosen sets} = \{i : x_i^* \geq 1/f\}}}$$

~~Choose sets with $x_i^* \geq$~~

Analysis: (This analysis is present in "Approximation Algorithms" — Vazirani Chap 14)

Let "f" be the max frequency among all items, of appearing in different sets

For example,

in the example set cover, the max frequency for item "4" $f = 3$

[it appears in sets: S_1, S_2, S_3].

This means A (the constraint matrix) has at most 3 ones in a row.

But since

$$\sum a_{ij} x_j^* \geq 1 \qquad \forall i \in \{1 \dots n\}$$

$$\Rightarrow \quad \exists x_j^* > 1/f \quad \text{for each } i \in \{1 \dots n\}.$$

So choose:

$$S = \{i : x_i^* \geq \frac{1}{f}\}$$

results in a valid set cover.

(Following MG, Pg 38)

$$S_{Rounding} = \{i : x_i^* > 1/f\}$$

Associated cost:

$$C = \sum_{i=1}^{n} w_i \, 1(x_i^* > 1/f)$$

$$1(x) = \begin{cases} 1 & \text{if } x = \text{True} \\ 0 & \text{if } x = \text{False} \end{cases}$$

$=$ (next page)

$$c = \sum_{i=1}^{n} w_i \, 1(x_i^* > 1/f) \geq \sum_{i : x_i^* > 1/f} w_i \cdot 1.$$

$$\geq \sum_{i : x_i^* > 1/f} w_i \cdot f x_i^* \qquad [\because f x_i^* > \tfrac{1}{f}]$$

$$\geq \sum_{i=1}^{n} w_i f x_i^* = \underline{f(\sum w_i x_i^*)}$$

$$= f \, c_{LP}$$

$\therefore \dfrac{c}{c_{LP}} = f \rightarrow$ This yields a f-approxi-mation

<u>Integrality gap</u>:

$$c = \sum w_i \, 1(x_i^* > 1/f)$$

$$\leq f \cdot \sum w_i x_i^* \qquad [\text{shown above}]$$

$$\leq f \sum w_i \bar{x}_i$$

where $\bar{x}_i$ is solution of ILP.

[$\because LP_{obj} \leq ILP_{obj}$ in case of minimization]

Integrality gap $= f$.

Q5: LP w/o GLPK

primal: $\min \quad 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5$

s.t.

$y_1: \quad x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4$

$y_2: \quad 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3.$

$x_i \geq 0.$

Dual:

$\max \quad 4y_1 + 3y_2$

s.t. $y_1 + 2y_2 \leq 2 \quad : x_1$

$y_1 - 2y_2 \leq 3 \quad : x_2$

$2y_1 + 3y_2 \leq 5 \quad : x_3$

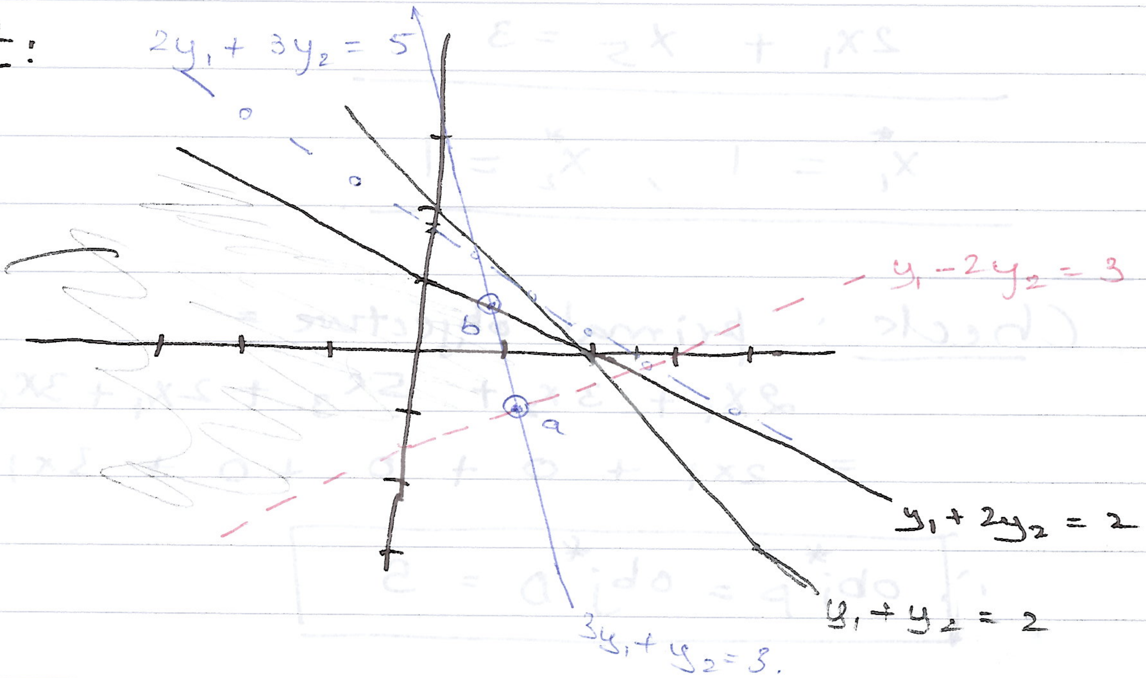
$y_1 + y_2 \leq 2 \quad : x_4$

$3y_1 + y_2 \leq 3 \quad : x_5$

$y_i \geq 0.$

Plot:

feasible region.



