Algorithms. Assignment 6

Problem 10

Suppose that a connected undirected graph $G = (V, E)$ models some villages and the roads connecting several pairs of these villages. The picturesque region has numerous creeks and ponds (and mosquitos), therefore the roads have bridges with weight restrictions: For every road $e \in E$ we are given a number $w(e)$ indicating that only trucks of weight at most $w(e)$ can use this road e.

Now the people from the Kruskal Freight Forwarders company want to find a route R between two villages u and v that is usable by trucks being as heavy as possible. That is, they want a path R connecting u and v that maximizes $\min\{w(e) | \, e \in R\}$. In contrast to the shortest-path problem, assume that the length of the route is not important at all (the region is not very large), but the weight restrictions are crucial.

There is a surprisingly simple and elegant algorithm that solves this problem even for all pairs u, v at once: Construct a maximum(!) spanning tree T, by applying Kruskal's algorithm "upside down". In detail: Start from T with an empty edge set, and in every step, add to T an edge e with *largest* $w(e)$ that does not create cycles in T.

Since T is a tree, it contains a unique path between any two given nodes u and v. And this path is an optimal route between u and v! Your task is to prove this claim.

We give a hint to get started with the proof. Consider any two nodes u, v and the path R in T that connects them. Let e be the edge of R with minimum $w(e)$ (in R). Assume there is a better path, and consider the moment when Kruskal's algorithm had inserted e in T. – Continue from here and derive a contradiction. There is not much to write, but you must find a proper argument.