

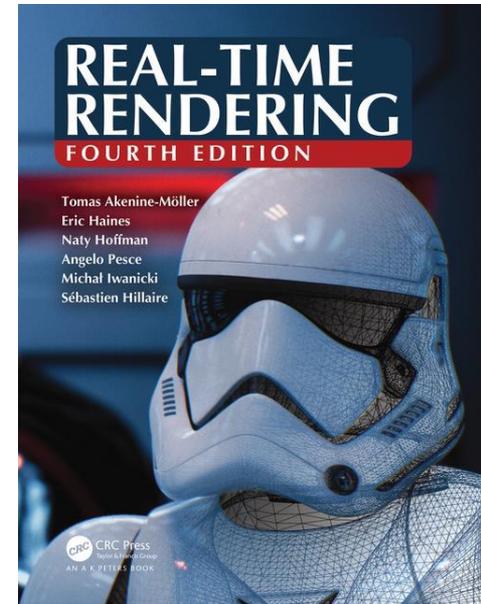


# Vectors and Transforms

In  
3D Graphics

# Course Structure

- 14 lectures
  - Book is the verbal format / more meticulous explanations
  - Lecture slides are only short summary
    - Perhaps not enough to fully understand
  - Exam (salstentamen):
    - I will only assume that you have studied the topics covered by the slides.
    - Reading instructions are pointers to more verbal descriptions in the book
    - May come a few “harder” questions, intended to force you to think beyond what’s in the slides (and that could of course accidentally be covered by the book).
- Tutorials – the practical experience
  - 1-6 “holds your hand”. Very fast. Intentionally lots of copy/paste. Do them in 2-3 weeks. No need to wait for their deadlines.
  - Project – Here, you apply the knowledge from tutorial 1-6, so you must have understood them.
    - You will need the 3-4 weeks for the project.



# The Bonus Material

- Bonus material on home page
  - <http://www.cse.chalmers.se/edu/course/TDA362/schedule.html>
  - Purpose: **only** to be of help in case lectures and course book is not enough for you to understand. Sometimes, it helps having same topics explained in a second way.
  - Skip the bonus material if you are not very interested.
  - No exam questions on bonus material!

# Quick Repetition of Vector Algebra

BarCharts, Inc.®

AMERICA'S #1 ACADEMIC OUTLINE!

PHYSICS

VECTORS AND COORDINATE SYSTEMS

### DEFINITIONS

**• Scalar and Vector Quantities.** Physical quantities such as temperature,  $T$ , distance,  $s$ , density,  $\rho$ , work,  $W$ , etc., that can be fully described by a single number are called **scalars**. Scalars are not associated with any direction. Physical quantities that have both **magnitude and direction** are called **vectors**, e.g. force,  $E$ , velocity,  $V$ , acceleration,  $a$ , momentum,  $E$ , etc.

**• Coordinate Systems.** A vector,  $E$ , can be described in reference to a **coordinate system**. Two-dimensional coordinate systems can be **cartesian or polar**. Three-dimensional coordinate systems can be **cartesian, cylindrical or spherical**.

### THREE-DIMENSIONAL (3-D) COORDINATE SYSTEMS:

**Cartesian Coordinates  $(x, y, z)$ .** A vector,  $E$ , in a 3-D cartesian coordinate system can be written as:

$$|E| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$E = V(V_x, V_y, V_z) = V_x i + V_y j + V_z k$$

3-D Cartesian Coordinates

**• Cylindrical Coordinates  $(r, \theta, z)$ .** A vector,  $E$ , in a cylindrical coordinate system can be written as:

$$E = V(r, \theta, z), \text{ where}$$

$$V_r = \sqrt{V_x^2 + V_y^2} \text{ and } z = z$$

Cylindrical Coordinates

**• Spherical Coordinates  $(r, \theta, \phi)$ .** A vector,  $E$ , in a spherical coordinate system is written as:  $E = V(r, \theta, \phi)$ , where:

$$V_r = \sqrt{V_x^2 + V_y^2 + V_z^2},$$

$$\theta = \tan^{-1} \left( \frac{V_y}{V_x} \right),$$

$$\phi = \cos^{-1} \frac{V_z}{\sqrt{V_x^2 + V_y^2 + V_z^2}}$$

Spherical Coordinates

**• Relation Between Cartesian and Spherical Coordinates**

$$V_x = |E| \sin \phi \cos \theta,$$

$$V_y = |E| \sin \phi \sin \theta,$$

$$V_z = |E| \cos \phi$$

$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

### VECTOR ALGEBRA

**• Vector Addition.** The sum of two vectors,  $E_A$  and  $E_B$ , in a 2-D cartesian coordinate system is a vector,  $E_R$ , defined as:

$$E_R = E_A + E_B$$

In component notation, the summation is given as:

$$V_{RX} = V_{AX} + V_{BX}, \quad V_{RY} = V_{AY} + V_{BY}$$

Vector Addition

**• Commutative Law of Vector Addition:**

$$E_A + E_B = E_B + E_A$$

**• Associative Law of Vector Addition:**

$$(E_A + E_B) + E_C = E_A + (E_B + E_C)$$

**• Distributive Law for Multiplication by a Scalar ( $\epsilon$ ):**

$$\epsilon(E_A + E_B) = \epsilon E_A + \epsilon E_B$$

**• Scalar or Dot Product:**

$$E_A \cdot E_B = E_A \cdot E_B = |E_A| |E_B| \cos \alpha$$

where  $\alpha$  is the angle between the two vectors. If the two vectors are perpendicular to each other then:

$$|E_A| |E_B| \cos \alpha = 0, \quad E_A \perp E_B$$

If the vectors are given in terms of their components, then in a 3-D cartesian coordinate system:

$$E_A \cdot E_B = V_{AX} V_{BX} + V_{AY} V_{BY} + V_{AZ} V_{BZ}$$

since:

$$i \cdot i = 1 \quad j \cdot j = 1 \quad k \cdot k = 1$$

$$i \cdot j = 0 \quad i \cdot k = 0 \quad j \cdot k = 0$$

$$j \cdot i = 0 \quad k \cdot i = 0 \quad k \cdot j = 0$$

Scalar Product

**• Vector or Cross Product**

$$E_A \times E_B = |E_A| |E_B| (\sin \alpha) \epsilon$$

where  $\epsilon$  is the unit vector perpendicular to the plane formed by vectors  $E_A$  and  $E_B$ .

**• Right-hand rule:** the direction of the vector  $\epsilon$  can be found by curling the fingers of the right hand around a hypothetical axis perpendicular to plane  $E_A - E_B$  so that the vector  $E_A$  rotates along the angle  $\alpha$  until it is aligned with vector  $E_B$ . The thumb then gives the direction of  $\epsilon$ .

Right-Handed Rule

$E_A \times E_B = -E_B \times E_A$

A cartesian coordinate system is called a **right-handed system** if  $i \times j = k$ .

If two vectors are parallel to each other:

$$E_A \times E_B = 0, \quad E_A \parallel E_B$$

If the vectors are given in terms of their components, then in a 3-D cartesian coordinate system:

$$E_A \times E_B = \begin{vmatrix} i & j & k \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{vmatrix}$$

Vector Product

**• Triple Scalar Product**

The magnitude of the triple scalar product is equal to the volume of the parallelepiped formed by the three vectors  $E_A, E_B, E_C$ :  $E_A \cdot (E_B \times E_C)$ .

Triple Scalar Product

**• Differentiation Formulas of Vectors**

$$\frac{d}{dt} [u(t) + v(t)] = \frac{du}{dt} + \frac{dv}{dt}$$

$$\frac{d}{dt} [c u(t)] = c \frac{du}{dt}$$

$$\frac{d}{dt} [f(t) u(t)] = \frac{df}{dt} u + f \frac{du}{dt}$$

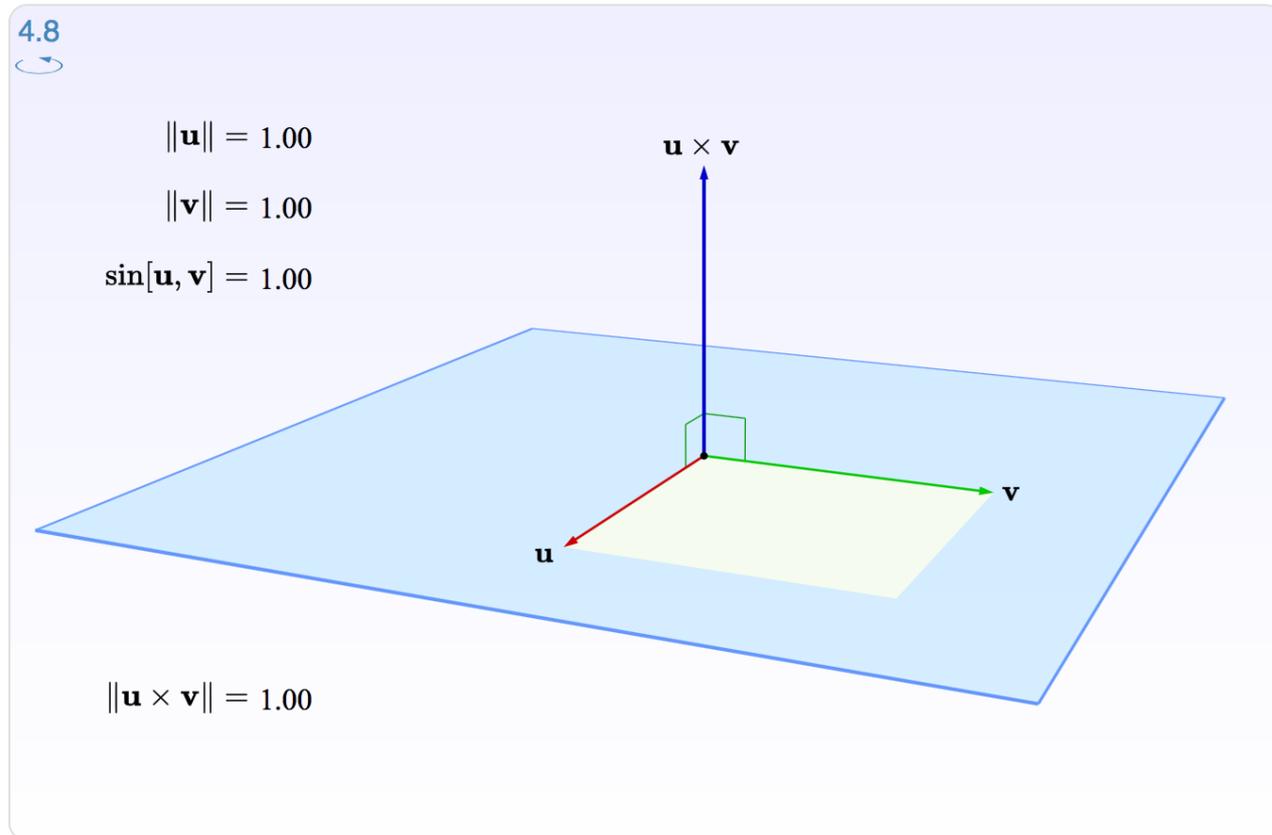
$$\frac{d}{dt} [u(t) \cdot v(t)] = \frac{du}{dt} \cdot v(t) + u(t) \cdot \frac{dv}{dt}$$

**• Integration of a Vector**

$$\int_a^b r(t) dt = [R(t)]_a^b = R(b) - R(a)$$

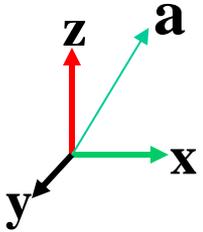
# Excellent interactive online linear algebra repetition:

- <http://immersivemath.com/ila/index.html>

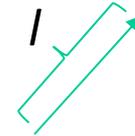


# Quick Repetition of Vector Algebra for 3D graphics

A 3D vector,  $\mathbf{a}$ , contains 3 elements:  $\mathbf{a} \equiv (a_x, a_y, a_z)$ , which are coordinates (or lengths) along the 3 coordinate axes.



The length,  $l$ , of a vector is:  $l = \|\mathbf{x}\| = \sqrt{(x^2 + y^2 + z^2)}$



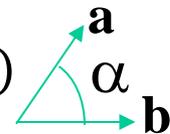
Normalizing a vector,  $\mathbf{n}$ , means to scale the vector such that it becomes a unit vector,  $\hat{\mathbf{n}}$ , i.e., its length = 1.

$$\text{E.g.,: } \hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{n}}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \left( \frac{n_x}{c}, \frac{n_y}{c}, \frac{n_z}{c} \right), \text{ where } c = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

The dot product is typically used to find the angle,  $\alpha$ , between two vectors.

If  $\mathbf{a}$  and  $\mathbf{b}$  are of unit length (normalized), then  $\cos \alpha = \mathbf{a} \cdot \mathbf{b}$

where  $\mathbf{a} \cdot \mathbf{b} = (a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = (a_x b_x + a_y b_y + a_z b_z)$



The definition of the dot product is:  $\cos \alpha = \frac{\mathbf{v}_a \bullet \mathbf{v}_b}{\|\mathbf{v}_a\| \|\mathbf{v}_b\|}$

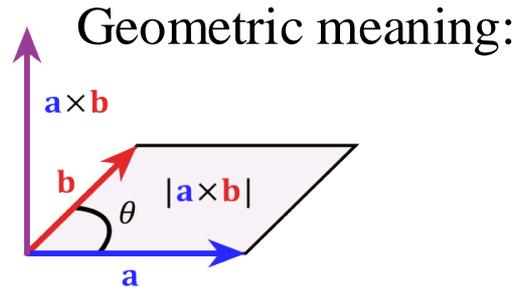
(so here for non-normalized  $\mathbf{a}$  and  $\mathbf{b}$ , we divide with their lengths, as you see.)

