Finite Automata and Formal Languages

TMV026/DIT321 – LP4 2010

Lecture 7

April 20th 2010

Overview of today's lecture:

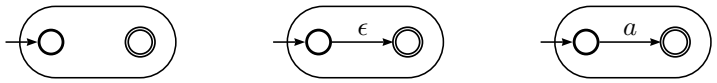
- **From Regular Expressions to Finite Automata**
- **Pumping Lemma for Regular Languages**
- **Closure Properties for Regular Languages**

From Regular Expressions to Finite Automata

Proposition: *Every language defined by a RE is also defined by a FA.*

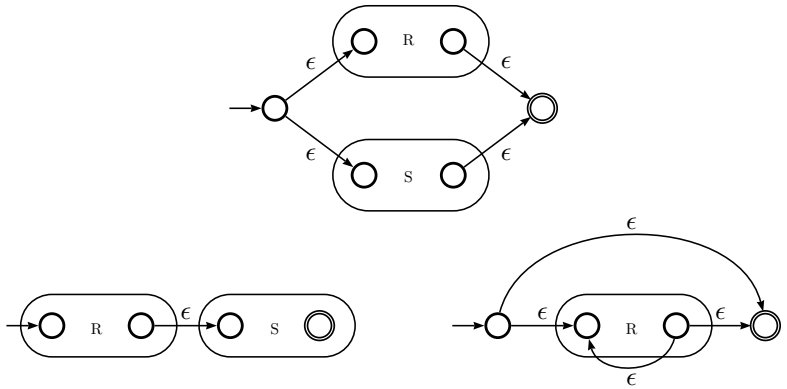
Proof: Let $\mathcal{L} = \mathcal{L}(R)$ for some RE R. By induction on R we construct a ϵ-NFA E with only one final state and no arcs into the initial state or out of the final state, and such that $\mathcal{L} = \mathcal{L}(E)$.

Base cases are $\emptyset$, ϵ and $a \in \Sigma$. The corresponding ϵ-NFA recognising the languages $\emptyset$, $\{\epsilon\}$ and $\{a\}$ respectively, are:



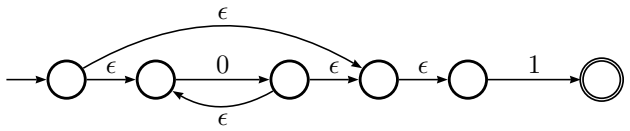
From RE to FA: Inductive Step

Given the RE R and S and FA for them, we construct the FA for $R + S$, RS and R^* recognising the languages $\mathcal{L}(R) \cup \mathcal{L}(S)$, $\mathcal{L}(R)\mathcal{L}(S)$ and $\mathcal{L}(R)^*$ respectively:

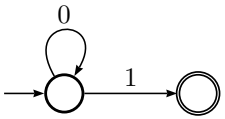


Example: From RE to FA

Let us follow this method to construct a FA for the RE 0^*1.



Compare it with the following FA:



How to identify Regular Languages?

We have seen that a language is regular iff there is a DFA that accepts the language.

Then we saw that DFA, NFA and ϵ-NFA are equivalent in the sense that we can convert between them.

Then FA accept all and only the regular languages (RL).

Now we have seen how to convert between FA and RE.

Hence RE also define all and only the RL.

How to Prove that a Language is NOT Regular?

In a FA with n states, any path

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \dots q_{m-1} \xrightarrow{a_m} q_m$$

has a loop if $m \geqslant n$.

That is, we have $i < j$ such that $q_i = q_j$ in the path above.

This can be seen as an application of the *Pigeonhole Principle*, which is an important reasoning technique in mathematics and computer science.

(See Wikipedia.)

The Pigeonhole Principle

"If you have more pigeons than pigeonholes and each pigeon flies into some pigeonhole, then there must be at least one hole with more than one pigeon."

