Finite Automata and Formal Languages
TMV026/DIT321 – LP4 2010

Formal Proofs, Alphabets and Words

Week 1

In these exercises, book sections and pages refer to those in the third edition of the course book.

Let $\mathbb{N}$ is the set of all positive integers $\{0, 1, 2, \ldots\}$ (see page 22 of the text book: "Integers as recursively defined concepts").

1. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by recursion as

   $$f(0) = 0 \qquad f(n+1) = f(n) + n$$

   What are the values of $f(2)$ and $f(3)$?

   Use mathematical induction to show that for all $n \in \mathbb{N}$ we have

   $$2f(n) = n^2 - n$$

2. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.

3. Let us define by recursion the following function:

   $$0! = 1 \qquad (n+1)! = (n+1) \times n!$$

   Show that $n! \geq 2^n$ for $n \geq 4$ by analogy with the proof of example 1.17, page 21 of the text book.

4. Let us define by recursion the following two functions $f, g : \mathbb{N} \to \mathbb{N}$

   $$\begin{array}{ll} f(0) = 0 & g(0) = 1 \\ f(n+1) = g(n) & g(n+1) = f(n) \end{array}$$

   What are the values of $g(2)$ and $f(4)$? Show by mathematical induction that for all $n \in \mathbb{N}$ we have

   $$f(n) + g(n) = 1 \qquad f(n)g(n) = 0$$

   Show by *mutual* induction that $f(n) = 0$ iff $g(n) = 1$ iff n is even and that $f(n) = 1$ iff $g(n) = 0$ iff n is odd, in analogy to the proof in pages 26–28 in the text book.

5. Let us define the Fibonacci function:

$$f(0) = 0 \qquad f(1) = 1 \qquad f(n+2) = f(n+1) + f(n)$$

We then define $s(0) = 0,\ s(n+1) = s(n) + f(n+1)$.

Prove by induction that we have

$$\forall n . s(n) = f(n+2) - 1.$$

Now we define

$$l(0) = 2,\ l(1) = 1,\ l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have $l(n+1) = f(n) + f(n+2)$.

6. Given x_1 and x_2 we define a_2 and a_1

$$\begin{array}{ll} a_1(0) = 0 & a_2(0) = 1 \\ a_1(t+1) = \neg(a_2(t) \vee x_1(t)) & a_2(t+1) = \neg(a_1(t) \vee x_2(t)) \end{array}$$

We may think of this as a digital circuit. The inputs are $x_1(t)$ and $x_2(t)$ and the output is $a_2(t)$. The predicate $a_1(t)$ represents an internal state of the circuit. Let 0 mean that the voltage is *low* and 1 mean that the voltage is *high*. The voltage at the input may take any value at any time.

Compute the values of $a_1(t)$ and $a_2(t)$ for the following sequences

$$x_1 = 000000000111000\ldots \qquad x_2 = 00111000000000\ldots$$

that is $x_1(0) = x_1(1) = \ldots = x_1(8) = 0,\ x_1(9) = x_1(10) = x_1(11) = \ 1, \ldots$

If we call a *pulse* of high voltage a sequence of three 1's, explain why this circuit can be called a *memory* (it *remembers* which input pulsed last).

7. If $\Sigma = \{a, b, c\}$, what are Σ^1, Σ^2 and Σ^0?

8. If $\Sigma = \{0, 1\}$, find a counterexample to the following alleged theorem: $\forall x, y \in \Sigma^*$ we have

$$x^2 y = xyx$$

(cf. section 1.3.4)

9. Let $\Sigma = \{0, 1\}$. We define $\phi : \Sigma^* \to \Sigma^*$ by recursion as follows

$$\phi(\epsilon) = \epsilon \qquad \phi(w0) = \phi(w)1 \qquad \phi(w1) = \phi(w)0$$

What are $\phi(1011)$ and $\phi(1101)$?

Show by induction on $|w|$ that

$$|\phi(w)| = |w|.$$

10. Let $\Sigma = \{0, 1\}$. We define the reverse function on Σ^* by the equations

$$rev(\epsilon) = \epsilon \qquad rev(ax) = rev(x)a$$

What are $rev(010)$ and $rev(10)$?

Show by induction on y that we have

$$rev(yx) \ = rev(x)rev(y).$$

Show by induction on $n \in \mathbb{N}$ that we have

$$rev(x^n) = (rev(x))^n.$$

11. Given a finite alphabet Σ, when can we have $x^2 = y^3$ with $x, y \in \Sigma^*$?