# Quick Repetition of Vector Algebra for 3D graphics

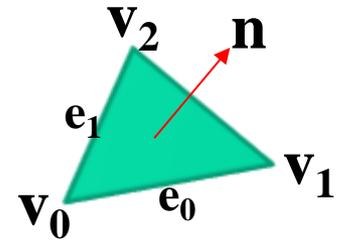
## Cross product

Definition:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$



The cross product is typically used to find a vector,  $\mathbf{a}$ , that is perpendicular to two others ( $\mathbf{b}$  and  $\mathbf{c}$ ):  $\mathbf{a} = \mathbf{b} \times \mathbf{c}$



Example to find a triangle normal:  $\mathbf{n} = \mathbf{e}_0 \times \mathbf{e}_1$ , where  $\mathbf{e}_0 = (\mathbf{v}_1 - \mathbf{v}_0)$  and  $\mathbf{e}_1 = (\mathbf{v}_2 - \mathbf{v}_0)$

In code: `n = cross(e0, e1);`

In maths:

$$\mathbf{n} = ((e_{0y}e_{1z} - e_{0z}e_{1y}), (e_{0z}e_{1x} - e_{0x}e_{1z}), (e_{0x}e_{1y} - e_{0y}e_{1x}))$$

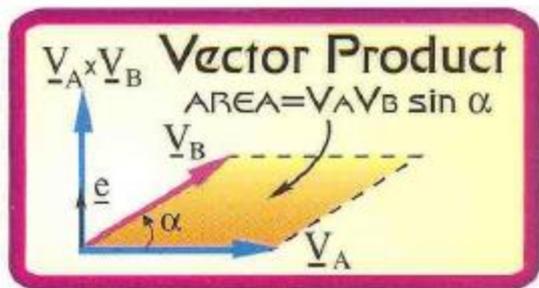
Note that the length of  $\mathbf{n}$  then is two times the size of the *triangle* area. (So the cross product can be used to find the area between two vectors).

We typically want *normals* to be of unit length (=1), and therefore we often normalize  $\mathbf{n}$ . In code: `n = normalize(n);`

# Quick Repetition of Vector Algebra – definitions

$$\underline{V}_A \times \underline{V}_B = \begin{Bmatrix} \underline{x} & \underline{y} & \underline{z} \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{Bmatrix} = (a,b,c), \text{ where } a = V_{AY}V_{BZ} - V_{AZ}V_{BY}$$

$$\begin{Bmatrix} \underline{x} & \underline{y} & \underline{z} \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{Bmatrix} \underline{x}$$



$$b = V_{AZ}V_{BX} - V_{AX}V_{BZ}$$

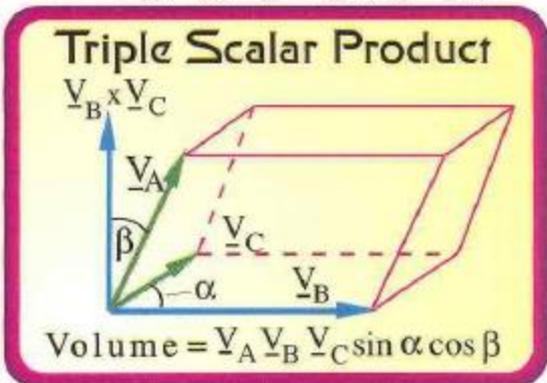
$$\begin{Bmatrix} \underline{x} & \underline{y} & \underline{z} \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{Bmatrix} \underline{V}_{BZ}$$

$$c = V_{AX}V_{BY} - V_{AY}V_{BX}$$

$$\underline{z} \begin{Bmatrix} \underline{x} & \underline{y} & \underline{z} \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{Bmatrix}$$

## • Triple Scalar Product

The magnitude of the triple scalar product is equal to the volume of the parallelepiped formed by the three vectors  $\underline{V}_A, \underline{V}_B, \underline{V}_C$ :  $\underline{V}_A \cdot (\underline{V}_B \times \underline{V}_C)$ .



For a 3x3 matrix, the determinant is computed as the tripple scalar product  
 $= \underline{V}_A \cdot (\underline{V}_B \times \underline{V}_C)$

# Structure of today's lecture

- Matrices
  - Matrix mult.
  - Transformation Pipeline
  - Practical usage of matrices
  - Rotations
  - Translations
  - Homogeneous coordinates
  - Shear / scale / normal matrix
  - Euler matrices
  - Quaternions
  - Projections
- Bresenham's line drawing algorithm

# Why transforms?

- We want to be able to **animate** objects and the camera
  - Translations
  - Rotations
  - Shears
  - ...
- We want to be able to use **projection** transforms

# How implement transforms?

- Matrices!
- Can you really do everything with a matrix?
- Not everything, but a lot!
- We use 3x3 and 4x4 matrices

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix}$$

# Matrix multiplication

Matrix-vector multiplication:

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} m_{00}p_x + m_{01}p_y + m_{02}p_z \\ m_{10}p_x + m_{11}p_y + m_{12}p_z \\ m_{20}p_x + m_{21}p_y + m_{22}p_z \end{pmatrix}$$

# Matrix multiplication

Matrix-vector multiplication:

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} m_{00}p_x + m_{01}p_y + m_{02}p_z \\ m_{10}p_x + m_{11}p_y + m_{12}p_z \\ m_{20}p_x + m_{21}p_y + m_{22}p_z \end{pmatrix}$$

# Matrix multiplication

Matrix-vector multiplication:

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} m_{00}p_x + m_{01}p_y + m_{02}p_z \\ m_{10}p_x + m_{11}p_y + m_{12}p_z \\ m_{20}p_x + m_{21}p_y + m_{22}p_z \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

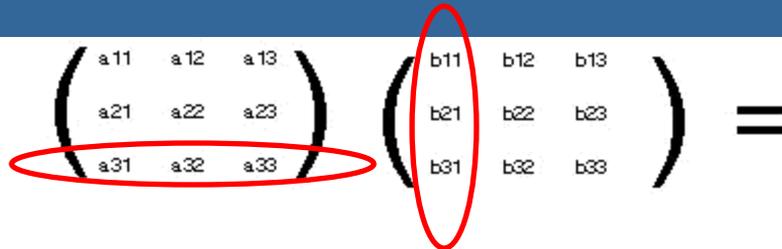
$$\begin{pmatrix} a_{11}.b_{11} + a_{12}.b_{21} + a_{13}.b_{31} & a_{11}.b_{12} + a_{12}.b_{22} + a_{13}.b_{32} & a_{11}.b_{13} + a_{12}.b_{23} + a_{13}.b_{33} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}.b_{12} + a_{22}.b_{22} + a_{23}.b_{32} & a_{21}.b_{13} + a_{22}.b_{23} + a_{23}.b_{33} \\ a_{31}.b_{11} + a_{32}.b_{21} + a_{33}.b_{31} & a_{31}.b_{12} + a_{32}.b_{22} + a_{33}.b_{32} & a_{31}.b_{13} + a_{32}.b_{23} + a_{33}.b_{33} \end{pmatrix}$$

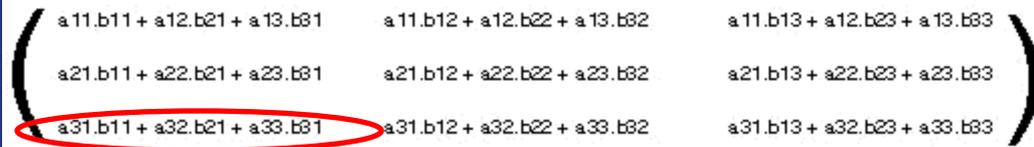
# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$


$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$


$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$

# Matrix multiplication

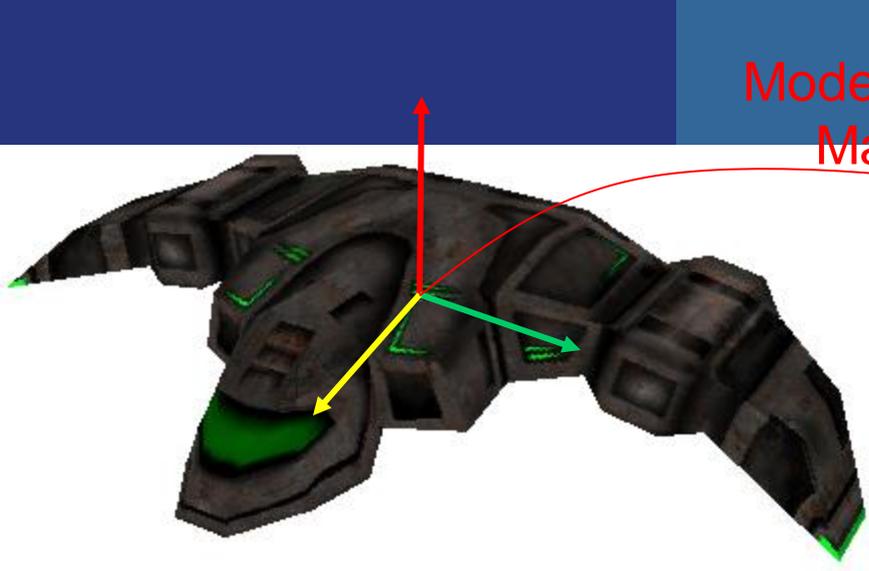
Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where  $c_{ij} = a_{row_i} \cdot b_{col_j}$

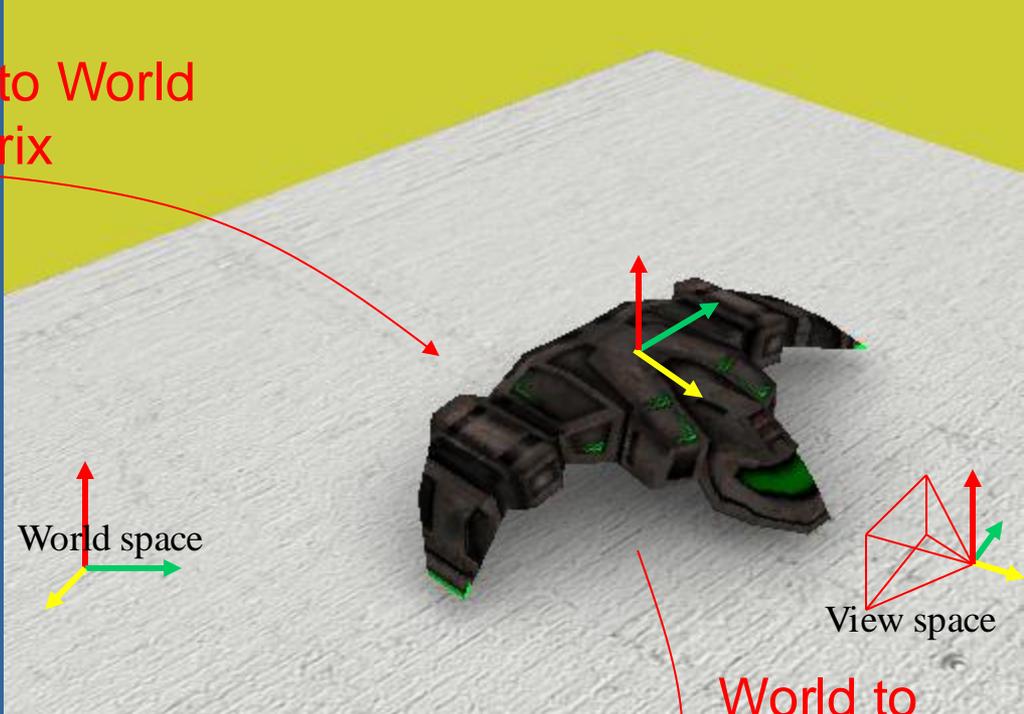
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{pmatrix}$$



Model space

Model to World Matrix



World space

View space

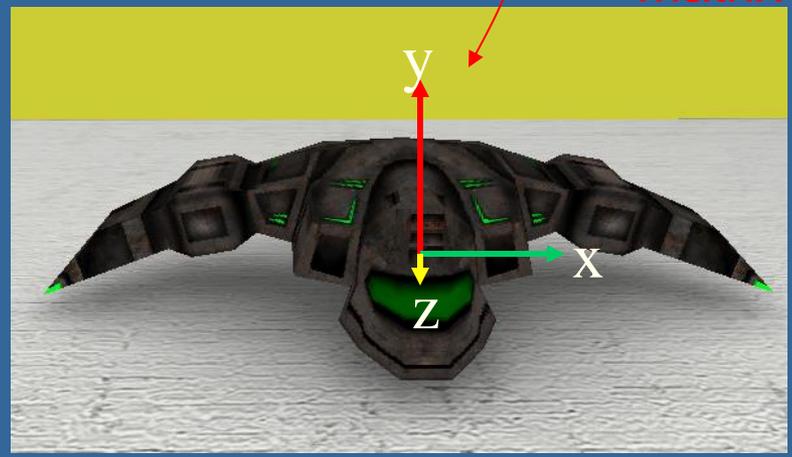
World to View Matrix

World space

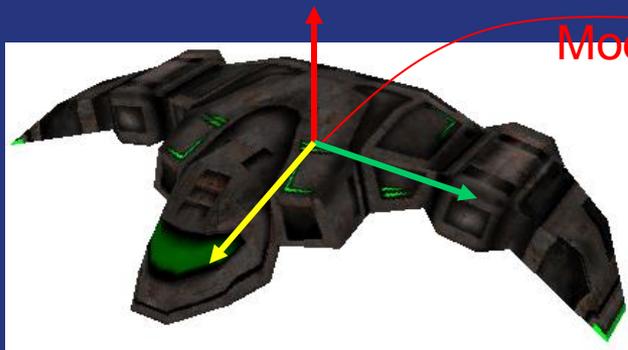
ModelViewMtx = "Model to View Matrix"

$$\text{ModelViewMtx} = (M_{V \leftarrow W} * M_{W \leftarrow M})$$

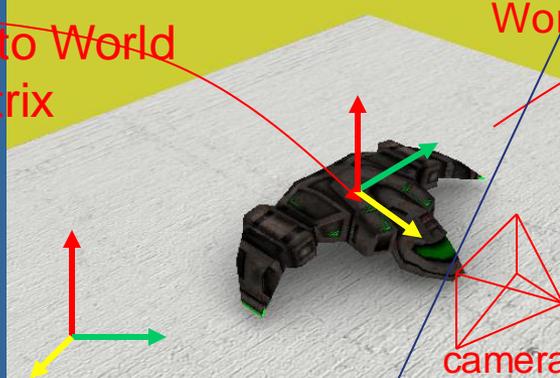
$$V_{\text{view\_space}} = \text{ModelViewMtx} * V_{\text{model\_space}}$$



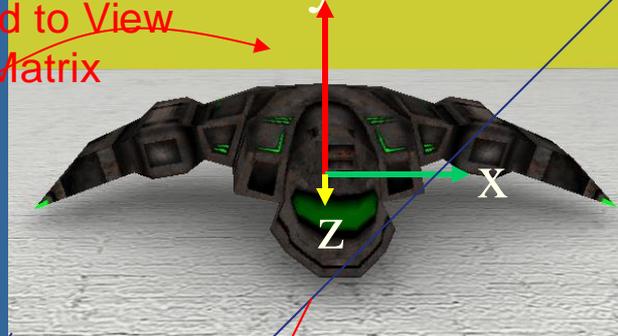
View space



Model space



World space



View space  
Camera space



Projection Matrix

After projection:  
Clip space  
(Unit space)  
(Normalized device coordinates)