(a): $3y_1 + y_2 = 3$

$y_1 - 2y_2 = 3$

$y_{1a} = \frac{9}{7} \quad y_{2a} = \frac{-5}{7}$

$obj_{(a)} = \frac{9}{7} \times 4 - \frac{5}{7} \times 3 = \frac{21}{7} = 3$

(b) $y_1 + 2y_2 = 2$

$3y_1 + y_2 = 3$

$y_{1(b)} = \frac{4}{5} \quad y_{2(b)} = \frac{3}{5}$

$obj_{(b)} = \frac{4}{5} \times 4 + \frac{3}{5} \times 3 = 5$

$\therefore$ The point $y^* = \left(\frac{4}{5}, \frac{3}{5}\right)$ maximizes the dual. with $obj_D^* = 5$.

<u>Note</u> constraints corresponding to x_2, x_3 & x_4 are not active.

$$\therefore \; x_2^* = 0 = x_3^* = x_4^*.$$

<u>Further</u>

$$\because \; y_1^* \neq 0 \; \& \; y_2^* \neq 0 \; \Rightarrow$$

$$x_1 + x_2 + 2x_3 + x_4 + 3x_5 = 4 \qquad (y_1^* \neq 0)$$

$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 = 3 \qquad (y_2^* \neq 0)$$

$$\Rightarrow \; \because \; x_2^* = x_3^* = x_4^* = 0$$

$$\Rightarrow \; x_1 + 3x_5 = 4$$

$$2x_1 + x_5 = 3$$

$$\underline{\overline{x_1^* = 1 \;,\; x_2^* = 1}}$$

<u>Check</u> : primal objective =

$$2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5$$

$$= 2\times 1 + 0 + 0 + 0 + 3\times 1 = 5.$$

$$\therefore \; \boxed{obj_P^* = obj_D^* = 5}$$

Q 6: Implementing Vertex Cover

primal: $\min \sum w_i x_i$

s.t.

$x_i + x_j \geq 1 \quad \forall (i,j) \in E$

$x_i \geq 0 \quad \forall i \in V.$

Standard notation used: $G = (V, E)$, $W = \{w_i : i \in V\}$, $n = |V|$

$m = |E|$

$\forall i \in V : \quad N(i) = \{ e \in E : i \in e \}$

the set of edges incident to vertex 'i'

ⓐ $y_{ij}^{(0)} = 0$; Rate $= 1$ $\Rightarrow y_{ij}^{(t)} = t$

if $y_{ij}^{(t)}$ is still active

ⓑ $w_i^{(t)} = w_i - \sum_{(i,j) \in E} y_{ij}^{(t)}$

$\Delta_i^{(t)}$ = active degree at time 't'.

$\therefore \delta t_i$ = time before vertex 'i' goes inactive at current rate

$= \dfrac{w_i^{(t)}}{\Delta_i^{(t)}}$

$\therefore \delta t = \min_{i \in V^{(t)}} \delta t_i = \min_{i \in V^{(t)}} \dfrac{w_i^{(t)}}{\Delta_i^{(t)}}$

where $V^{(t)}$ denotes the currently active vertices.

(c) Algorithm:

Input: $G = (V, E)$; $W = \{w_i : i \in V\}$.

Initialize: $t = 0$; $x_v = 0 \ \forall v \in V$

$V^{(t)} \leftarrow V$

$y_{ij}^{(t)} \leftarrow 0 \ \forall (i, j) \in E$

Loop: While $V^{(t)} \neq \phi$.

For $v \in V^{(t)}$

Compute $w_i^{(t)} = w_i - \sum_{(i,j) \in E} y_{ij}^{(t)}$

Compute $\Delta_i^{(t)} = |N(i) \cap V^{(t)}|$

Compute $\delta t_i = \dfrac{w_i^{(t)}}{\Delta_i^{(t)}}$

end.

Finding correct vertex to freeze → Find $\underset{V^{(t)}}{\underset{\Uparrow}{v'}}$ having min δt_i

freezing the vertex → Set $x_{v'} = 1$ ~~& $V^{(t+\delta t_{v'})} = V^{(t)} \setminus \{v'\}$~~

for $e \in N(v^{(t)})$ ← all active edges

updating active edges by ~~[illegible]~~ $\delta t \times$ rate → $y_{ij}^{(t+\delta t)} = t + \delta t$.

end

$t \leftarrow t + \delta t$.

Set $V^{(t)} \leftarrow V^{(t)} \setminus \{v'\}$.

end.

Return X_v.