More formally: if $f : X \to Y$ and $|X| > |Y|$ then f cannot be *injective* and there must exist at least 2 different elements with the same image, that is, there must exist $x, z \in X$ such that $x \neq z$ and $f(x) = f(z)$.

This principle is often used to show the existence of an object without building this object explicitly.

Example: In a room with at least 13 people, at least 2 of them are born the same month (maybe on different years).

We know the existence of these 2 people, maybe without being able to know exactly who they are.

How to Prove that a Language is Not Regular?

Example: Let us prove that $\mathcal{L} = \{0^m 1^m | m \geqslant 0\}$ is not a RL.

Let us assume it is: then $\mathcal{L} = \mathcal{L}(A)$ for some FA A with n states.

Let $k \geqslant n$ and let $w = 0^k 1^k \in \mathcal{L}$.

Then there must be an accepting path $q_0 \xrightarrow{w} q \in F$.

Since there are only n states and $k \geqslant n$ we know there is a loop (by the pigeonhole principle) at some point in the first part.

Then $w = xyz$ with $|xy| = j \leqslant n$, $y \neq \epsilon$ and $z = 0^{k-j} 1^k$ such that

$$q_0 \xrightarrow{x} q_l \xrightarrow{y} q_l \xrightarrow{z} q \in F$$

Observe that the following path is also an accepting path

$$q_0 \xrightarrow{x} q_l \xrightarrow{z} q \in F$$

However y must be of the form 0^i with $i > 0$ hence $xz = 0^{k-i} 1^k \notin \mathcal{L}$.

This contradicts the fact that A accepts $\mathcal{L}$.

The Pumping Lemma for Regular Languages

Theorem: *Let $\mathcal{L}$ be a RL. Then, there exists a constant n (which depends on $\mathcal{L}$) such that for every string $w \in \mathcal{L}$ and $|w| \geqslant n$, we can break w into 3 strings $w = xyz$ such that*

1. $y \neq \epsilon$
2. $|xy| \leqslant n$
3. $\forall k \geqslant 0, xy^k z \in \mathcal{L}$

Proof of the Pumping Lemma

Assume we have a FA A that accepts the language, then $\mathcal{L} = \mathcal{L}(A)$.
Let n be the number of states in A.

Then any path of length $m \geqslant n$ has a loop.
Let us consider $w = a_1 a_2 \ldots a_m \in \mathcal{L}$.

We have an accepting path and a loop such that

$$q_0 \xrightarrow{x} q_l \xrightarrow{y} q_l \xrightarrow{z} q \in F$$

with $w = xyz \in \mathcal{L}$, $y \neq \epsilon$, $|xy| \leqslant n$.

Then we also have

$$q_0 \xrightarrow{x} q_l \xrightarrow{y^k} q_l \xrightarrow{z} q \in F$$

for any k, that is, $\forall k \geqslant 0, xy^k z \in \mathcal{L}$.

Application of the Pumping Lemma

Example: Let us use the Pumping lemma to prove that $\{0^m 1^m | m \geqslant 0\}$ is not a RL.

We assume it is.
So let n be given by the lemma and let $w = 0^n 1^n$, hence $|w| \geqslant n$.

By the lemma we know that $w = xyz$ with $y \neq \epsilon$, $|xy| \leqslant n$ and $\forall k \geqslant 0, xy^k z \in \mathcal{L}$.

Since $y \neq \epsilon$, we know that $y = 0^i$ with $i \geqslant 1$.

However, we have a contradiction since $xy^k z \notin \mathcal{L}$ for $k \neq 1$.

Note: The Pumping lemma is connected to the fact that a FA has *finite memory*! If we could build a machine with infinitely many states it would be able to recognise the language.

Another Application of the Pumping Lemma

Example: Let us prove that $\mathcal{L} = \{0^i 1^j | i \leqslant j\}$ is not a RL.

Let n be given by the Pumping lemma and let $w = 0^n 1^{n+1} \in \mathcal{L}$ and $|w| \geqslant n$.