Then, screen mapping to window coordinates =>  
Screen space:



ModelViewMtx = "Model to View Matrix"

$$\text{ModelViewMtx} * v = (M_{V \leftarrow W} * M_{W \leftarrow M}) * v$$

Full projection to Clip space:

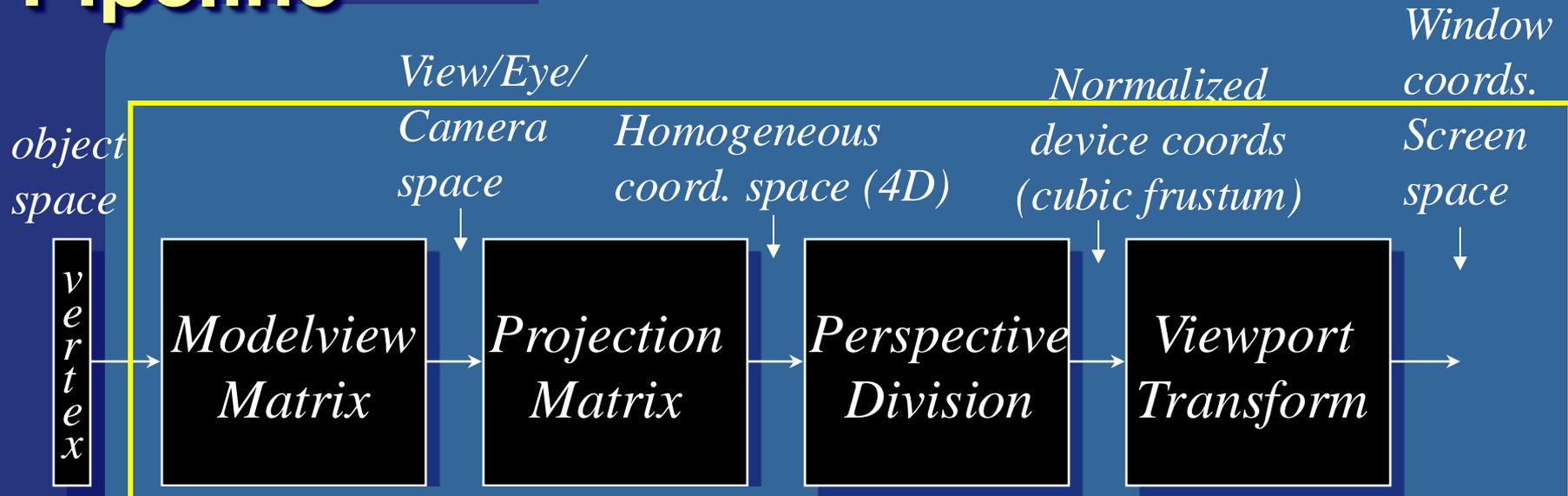
$$M_{\text{ModelViewProjection\_Matrix}} = \text{projectionMatrix} * \text{ModelViewMatrix}$$

$$\text{E.g.: } v_{\text{clip\_space}} = M_{\text{MVP}} * v_{\text{model\_space}} ; \quad // M_{\text{MVP}} = (M_P * M_{V \leftarrow W} * M_{W \leftarrow M})$$

## Lecture 2:

# Transformation Pipeline

Clip space: clipping is nowadays typically done in homogeneous space. However, it used to be done in unit-cube space. Both terminologies are still used.



What we do in the vertex shader:

```
gl_Position = modelViewProjectionMatrix*vec4(vertex,1);
```

The perspective division is then done automatically by the GPU, and same with the screen mapping (aka viewport transform).

# Simple transform example

- Say you have a circle with origin at (0,0,0) and with radius 1, i.e., a unit circle 
- `mat4 m = translate({8,0,0}); // create translation matrix`
- `RenderCircle(m); // Draw circle using m as  
// model-to-world matrix`
- `mat4 s = scale({2,2,2}); // create scaling matrix`
- `mat4 t = translate({3,2,0}); // create translation matrix`
- `RenderCircle(t*s); // use matrix (t*s)`

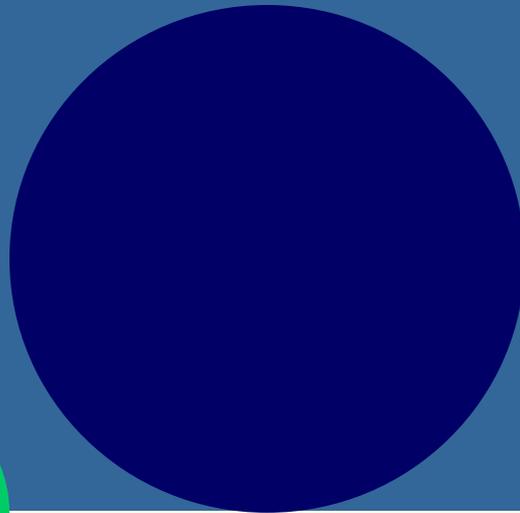
What happens?  
See next slide...

# Simple transform example

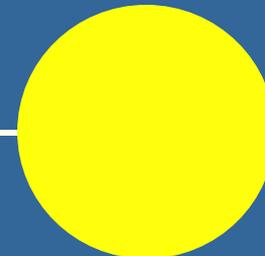
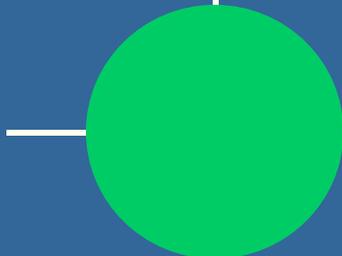
- A circle

y

```
mat4 s = scale({2,2,2});  
mat4 t = translate({3,2,0});  
RenderCircle(t*s); // Effect= first scaling, then translation
```



```
mat4 m = translate({8,0,0});  
RenderCircle(m);
```



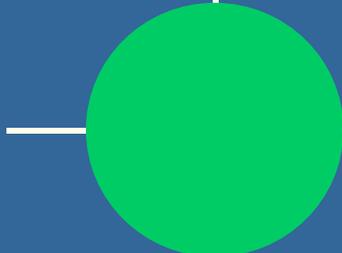
x

# Simple transform example

- A circle

y

```
mat4 s = scale({2,2,2});  
mat4 t = translate({3,2,0});  
RenderCircle(s*t); // Effect= first translation, then scaling.  
// Each vertex in the sphere will first  
// be translated (3,2,0) and then have its  
// coordinate doubled in x,y,z.  
// I.e., scaling is applied also on the  
// translation.  
// This is less intuitive so humans  
// prefer to do scaling first and then  
// translation.
```



Matrix multiplication is order dependent:  $T S \neq S T$

# Example of a simple GfxObject class

```
class GfxObject { Code on the CPU side
public:
    load("filename"); // Creates m_shaderProgram + m_vertexArrayObject
    render(mat4 projectionMatrix, mat4 viewMatrix)
    {
        mat4 modelViewProjectionMatrix = projectionMatrix * viewMatrix *
                                         m_modelMatrix;
        int loc = glGetUniformLocation(shaderProgram, "modelViewProjectionMatrix");
        glUniformMatrix4fv(loc, 1, false, &modelViewProjectionMatrix[0].x);

        glEnableVertexAttribArray(0); // the array with an x,y,z per vertex
        glEnableVertexAttribArray(1); // e.g., the array for rgb colors per vertex
        glUseProgram(m_shaderProgram); // use our vertex and fragment shader
        glBindVertexArray(m_vertexArrayObject); // tell which arrays to use
        glDrawArrays( GL_TRIANGLES, 0, m_numVertices); // draw all triangles
    };
private:
    mat4    m_modelMatrix;
    uint    m_numVertices;
    GLuint  m_shaderProgram;
    GLuint  m_vertexArrayObject;
};
```

```
#version 420 VERTEX SHADER on the GPU
layout(location = 0) in vec3 position;
layout(location = 1) in vec3 color;

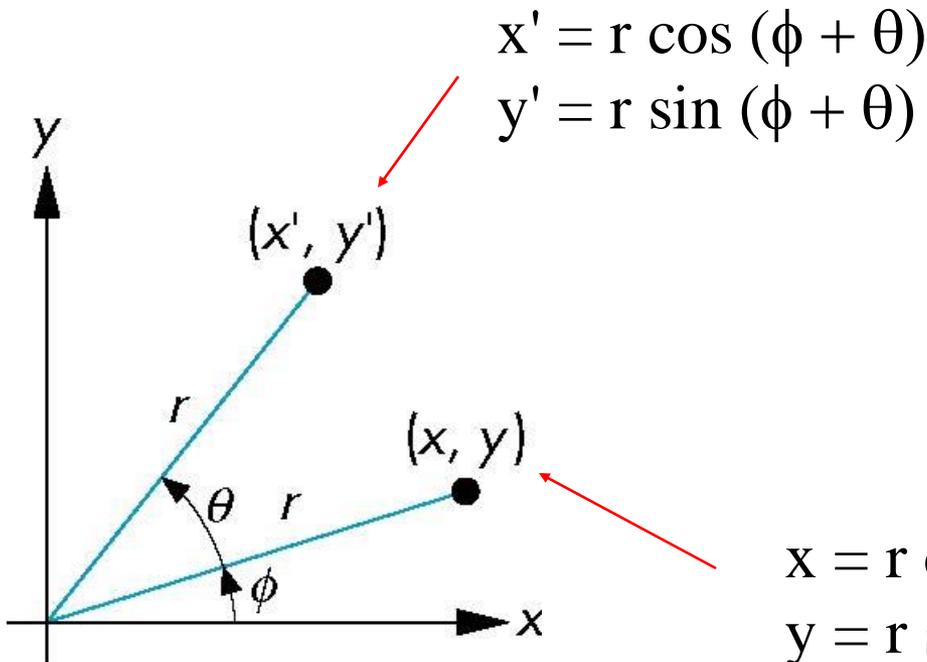
out vec4 outColor;
uniform mat4 modelViewProjectionMatrix;

void main()
{
    gl_Position = modelViewProjectionMatrix *
                  vec4(position, 1.0);
    outColor = vec4(color, 1.0);
}
```

# Rotation (2D)

Let's derive the 2D-rotation matrix,  $\mathbf{R}$ ,... Consider rotation about the origin by  $\theta$  degrees

– radius stays the same, angle increases by  $\theta$



Matrix for 2D rotation  $\theta$  radians:

Answer:  $\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

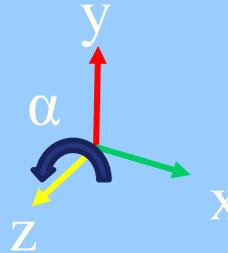
$$\underbrace{\begin{pmatrix} r \cos(\phi + \theta) \\ r \sin(\phi + \theta) \end{pmatrix}}_{(x', y')} = \begin{pmatrix} r (\cos \theta \cos \phi - \sin \theta \sin \phi) \\ r (\sin \theta \cos \phi + \cos \theta \sin \phi) \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} r \cos \phi \\ r \sin \phi \end{bmatrix}$$

Using trigonometric formulas

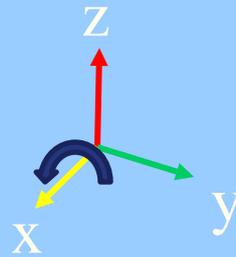
$\mathbf{R}$   $(x, y)$

# Rotations in 3D

- Same as in 2D for Z-rotations, but with a 3x3 matrix

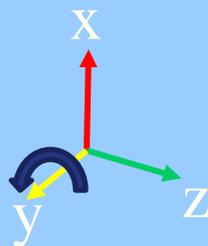
$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \Rightarrow \mathbf{R}_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$


- For X



$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}$$

- For Y



$$\mathbf{R}_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

# Translations must be simple?

Translation

Rotation

$$\begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \mathbf{p} = \mathbf{p} + \mathbf{t} \quad \mathbf{n} = \mathbf{R}\mathbf{p}$$

- Rotation is matrix mult, translation is add
- Would be nice if we could only use matrix multiplications...
- Turn to **homogeneous coordinates**
- Add a new component to each vector

# Homogeneous notation

- A point:  $\mathbf{p} = (p_x \ p_y \ p_z \ 1)^T$
- Translation becomes:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{T}(\mathbf{t})} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{pmatrix}$$

- A vector (direction):  $\mathbf{d} = (d_x \ d_y \ d_z \ 0)^T$
- Translation of vector:  $\mathbf{T}\mathbf{d} = \mathbf{d}$
- Homogeneous notation also allows for projections (later)

# Rotations in 4x4 form

- Just add a row at the bottom, and a column at the right:

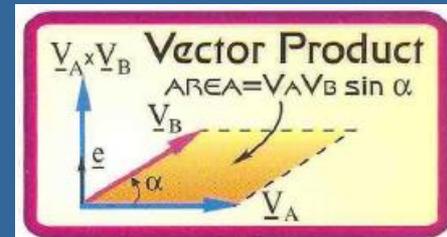
$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Similarly for X and Y

- Determinant = volume change when the transform is applied to a unit cube
  - $\det(R) = 1$  for all rot. matrices (= tripple scal. prod for 3x3 mtx)

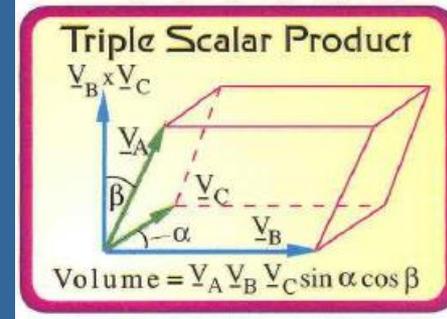
- For 3x3 rot-matrices:

- Trace( $R$ ) =  $1 + 2\cos(\alpha)$ , where  $\alpha$  is the rotation angle around the rotation axis.
- Eigenvector( $R$ ) is the rotation axis. I.e., if  $R\hat{v} = \lambda\hat{v}$ , then the eigenvalue  $\lambda$  must be 1 (because a pure rot does no scaling) and eigenvector  $\hat{v}$  is  $R$ 's rotation axis since  $\hat{v}$  is unaffected by the rotation  $R$ .