Then we know that $w = xyz$ with $y \neq \epsilon$, $|xy| \leqslant n$ and $\forall k \geqslant 0, xy^k z \in \mathcal{L}$.

Since $y \neq \epsilon$, we know that $y = 0^r$ with $r \geqslant 1$.

However, we have a contradiction since $xy^k z \notin \mathcal{L}$ for $k > 2$.
(Even for $k = 2$ if $r > 1$.)

Example: What about the languages $\{0^i 1^j \mid i \geqslant j\}$, $\{0^i 1^j \mid i > j\}$ and $\{0^i 1^j \mid i \neq j\}$? Does the Pumping lemma help?

Pumping Lemma is not a Necessary Condition

By showing that the Pumping lemma does not apply to a certain language $\mathcal{L}$ we prove that $\mathcal{L}$ is not regular.

However, if the Pumping lemma *does* apply to $\mathcal{L}$, I *cannot* conclude whether $\mathcal{L}$ is regular or not!

Example: We know $\mathcal{L} = \{b^m c^m \mid m \geqslant 0\}$ is not regular.

Let us consider $\mathcal{L}' = a^+\mathcal{L} \cup (b+c)^*$.

$\mathcal{L}'$ is not regular. If $\mathcal{L}'$ would be regular, then we can prove that $\mathcal{L}$ is regular (with the properties we will see in the rest of this lecture, see slide 25).

However, the Pumping lemma does apply for $\mathcal{L}'$ with $n = 1$.

This shows the Pumping lemma is not a necessary condition for a language to be regular.

Closure Properties of the Regular Languages

Let $\mathcal{L}$ and $\mathcal{M}$ be RL. Then $\mathcal{L} = \mathcal{L}(R) = \mathcal{L}(D)$ and $\mathcal{M} = \mathcal{L}(S) = \mathcal{L}(F)$ for RE R and S, and DFA D and F.

We have seen that RL are closed under the following operations:

- union : $\mathcal{L} \cup \mathcal{M} = \mathcal{L}(R + S)$ or $\mathcal{L} \cup \mathcal{M} = \mathcal{L}(D \oplus F)$ (slide 27, lect. 3).
- complement : $\overline{\mathcal{L}} = \mathcal{L}(\overline{D})$ (slide 28, lect. 3).
- intersection : $\mathcal{L} \cap \mathcal{M} = \overline{\overline{\mathcal{L}} \cup \overline{\mathcal{M}}}$ or $\mathcal{L} \cap \mathcal{M} = \mathcal{L}(D \times F)$ (slide 22, lect. 3).
- difference : $\mathcal{L} - \mathcal{M} = \mathcal{L} \cap \overline{\mathcal{M}}$
- concatenation : $\mathcal{L}\mathcal{M} = \mathcal{L}(RS)$
- closure ("star" operation) : $\mathcal{L}^* = \mathcal{L}(R^*)$
- prefix : $\mathsf{Prefix}(\mathcal{L})$ See exercise 2 on DFA.
  (Hint: in D, make final all states in a path from the start state to final state)

Closure under Prefix

Another way to prove that the language of prefixes of a RL is regular is as follows.

Define the following function over RE:

$$
\begin{aligned}
pre(\emptyset) &= \emptyset \\
pre(\epsilon) &= \epsilon \\
pre(a) &= \epsilon + a \\
pre(R_1 + R_2) &= pre(R_1) + pre(R_2) \\
pre(R_1 R_2) &= pre(R_1) + R_1 pre(R_2) \\
pre(R^*) &= R^* pre(R)
\end{aligned}
$$

and prove that $\mathcal{L}(pre(R)) = \mathsf{Prefix}(\mathcal{L}(R))$.

Then, if $\mathcal{L} = \mathcal{L}(R)$ for some RE R then $\mathsf{Prefix}(\mathcal{L}) = \mathsf{Prefix}(\mathcal{L}(R)) = \mathcal{L}(pre(R))$.