## • Triple Scalar Product

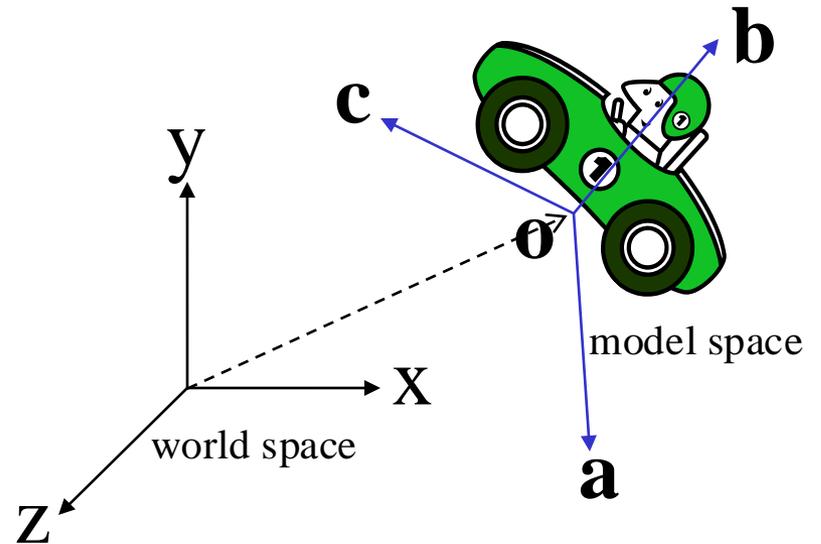
The magnitude of the triple scalar product is equal to the volume of the parallelepiped formed by the three vectors  $\underline{V}_A, \underline{V}_B, \underline{V}_C$ :  $\underline{V}_A \cdot (\underline{V}_B \times \underline{V}_C)$ .



# Change of Frames

- How to get the  $M_{\text{model-to-world}}$  matrix:

$$M_{\text{model-to-world}} = \begin{bmatrix} a_x & b_x & c_x & o_x \\ a_y & b_y & c_y & o_y \\ a_z & b_z & c_z & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The basis vectors **a**, **b**, **c** are expressed in the world-space coordinate system

(Both coordinate systems are right-handed)

# World-to-view Matrix

How to compute the world-to-view matrix, given the camera's world-space position  $\mathbf{o}$ , camera\_up\_vector, and camera\_view\_vector:

$$\mathbf{M}_{\text{view-to-world}} = \begin{bmatrix} a_x & b_x & c_x & o_x \\ a_y & b_y & c_y & o_y \\ a_z & b_z & c_z & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The camera axes  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are expressed in the world coordinate system

How to compute camera axes  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and  $\mathbf{o}$ :

$\mathbf{c} = -\text{camera\_view\_vector}$

$\mathbf{b} = \text{camera\_up\_vector}$  (sometimes same as worldspace up vector, so  $(0,1,0)$ ).

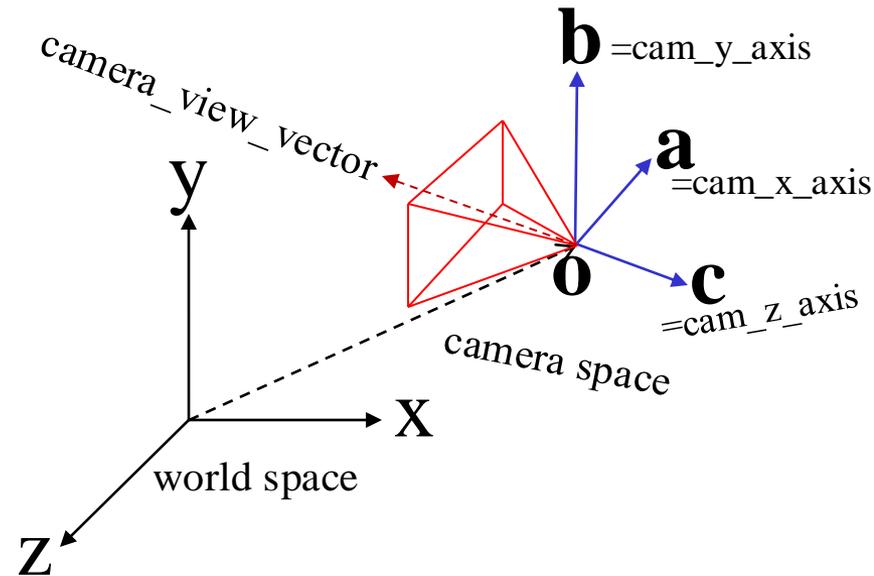
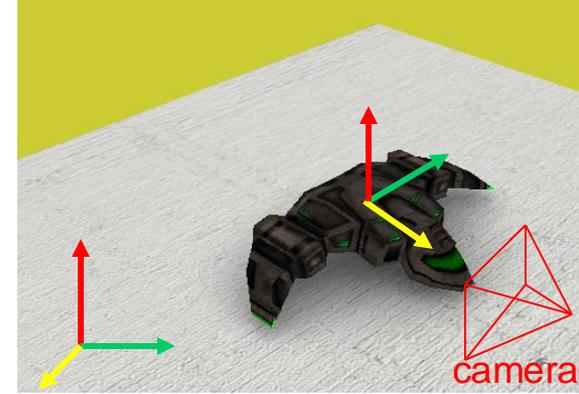
$\mathbf{a} = (\mathbf{b} \times \mathbf{c})$ .

Now normalize  $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$\mathbf{o} = \text{camera\_pos}$

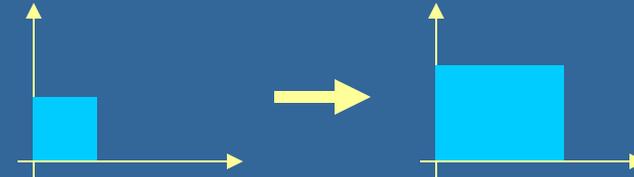
Finally:

$$\mathbf{M}_{\text{world-to-view}} = (\mathbf{M}_{\text{view-to-world}})^{-1}$$



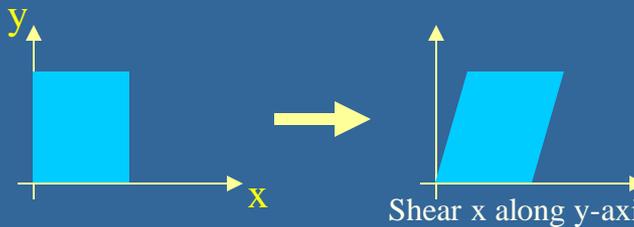
# More basic transforms

- Scaling



$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Shear



$$\begin{bmatrix} 1 & s_{x_y} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rigid-body transform:

- Rotation and/or then translation:

$$\mathbf{X} = \mathbf{TR}$$

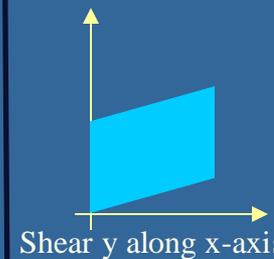
- Concatenation of matrices:

- Not commutative, i.e.,  $\mathbf{RT} \neq \mathbf{TR}$

- In  $\mathbf{X} = \mathbf{TR}$ , the rotation is done first

- Inverses and rotation about arbitrary axis:

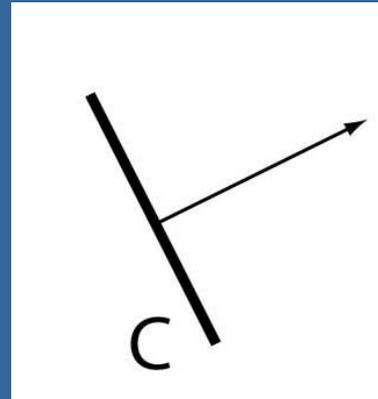
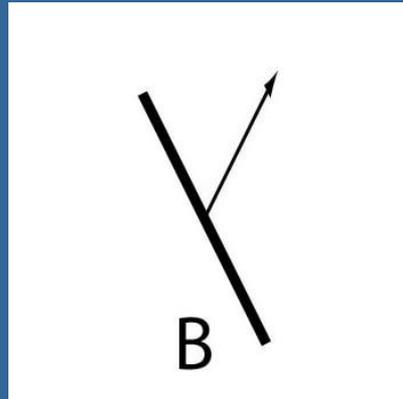
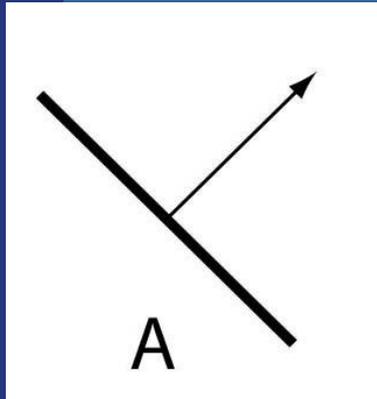
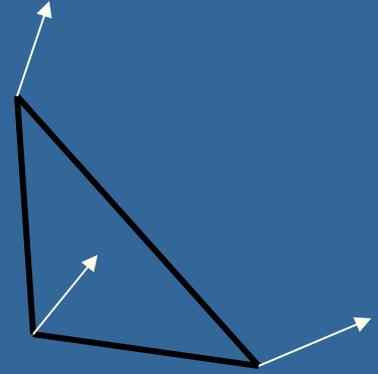
- Rigid body:  $\mathbf{X}^{-1} = \mathbf{X}^T$  for 3x3 matrices



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ s_{y_x} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Normal transforms

## Not so normal...



- Cannot use same matrix to transform normals  
Assume  $\mathbf{M}$  is the modelview matrix.

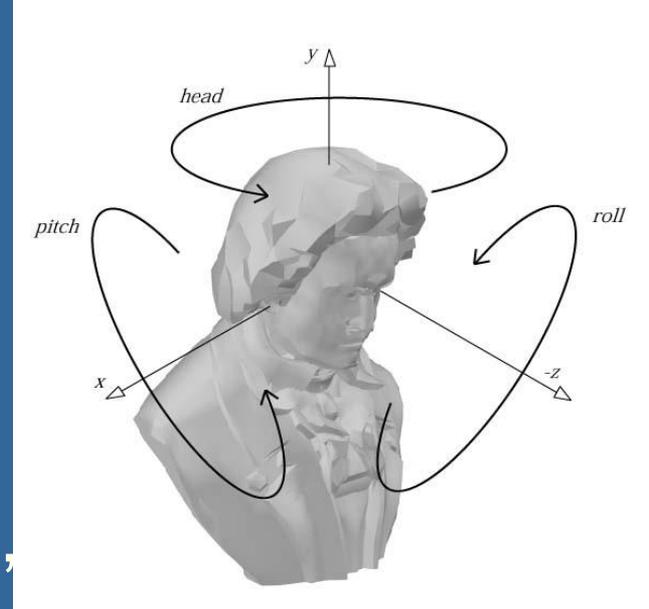
Use :  $\mathbf{N} = (\mathbf{M}^{-1})^T$  instead of  $\mathbf{M}$

for the normals.

- $\mathbf{M}$  works for pure rotations and translations, though

# The Euler Transform

- Assume the camera or object looks down the negative z-axis, with up in the y-direction, x to the right



- $h$ =head

- $p$ =pitch

- $r$ =roll

- Optional

- You may read about Gimbal lock in book, p: 67

- See also

- <http://mathworld.wolfram.com/EulerAngles.html>

To avoid problems with Euler transforms, for every rotation, also rotate remaining rotation axes

# Using Euler transforms

Head:

- Rotate around  $y$ -axis
- Recompute  $x$ - and  $z$ -axes
  - By rotating them as vectors



Pitch:

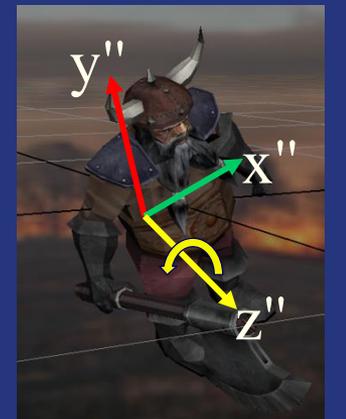
- Rotate around  $x'$ -axis
- Recompute  $z$  axis again



Roll:

- Rotate around  $z''$ -axis

How do we rotate vectors (axes) and points (vertices) around an **arbitrary** axis?



How do we rotate vectors and points around an **arbitrary** normalized 3D axis  $\mathbf{u}$ ?

## Quaternions

$$\begin{aligned}\hat{\mathbf{q}} &= (\mathbf{q}_v, q_w) = (q_x, q_y, q_z, q_w) \\ &= iq_x + jq_y + kq_z + q_w\end{aligned}$$

- Extension of imaginary numbers
- Compact+fast representation of rotations
- Focus on unit quaternions:

- Norm (or length):  $n(\hat{\mathbf{q}}) = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2} = 1$

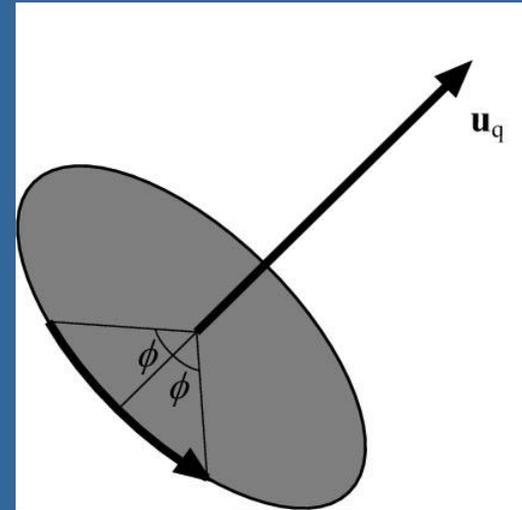
- A unit quaternion can be written as:

$$\hat{\mathbf{q}} = (\sin \phi \mathbf{u}_q, \cos \phi) \quad \text{where } \|\mathbf{u}_q\| = 1$$