Closure Properties of the Regular Languages

We shall now see that RL are also closed under the following operations:

- reversal
  Recall that intuitively, $\mathsf{rev}(a_1 \ldots a_n) = a_n \ldots a_1$.
  See formal definition in slide 18, lecture 2.
  Recall also that $\forall x, \mathsf{rev}(\mathsf{rev}(x)) = x$ (see slide 19, lecture 2).
  Given $\mathcal{L}$, let $\mathcal{L}^{\mathsf{r}} = \{\mathsf{rev}(x) \mid x \in \mathcal{L}\}$.

- homomorphism (substitution of string by symbols)

- inverse homomorphism

Closure under Reversal

We define the following function over RE:

$$\emptyset^r = \emptyset \qquad \epsilon^r = \epsilon \qquad a^r = a$$
$$(R_1 + R_2)^r = R_1^r + R_2^r$$
$$(R_1 R_2)^r = R_2^r R_1^r$$
$$(R^*)^r = (R^r)^*$$

Theorem: *If $\mathcal{L}$ is regular so is $\mathcal{L}^r$.*

Proof: (See theo. 4.11, pages 139–140). Let R be a RE such that $\mathcal{L} = \mathcal{L}(R)$.

We need to prove by structural induction on R that $\mathcal{L}(R^r) = (\mathcal{L}(R))^r$.

Hence $\mathcal{L}^r = (\mathcal{L}(R))^r = \mathcal{L}(R^r)$ and $\mathcal{L}^r$ is regular.

Example: The reverse of the language defined by $(0+1)^*0$ can be defined by $0(0+1)^*$

Closure under Reversal

Another way to prove this result is by constructing a ϵ-NFA for $\mathcal{L}^r$.

Proof: Let $N = (Q, \Sigma, \delta_N, q_0.F)$ be a NFA such that $\mathcal{L} = \mathcal{L}(N)$.

Define $E = (Q \cup \{q\}, \Sigma, \delta_E, q.\{q_0\})$ with $q \notin Q$ and δ_E such that

$$r \in \delta_E(s, a) \text{ iff } s \in \delta_N(r, a) \text{ for } r, s \in Q$$
$$r \in \delta_E(q, \epsilon) \text{ iff } r \in F$$

See Bassel's notes added under "News" in the web page for the formal proof that E recognises $\mathcal{L}^r$.

Recall: Functions between Languages

(from slide 26, lecture 2)

Definition: A function $f : \Sigma^* \to \Delta^*$ between 2 languages should be such that it satisfies

$$f(\epsilon) = \epsilon$$
$$f(xy) = f(x)f(y)$$

Intuitively, $f(a_1 \dots a_n) = f(a_1) \dots f(a_n)$.

Notice that $f(a) \in \Delta^*$ if $a \in \Sigma$.

In addition, f is called ***coding*** iff f is *injective*.

Definition: $f(\mathcal{L}) = \{f(x) \mid x \in \mathcal{L}\}$.

Languages are Monoids

Definition: A ***monoid*** is an algebraic structure with an associative binary operation and an identity element.

Let Σ be an alphabet.

Then Σ^* is a monoid if we consider the concatenation as binary operation and ϵ as the identity element with respect to the binary operation.

Recall:

- Concatenation is associative: $(xy)z = x(yz)$
- $x\epsilon = \epsilon x = \epsilon$
- Concatenation is in general not commutative (but this is not required in the definition of a monoid)

Homomorphisms

Definition: A ***homomorphism*** is a structure-preserving map between 2 algebraic structures.

Note: A function $h : \Sigma^* \to \Delta^*$ satisfying

$$h(\epsilon) = \epsilon$$
$$h(xy) = h(x)h(y)$$

can be seen as a homomorphism between the monoids (languages) Σ^* and Δ^*.

Recall we have then that $h(a_1 \dots a_n) = h(a_1) \dots h(a_n)$.