# Unit quaternions are perfect for rotations!

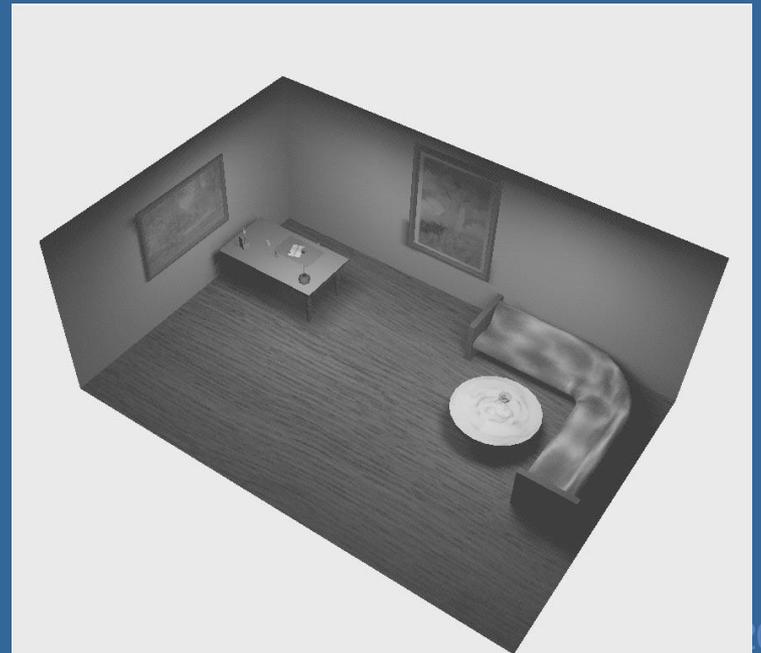
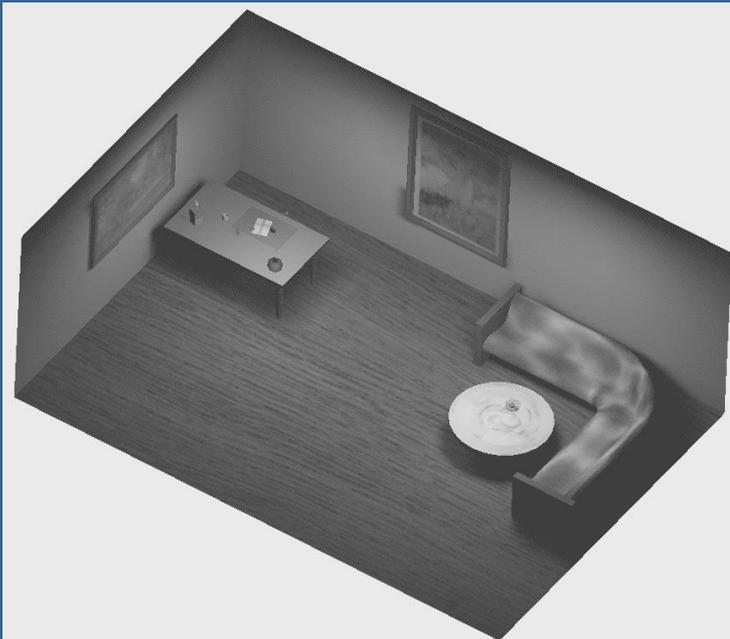
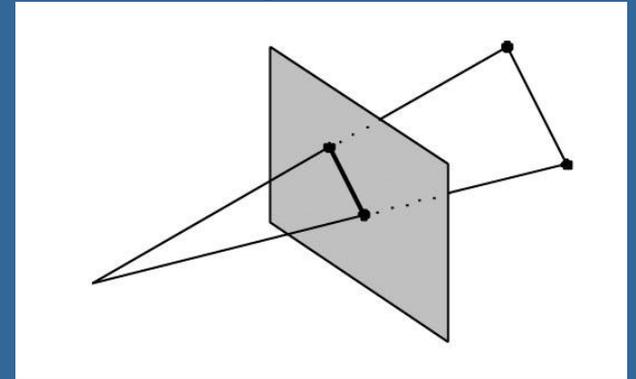
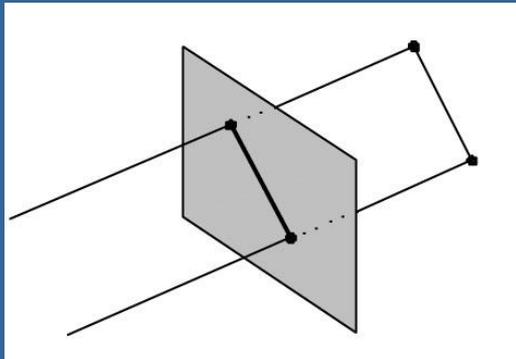
$$\hat{\mathbf{q}} = (\sin \phi \mathbf{u}_q, \cos \phi)$$

- Compact (4 components)
- Can show that  $\hat{\mathbf{q}}\hat{\mathbf{p}}\hat{\mathbf{q}}^{-1}$
- ...represents a rotation of  $2\phi$  radians around  $\mathbf{u}_q$  of  $\mathbf{p}$
- That is: a unit quaternion represents a rotation as a **rotation axis** and an **angle**
- `rotate(ux, uy, uz, angle);`
  - See p:80 how to convert  $q$  to matrix.
- Interpolation from one quaternion to another is much simpler (vs matrices), and gives optimal results (p:77)

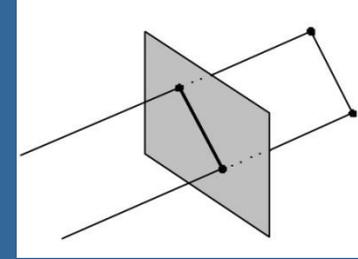


# Projections

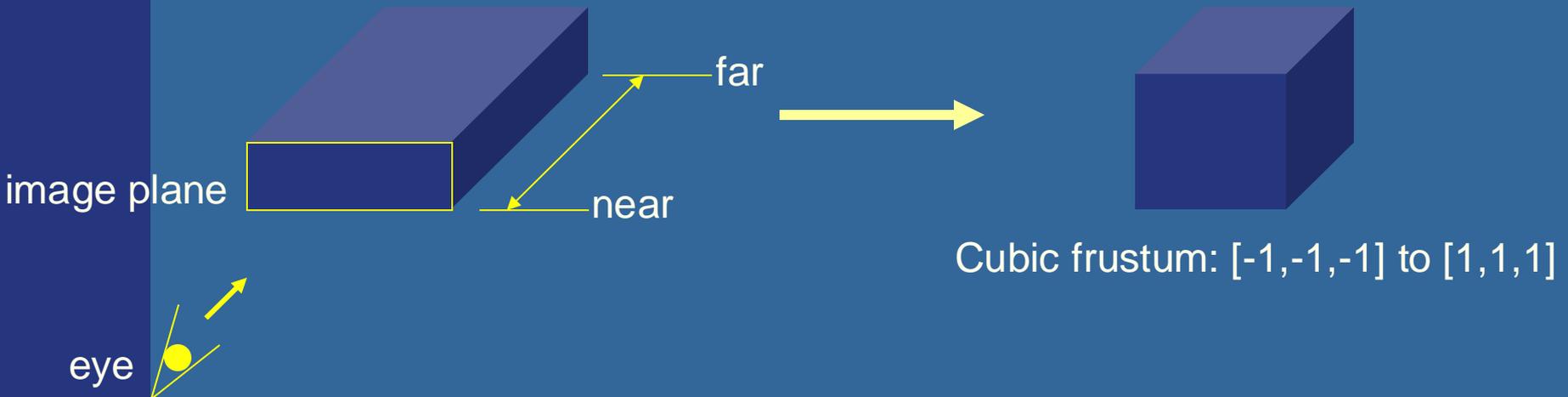
- Orthogonal (parallel) and Perspective



# Orthogonal projection

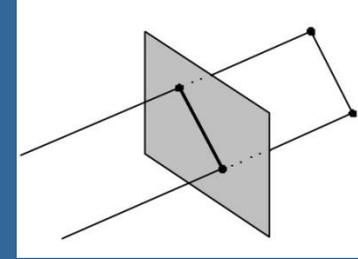


- However, by projecting to a plane, the depth information is lost
  - The matrix is not invertible. Its determinant is zero.
- For Z-buffering
  - It is not sufficient to project on to a plane
  - Rather, we need to "project" into a cube

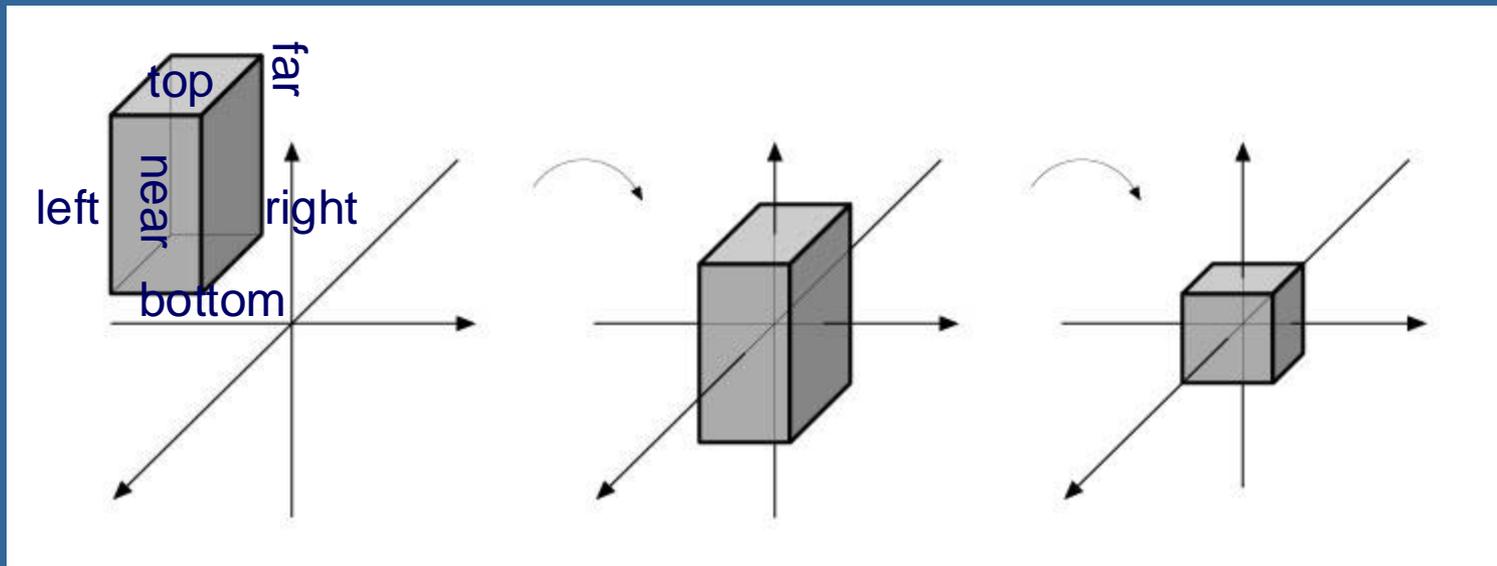


- This cube is also used for perspective proj.
- Simplifies clipping

# Orthogonal projection



- The projection into a cube, instead of a plane, is invertible
  - i.e., depth information is kept.
- Simple to derive:
  - Just a translation and scale



# What about those homogenous coordinates?

$$\mathbf{p} = (p_x \quad p_y \quad p_z \quad p_w)^T$$

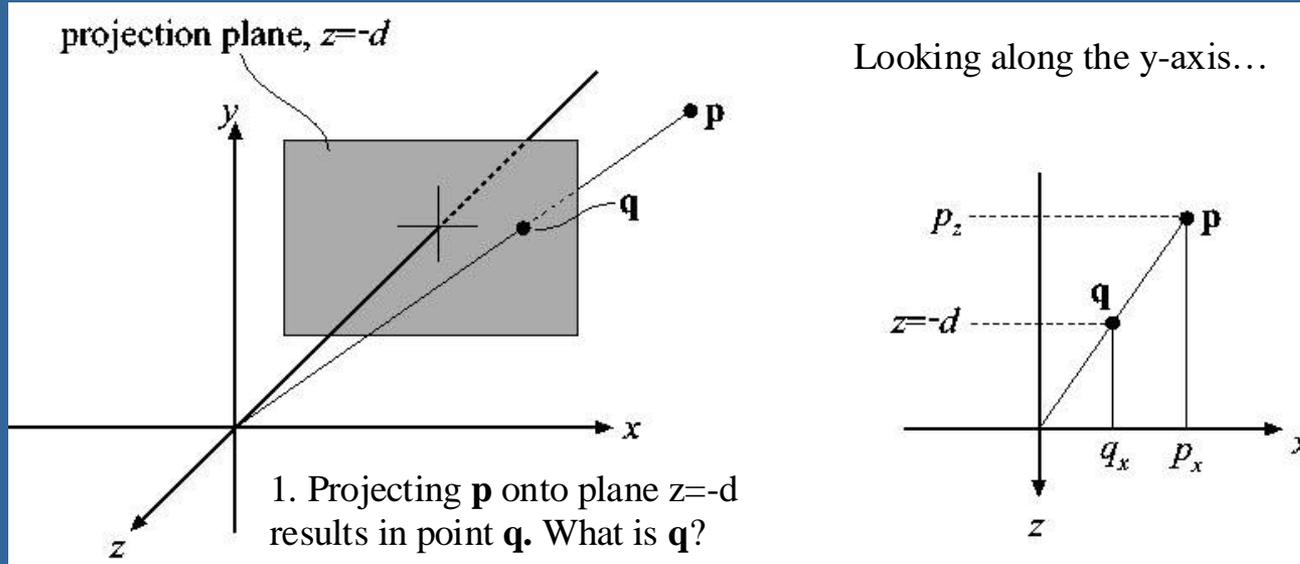
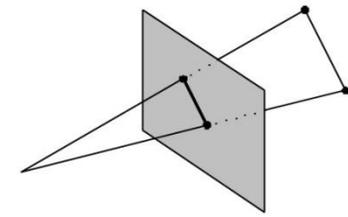
- Homogeneous coordinates is a 4D space where 3D points becomes infinite lines. This space is essential to robustly handle points projected to infinity, in order to render objects that stretch to the horizon or go beyond the visible scene bounds.
- To transform from 3D to 4D homogeneous coordinates, set  $p_w=1$  for points (and  $p_w=0$  for vectors).
- To convert back to 3D space, perform the *perspective division* (aka *de-homogenization* process):
  - i.e., divide all components by  $p_w$

$$\mathbf{p} = (p_x / p_w \quad p_y / p_w \quad p_z / p_w \quad 1)^T$$

- Gives a point again!
- Can be used for perspective projections, as we will see

What about division by 0 for vectors, since their  $w=0$ ? Note, vectors, like normals, are not multiplied with the projection matrix so perspective division is not done on them!

# Perspective projection



Symmetry gives that:

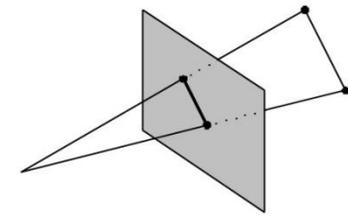
$$\frac{q_x}{p_x} = \frac{-d}{p_z} \Rightarrow q_x = -d \frac{p_x}{p_z}$$

and For y:  $q_y = -d \frac{p_y}{p_z}$

This is the perspective projection matrix of  $\mathbf{p}$  onto plane  $z = -d$ . Proof next slide...

$$\mathbf{P}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix}$$

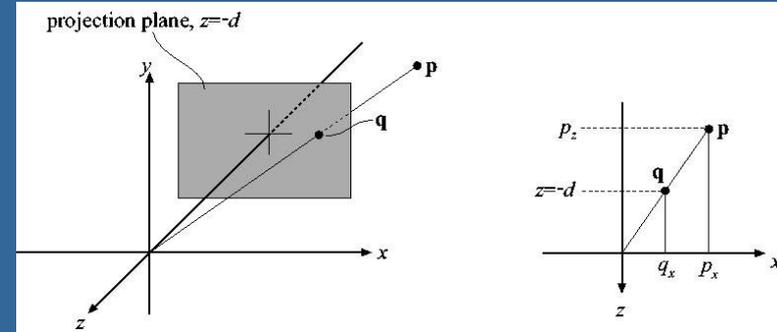
# Perspective projection



$$\mathbf{P}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix}$$

Projection matrix onto plane  $z = -d$

Proof: by multiplying  $\mathbf{P}_p \mathbf{p}$  with  $\mathbf{p}$   $\mathbf{P}_p \mathbf{p} = \mathbf{q}$  we should get:  $q_x = -d \frac{p_x}{p_z}$   $q_y = -d \frac{p_y}{p_z}$   $q_z = -d$

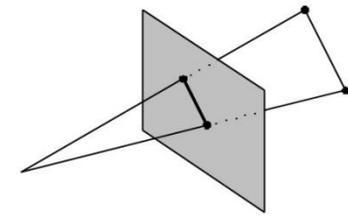


And we can clearly see that this is the case:

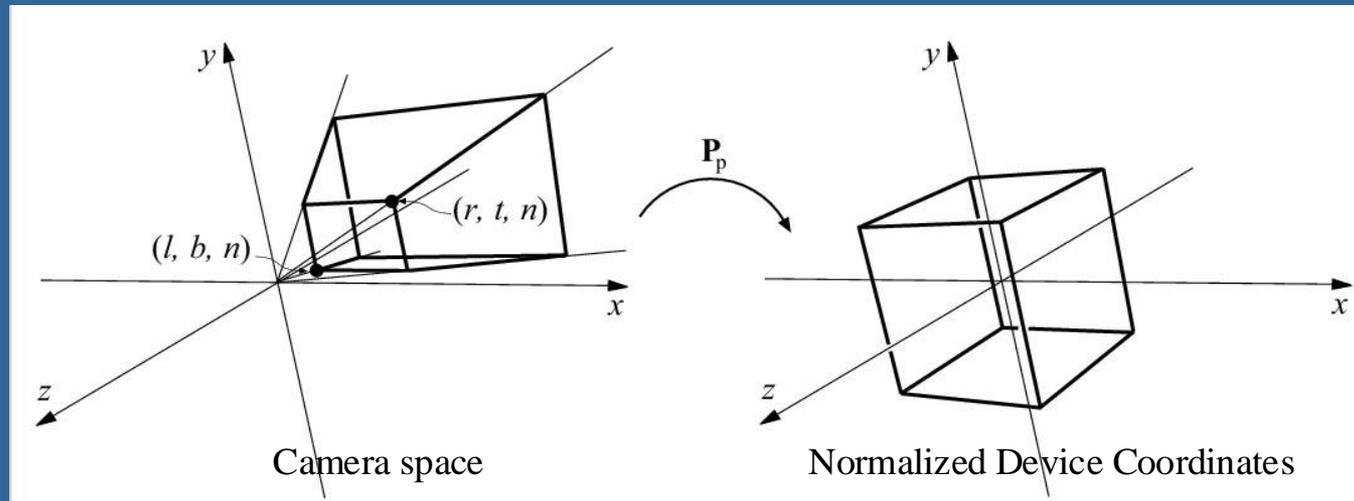
$$\mathbf{P}_p \mathbf{p} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ -p_z/d \end{pmatrix} \Rightarrow \mathbf{q} = \begin{pmatrix} -dp_x/p_z \\ -dp_y/p_z \\ -dp_z/p_z \\ 1 \end{pmatrix} = \begin{pmatrix} -dp_x/p_z \\ -dp_y/p_z \\ -d \\ 1 \end{pmatrix}$$

The "arrow" is the *perspective division* or *de-homogenization* process, i.e., division with  $w$ . This creates the foreshortening effect of  $x, y$  with distance from camera.

# Perspective projection

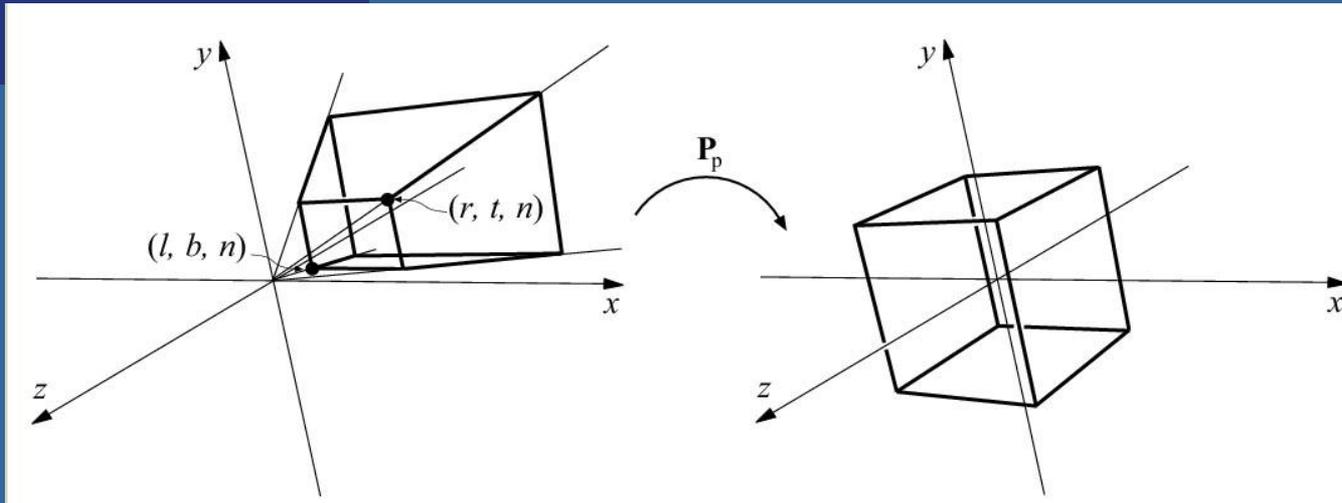


- Again, the determinant is 0 (not invertible)
  - i.e., we lost the depth information
- To make the rest of the pipeline the same as for orthogonal projection:
  - project into the same cube instead (by multiplying with projection matrix and doing perspective divide)
  - This cubic space is called Normalized Device Coordinates (NDC) and sometimes “unit space”.



- This projection matrix is not that much different from  $\mathbf{P}_p$  but does not collapse z-coord to a plane

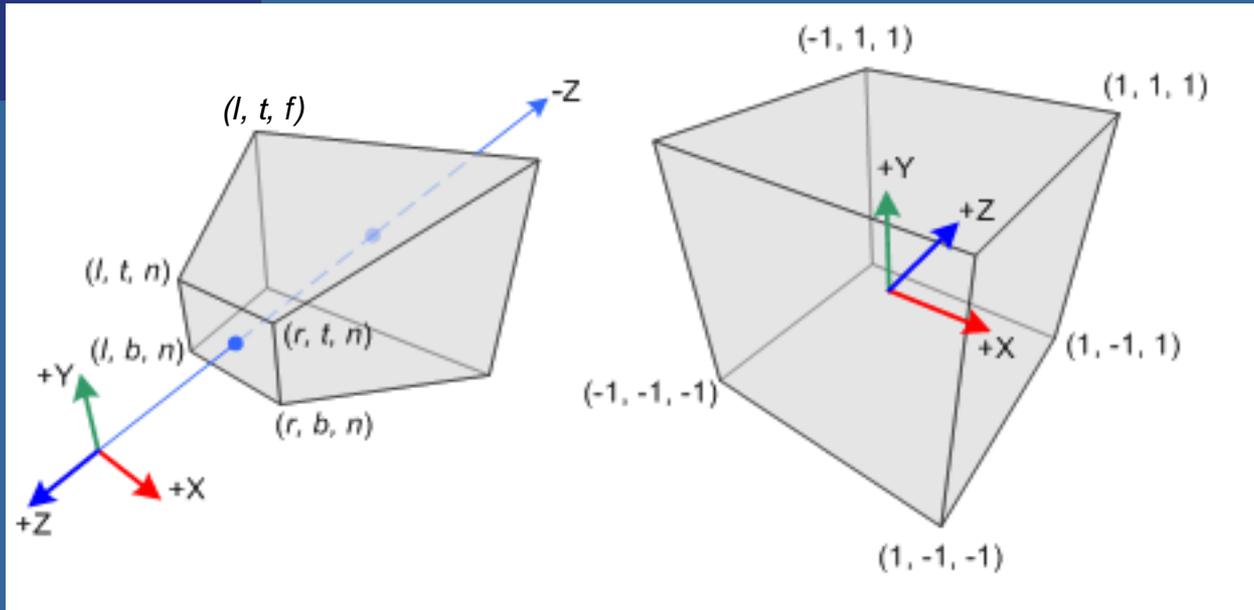
# Understanding the projection matrix



$$\mathbf{P}_p \mathbf{p} = \begin{pmatrix} s_x & 0 & a & 0 \\ 0 & s_y & b & 0 \\ 0 & 0 & s_z & c \\ 0 & 0 & -1/d & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x p_x + a p_z \\ s_y p_y + b p_z \\ s_z p_z + c \\ -p_z / d \end{pmatrix} \Rightarrow \mathbf{q} = \begin{pmatrix} -d(s_x p_x / p_z + a) \\ -d(s_y p_y / p_z + b) \\ -d(s_z p_z + c) / p_z \\ 1 \end{pmatrix}$$

- $s_x, s_y, s_z$  – Scaling
- $a, b$  – Due to homogenization, this controls asymmetry of the frustum
- $c$  – Keeps z-info
- $-1/d$  – Perspective division based on  $p_z$

# OpenGL projection matrix



$$P_{OpenGL} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

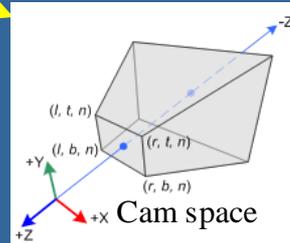
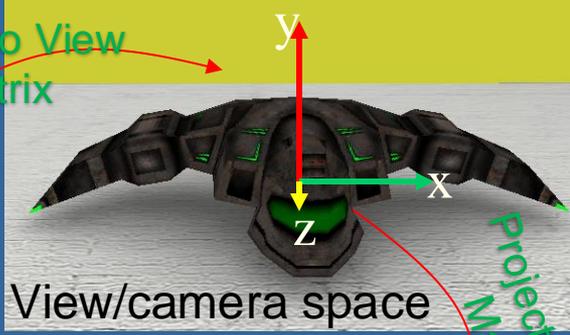
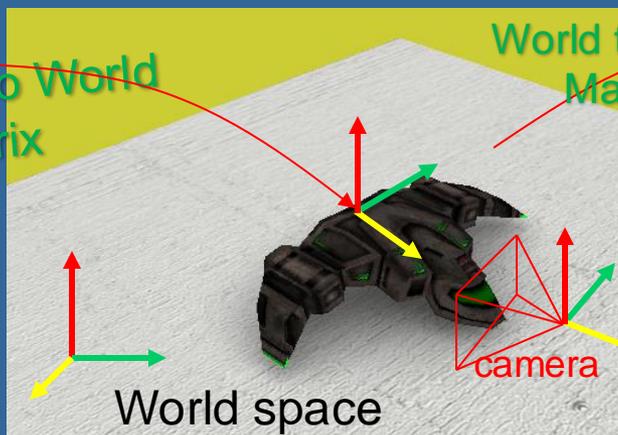
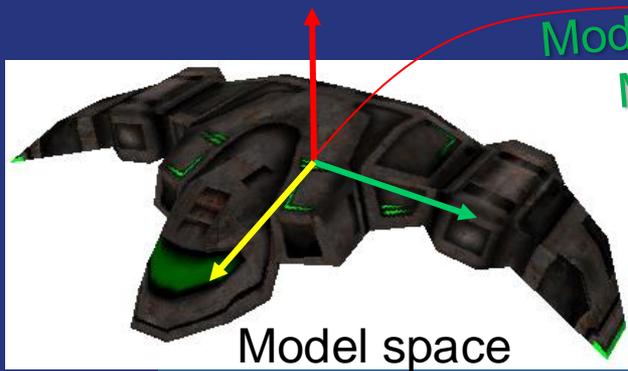
$l$  = left  
 $r$  = right  
 $t$  = top  
 $b$  = bottom  
 $n$  = near  
 $f$  = far  
 Values in camera space

```
mat4 projMtx = perspective(fov, width / height, near, far);
```

# Summary Vertex Transforms

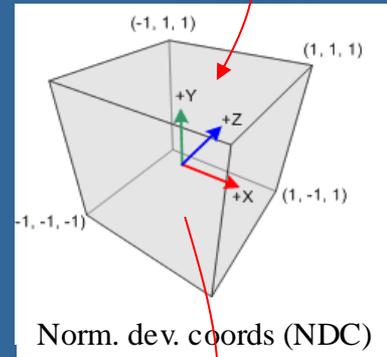
- Next slide is a summary of the vertex transformations that occur during the geometry stage, mostly by the vertex shader.
  - Nevertheless, most matrices are computed on the CPU side during the application stage and are sent to the vertex shader, one by one or concatenated,
    - e.g. as `modelViewMatrix` and `ProjectionMatrix`
    - or `modelViewProjectionMatrix`,
  - before the CPU call to `glDrawArrays()` that launches the GPU pipeline for the triangles.

# A summary:



Homogeneous coords (x,y,z,w)  
Clip space.