Closure under Homomorphisms

Theorem: *If $\mathcal{L}$ is a RL over Σ and $h : \Sigma^* \to \Delta^*$ is an homomorphism on Σ then $h(\mathcal{L})$ is also regular.*

Proof: We define the following function over RE:

$$f_h(\emptyset) = \emptyset \qquad f_h(\epsilon) = \epsilon \qquad f_h(a) = h(a)$$
$$f_h(R_1 + R_2) = f_h(R_1) + f_h(R_2)$$
$$f_h(R_1 R_2) = f_h(R_1) f_h(R_2)$$
$$f_h(R^*) = (f_h(R))^*$$

We need to prove by structural induction on R that $\mathcal{L}(f_h(R)) = h(\mathcal{L}(R))$.

Now, if RE R is such that $\mathcal{L} = \mathcal{L}(R)$ then we have that $h(\mathcal{L})$ is regular since $h(\mathcal{L}) = h(\mathcal{L}(R)) = \mathcal{L}(f_h(R))$.

(See Theorem 4.14, pages 141–142.)

Closure under Homomorphisms

Let $h : \Sigma^* \to \Delta^*$ be a homomorphism and $\mathcal{L}$ a RL over Σ.

By the previous theorem and the definition of RL, we know that there exists a DFA D over Σ and a DFA F over Δ such that

$$\mathcal{L} = \mathcal{L}(D) \quad \text{and} \quad h(\mathcal{L}) = \mathcal{L}(F)$$

F can be constructed from the RE for $\mathcal{L}$ (via an ϵ-NFA).

Not obvious though how to construct the DFA directly.

Inverse Homomorphisms

Definition: If $h : \Sigma^* \to \Delta^*$ is a homomorphism and $\mathcal{L}$ is a language over Δ^*, $h^{-1}(\mathcal{L})$ (read h ***inverse of*** $\mathcal{L}$) is the set of strings w such that $h(w) \in \mathcal{L}$.

In other words, $h^{-1}(\mathcal{L}) = \{w \mid h(w) \in \mathcal{L}\}$.

Note: h^{-1} does not necessarily correspond to a function!

Example: Imagine we have that $h(a) = c$, $h(b) = c$ and $\mathcal{L} = \{c\}$.

Then $h^{-1}(\mathcal{L}) = \{a, b\}$ but h^{-1} itself is not a function.

Closure under Inverse Homomorphisms

Theorem: *Let $h : \Sigma^* \to \Delta^*$ be a homomorphism. If $\mathcal{L}$ is a RL over Δ then $h^{-1}(\mathcal{L})$ is a RL over Σ.*

Proof: Let $D = (Q, \Delta, \delta, q_0, F)$ be a DFA such that $\mathcal{L} = \mathcal{L}(D)$.

We define the DFA $D' = (Q, \Sigma, \delta', q_0, F)$ over Σ such that

$$\delta'(q, a) = \hat{\delta}(q, h(a))$$

By induction on $|w|$ we prove that $\hat{\delta}'(q, w) = \hat{\delta}(q, h(w))$

(Recall that $\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y)$.)

Then D' accepts w iff D accepts $h(w)$ (since set of accepting states is the same in both DFA).

Note: Since h^{-1} might not be a function it seems difficult to define directly the RE that corresponds to the h inverse of $\mathcal{L}$.

Example: $\mathcal{L}'$ from Slide 12

Example: We know $\mathcal{L} = \{b^m c^m \mid m \geqslant 0\}$ is not regular.

Let us consider $\mathcal{L}' = a^+\mathcal{L} \cup (b + c)^*$.

We will prove that $\mathcal{L}'$ is not regular. Let us assume it is.

Then $a^+\mathcal{L} = \mathcal{L}' \cap \overline{(b + c)^*}$ must be regular.

Then, $\mathcal{L} = h(a^+\mathcal{L})$ must also be regular, where h is the following homomorphism: $h(a) = \epsilon, h(b) = b, h(c) = c$.

We arrive at a contradiction, hence $\mathcal{L}'$ cannot be regular.