Perspective division 1/w



Matrices for translation, rotation, scaling:

Transl. Rot

$$T \text{ matrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R \text{ matrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$S \text{ matrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

World-to-view matrix:

**World-to-view Matrix**

How to compute the world-to-view matrix, given the camera's world-space position  $\mathbf{o}$ , camera up vector, and camera view vector:

$$M_{\text{view-to-world}} = \begin{bmatrix} a_x & b_x & c_x & o_x \\ a_y & b_y & c_y & o_y \\ a_z & b_z & c_z & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The camera axes  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are expressed in the world coordinate system

How to compute camera axes  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and  $\mathbf{o}$ :

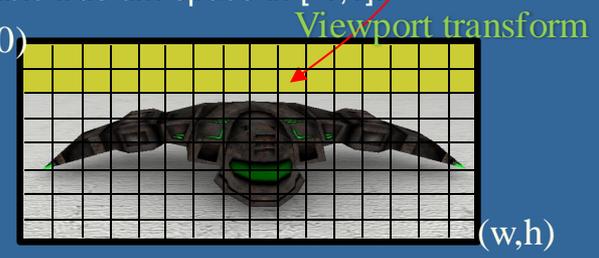
- $\mathbf{o}$  = camera\_pos\_vector
- $\mathbf{b}$  = camera\_up\_vector (sometimes same as worldspace up vector, so (0,1,0))
- $\mathbf{a}$  =  $\mathbf{b} \times \mathbf{v}$
- Note: normalized  $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- $\mathbf{o}$  = camera\_pos

Finally:  $M_{\text{world-to-view}} = (M_{\text{view-to-world}})^{-1}$

Projection matrix:

$$P_{\text{OpenGL}} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- The clipping is often done in homogeneous coordinates (aka clip space).
- The perspective division by  $w$  (homogenization step) takes us to normalized device coordinates (NDC), i.e., a cubic frustum space in  $[-1,1]$ .
- Then, the viewport transform (screen mapping) takes us to screen space or window coords, with  $x, y$  in  $[0, w] \times [0, h]$  and  $z$  in  $[0, 1]$ .
- Rasterization then occurs in this space.



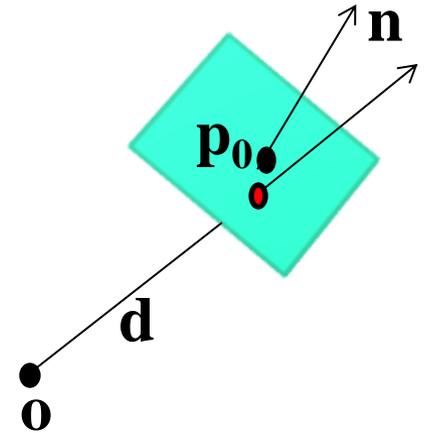
# Ray/Plane Intersections

- Ray:  $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$
- Plane:  $\mathbf{n} \cdot \mathbf{x} + d = 0$ ; ( $d = -\mathbf{n} \cdot \mathbf{p}_0$ )
- Set  $\mathbf{x} = \mathbf{r}(t)$ :

$$\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d}) + d = 0$$

$$\mathbf{n} \cdot \mathbf{o} + t(\mathbf{n} \cdot \mathbf{d}) + d = 0$$

$$t = (-d - \mathbf{n} \cdot \mathbf{o}) / (\mathbf{n} \cdot \mathbf{d})$$



```
Vec3f rayPlaneIntersect(vec3f o,dir, n, d)
{
    float t=(-d-n.dot(o)) / (n.dot(dir));
    return o + dir*t;
}
```

From book, p: 987

## Line/Line intersection in 2D

- $r_1(s) = \mathbf{o}_1 + s\mathbf{d}_1$

- $r_2(t) = \mathbf{o}_2 + t\mathbf{d}_2$



- $r_1(s) = r_2(t) \quad (1)$

- $\mathbf{o}_1 + s\mathbf{d}_1 = \mathbf{o}_2 + t\mathbf{d}_2 \quad (2)$

noting that  $\mathbf{d} \cdot \mathbf{d}^\perp = 0$ , [ $\mathbf{d} = (a, b) \rightarrow \mathbf{d}^\perp = (b, -a)$ ]

$$s\mathbf{d}_1 \cdot \mathbf{d}_2^\perp = (\mathbf{o}_2 - \mathbf{o}_1) \cdot \mathbf{d}_2^\perp$$

$$s = \frac{(\mathbf{o}_2 - \mathbf{o}_1) \cdot \mathbf{d}_2^\perp}{(\mathbf{d}_1 \cdot \mathbf{d}_2^\perp)}$$

$$t\mathbf{d}_2 \cdot \mathbf{d}_1^\perp = (\mathbf{o}_1 - \mathbf{o}_2) \cdot \mathbf{d}_1^\perp$$

$$t = \frac{(\mathbf{o}_1 - \mathbf{o}_2) \cdot \mathbf{d}_1^\perp}{(\mathbf{d}_2 \cdot \mathbf{d}_1^\perp)}$$

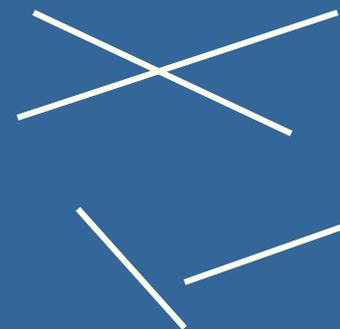
From book, p: 988

## Line/Line intersection in 3D

- $r_1(s) = \mathbf{o}_1 + s\mathbf{d}_1$

- $r_2(t) = \mathbf{o}_2 + t\mathbf{d}_2$

$s, t$  correspond to  
closest points



- $r_1(s) = r_2(t)$  (1)

- $\mathbf{o}_1 + s\mathbf{d}_1 = \mathbf{o}_2 + t\mathbf{d}_2$  (2)

noting that  $\mathbf{d} \times \mathbf{d} = 0$

$\|(\mathbf{d}_1 \times \mathbf{d}_2)\|^2 = 0$  means parallel lines

$s\mathbf{d}_1 \times \mathbf{d}_2 = (\mathbf{o}_2 - \mathbf{o}_1) \times \mathbf{d}_2$  (i.e., cross mult. both sides with  $\mathbf{d}_2$  to drop  $t$ )

$t\mathbf{d}_2 \times \mathbf{d}_1 = (\mathbf{o}_1 - \mathbf{o}_2) \times \mathbf{d}_1$  (i.e., cross mult. both sides with  $\mathbf{d}_1$  to drop  $s$ )

$\Rightarrow$

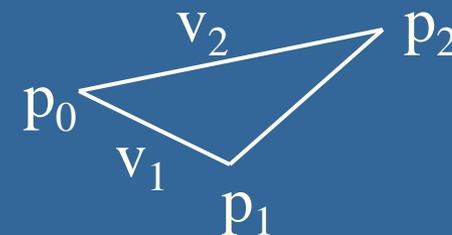
$$s (\mathbf{d}_1 \times \mathbf{d}_2) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = ((\mathbf{o}_2 - \mathbf{o}_1) \times \mathbf{d}_2) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)$$

$$t (\mathbf{d}_2 \times \mathbf{d}_1) \cdot (\mathbf{d}_2 \times \mathbf{d}_1) = ((\mathbf{o}_1 - \mathbf{o}_2) \times \mathbf{d}_1) \cdot (\mathbf{d}_2 \times \mathbf{d}_1)$$

$$s = \frac{\det(\mathbf{o}_2 - \mathbf{o}_1, \mathbf{d}_2, \mathbf{d}_1 \times \mathbf{d}_2)}{\|(\mathbf{d}_1 \times \mathbf{d}_2)\|^2}$$

$$t = \frac{\det(\mathbf{o}_2 - \mathbf{o}_1, \mathbf{d}_1, \mathbf{d}_1 \times \mathbf{d}_2)}{\|(\mathbf{d}_1 \times \mathbf{d}_2)\|^2}$$

# Area and Perimeter



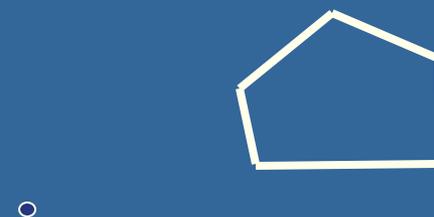
For polygon  $p_0, p_1 \dots p_n$

Perimeter = omkrets = sum of length of each edge in 2D and 3D:

$$O = \sum_{i=0}^{n-1} \|p_{i+1} - p_i\| = \sum_{i=0}^{n-1} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 + (z_{i+1} - z_i)^2}$$

Area in 2D:

$$A = \frac{1}{2} \left| \sum_{i=1}^{n-1} \langle x_i y_{i+1} - x_{i+1} y_i \rangle \right|$$



We can understand the formula from using Greens theorem: integrating over border to get area

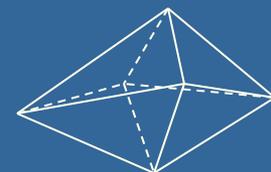
Choose arbitrary point to integrate from, e.g. Origin (0,0,0)

$$A_{triangle} = \frac{1}{2} (v_1 \times v_2)$$

Works for non-convex polygons as well

# Volume in 3D

The same trick for computing area in 2D can be used to easily compute the volume in 3D for triangulated objects



Again, choose arbitrary point-of-integration, e.g. Origin (0,0,0)

With respect to point-of-integration

- For all backfacing triangles, add volume
- For all frontfacing triangles, subtract volume

Works for non-convex objects as well

$$V_{tetrahedron} = \frac{1}{3!} |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| = \frac{1}{3!} |\det(\mathbf{a}, \mathbf{b}, \mathbf{c})|$$

where

$\mathbf{a} = \mathbf{p}_1 - \text{origin}$

$\mathbf{b} = \mathbf{p}_2 - \text{origin}$

$\mathbf{c} = \mathbf{p}_3 - \text{origin}$

$$V_{object} = \frac{1}{3!} \sum_{i=1}^n \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

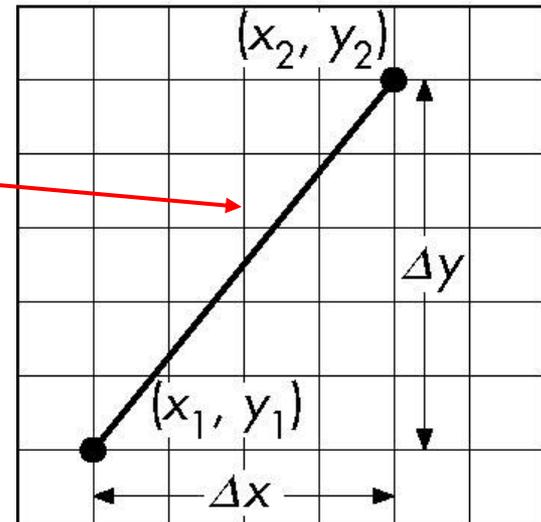
The sign of the determinant will automatically handle positive and negative contribution

# Scan Conversion of Line Segments

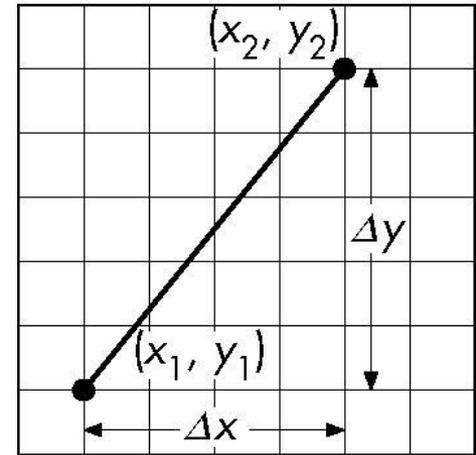
- Start with line segment in window coordinates with integer values for endpoints
- Assume implementation has a **write\_pixel** function

$$k = \frac{\Delta y}{\Delta x}$$

$$y = kx + m$$



# DDA Algorithm



- Digital Differential Analyzer

- DDA was a mechanical device for numerical solution of differential equations

- Line  $y=kx+m$  satisfies differential equation

$$dy/dx = k = \Delta y / \Delta x = y_2 - y_1 / x_2 - x_1$$

- Along scan line  $\Delta x = 1$

```
y=y1;
```

```
For (x=x1; x<=x2, ix++) {
```

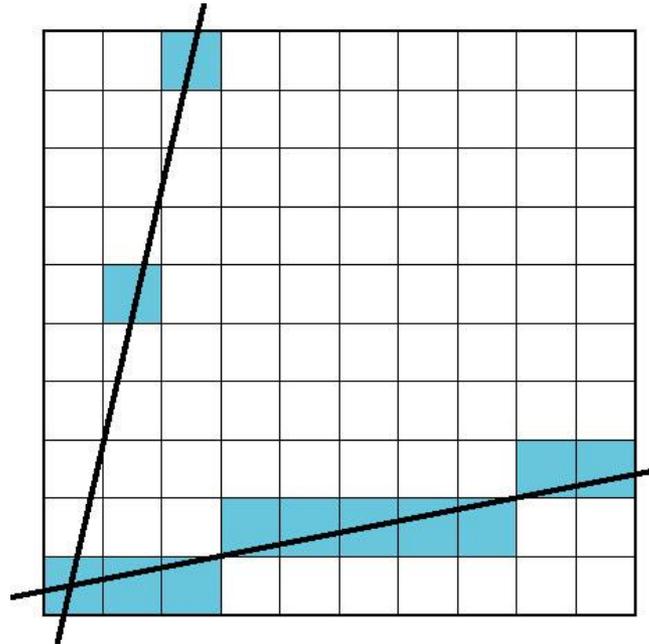
```
    write_pixel(x, round(y), line_color)
```

```
    y+=k;
```

```
}
```

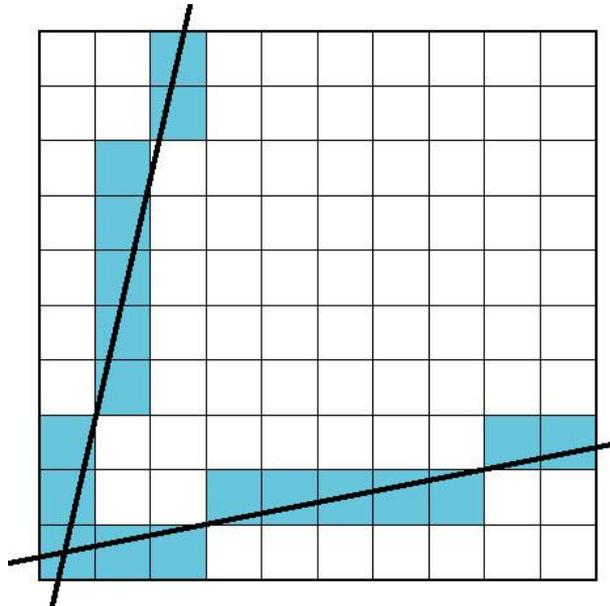
# Problem

- DDA = for each x plot pixel at closest y
  - Problems for steep lines



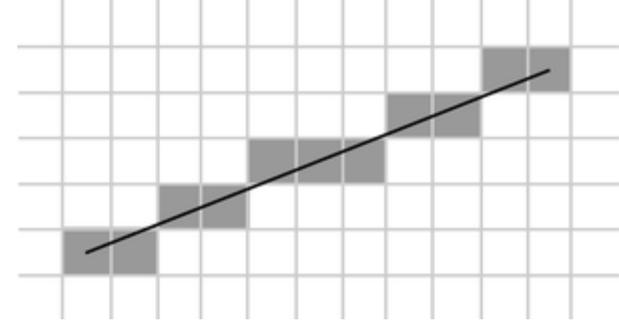
# Using Symmetry

- Use for  $1 \geq k \geq 0$
- For  $k > 1$ , swap role of  $x$  and  $y$ 
  - For each  $y$ , plot closest  $x$



- The problem with DDA is that it uses floats which was slow in the old days
- Bresenham's algorithm only uses integers

# Bresenham's line drawing algorithm



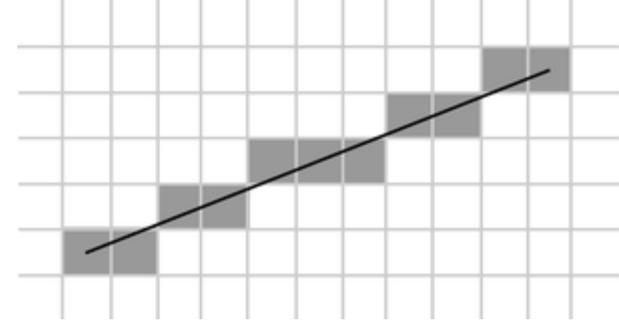
- The line is drawn between two points  $(x_0, y_0)$  and  $(x_1, y_1)$
- Slope  $k = \frac{(y_1 - y_0)}{(x_1 - x_0)}$  ( $y = kx + m$ )
- Each time we step 1 in x-direction, we should increment y with  $k$ . Otherwise the error in y increases with  $k$ .
- If the error surpasses 0.5, the line has become closer to the next y-value, so we add 1 to  $y$ , simultaneously decreasing the error by 1

```
function line(x0, x1, y0, y1)
  int deltax := abs(x1 - x0)
  int deltay := abs(y1 - y0)
  real error := 0
  real deltaerr := deltay / deltax
  int y := y0
  for x from x0 to x1
    plot(x,y)
    error := error + deltaerr
    if error ≥ 0.5
      y := y + 1
      error := error - 1.0
```

See also

[http://en.wikipedia.org/wiki/Bresenham's\\_line\\_algorithm](http://en.wikipedia.org/wiki/Bresenham's_line_algorithm)

# Bresenham's line drawing algorithm



- Now, convert algorithm to only using integer computations
- Trick: multiply the fractional number, *deltaerr*, by *deltax*
  - enables us to express *deltaerr* as an integer.
  - The comparison *if error*  $\geq 0.5$  is multiplied on both sides by  $2 * \textit{deltax}$

Old float version:

```
function line(x0, x1, y0, y1)
  int deltax := abs(x1 - x0)
  int deltay := abs(y1 - y0)
  real error := 0
  real deltaerr := deltay / deltax
  int y := y0
  for x from x0 to x1
    plot(x,y)
    error := error + deltaerr
    if error  $\geq 0.5$ 
      y := y + 1
      error := error - 1.0
```

New integer version:

```
function line(x0, x1, y0, y1)
  int deltax := abs(x1 - x0)
  int deltay := abs(y1 - y0)
  real error := 0
  real deltaerr := deltay ← Multiply by deltax
  int y := y0
  for x from x0 to x1
    plot(x,y)
    error := error + deltaerr
    if 2*error  $\geq$  deltax ← Multiply by 2 deltax
      y := y + 1
      error := error - deltax ← Multiply by deltax
```

# Complete Bresenham's line drawing algorithm

```
function line(x0, x1, y0, y1)
  boolean steep := abs(y1 - y0) > abs(x1 - x0)
```

```
  if steep then
```

```
    swap(x0, y0)
```

```
    swap(x1, y1)
```

```
  if x0 > x1 then
```

```
    swap(x0, x1)
```

```
    swap(y0, y1)
```

```
  int deltax := x1 - x0
```

```
  int deltay := abs(y1 - y0)
```

```
  int error := 0
```

```
  int ystep
```

```
  int y := y0
```

```
  if y0 < y1 then ystep := 1 else ystep := -1
```

```
  for x from x0 to x1
```

```
    if steep then plot(y,x) else plot(x,y)
```

```
    error := error + deltay
```

```
    if 2*error ≥ deltax
```

```
      y := y + ystep
```

```
      error := error - deltax
```

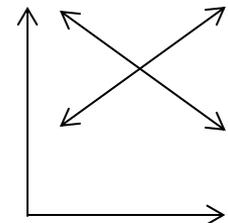
Swap loop axis

Swap start and end points

The first case is allowing us to draw lines that still slope downwards, but head in the opposite direction. I.e., swapping the initial points if  $x_0 > x_1$ .

To draw lines that go up, we check if  $y_0 \geq y_1$ ; if so, we step  $y$  by  $-1$  instead of  $1$ .

To be able to draw lines with a slope less than one, we take advantage of the fact that a steep line can be reflected across the line  $y=x$  to obtain a line with a small slope. The effect is to switch the  $x$  and  $y$  variables.



# You need to know

- How to create a simple Scaling matrix, rotation matrix, translation matrix and orthogonal projection matrix
- Change of frames (creating model-to-view matrix)
- Understand how quaternions are used
- Understanding of Euler transforms
- DDA line drawing algorithm
- Understand what is good with Bresenham's line drawing algorithm, i.e., uses only integers.

All the following slides are simply extra non-compulsory material that explains the content of the lecture in a different way.

Most of the following slides are  
from

Ed Angel

Professor of Computer Science,  
Electrical and Computer Engineering,  
and Media Arts

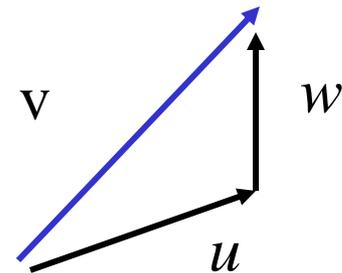
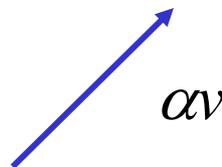
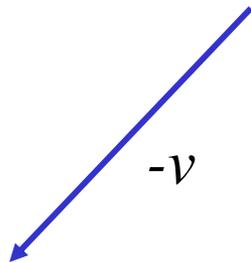
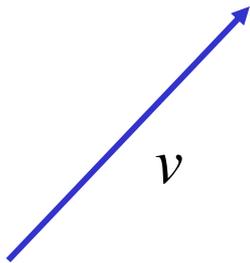
University of New Mexico

# Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

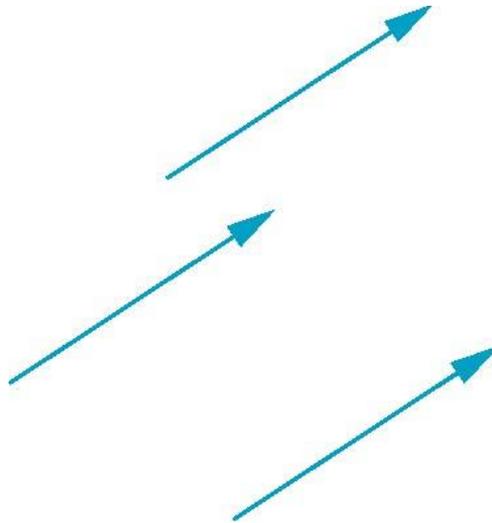
# Vector Operations

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types. Every vector can be multiplied by a scalar.
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector



# Vectors Lack Position

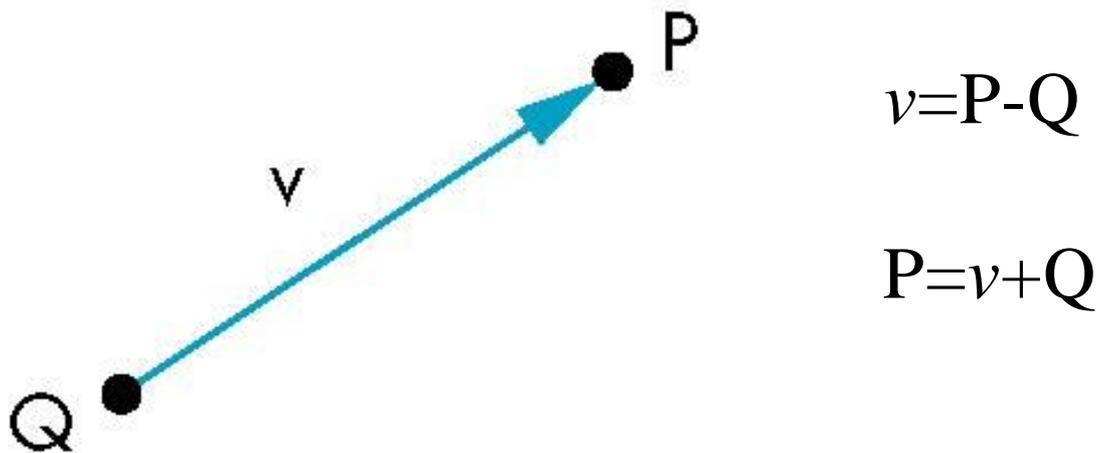
- These vectors are identical
  - Same length and magnitude



- Vectors insufficient for geometry
  - Need points

# Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



# Affine Spaces

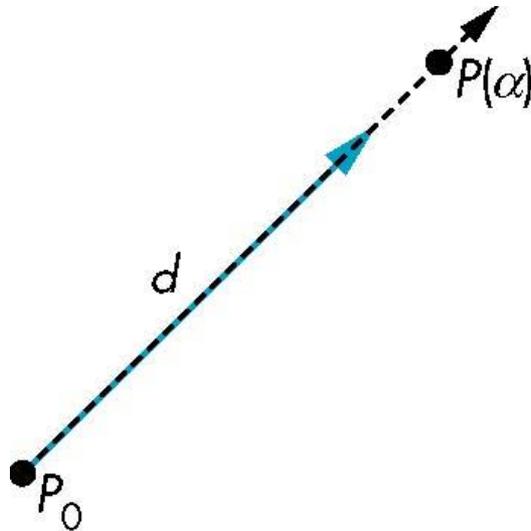
- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \cdot P = P$
  - $0 \cdot P = \mathbf{0}$  (zero vector)

# Lines

- Consider all points of the form

- $P(\alpha) = P_0 + \alpha \mathbf{d}$

- Set of all points that pass through  $P_0$  in the direction of the vector  $\mathbf{d}$



# Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit:  $y = kx + m$
  - Implicit:  $ax + by + c = 0$
  - Parametric:
    - $x(\alpha) = \alpha x_0 + (1-\alpha)x_1$
    - $y(\alpha) = \alpha y_0 + (1-\alpha)y_1$

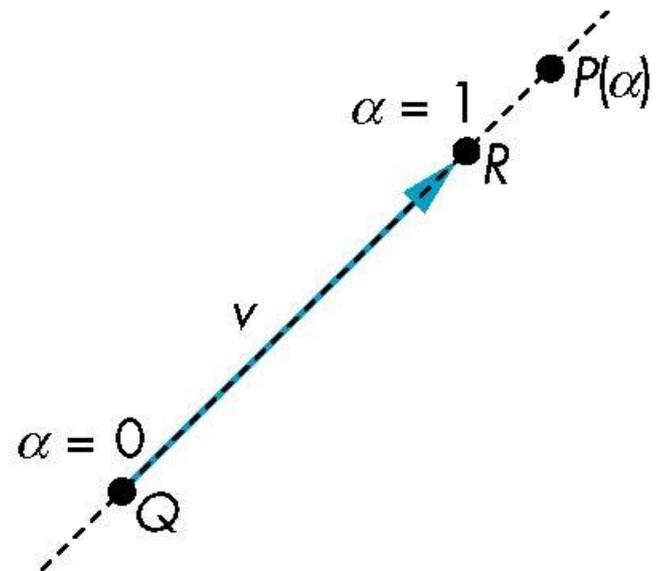
# Rays and Line Segments

- If  $\alpha \geq 0$ , then  $P(\alpha)$  is the *ray* leaving  $P_0$  in the direction  $\mathbf{d}$

If we use two points to define  $\mathbf{v}$ , then

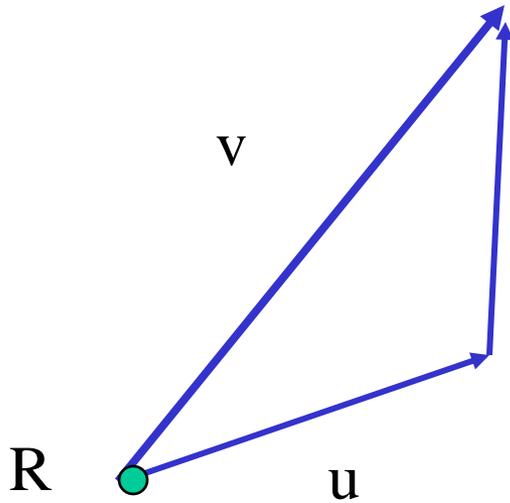
$$\begin{aligned} P(\alpha) &= Q + \alpha (R - Q) = Q + \alpha \mathbf{v} \\ &= \alpha R + (1 - \alpha)Q \end{aligned}$$

For  $0 \leq \alpha \leq 1$  we get all the points on the *line segment* joining  $R$  and  $Q$

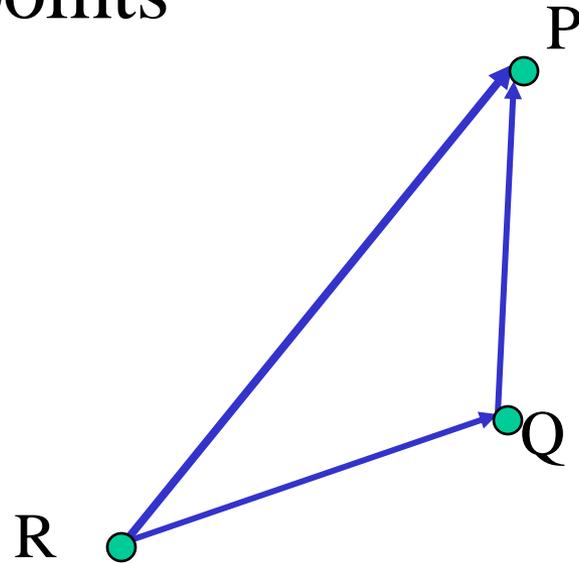


# Planes

- A plane can be defined by a point and two vectors or by three points

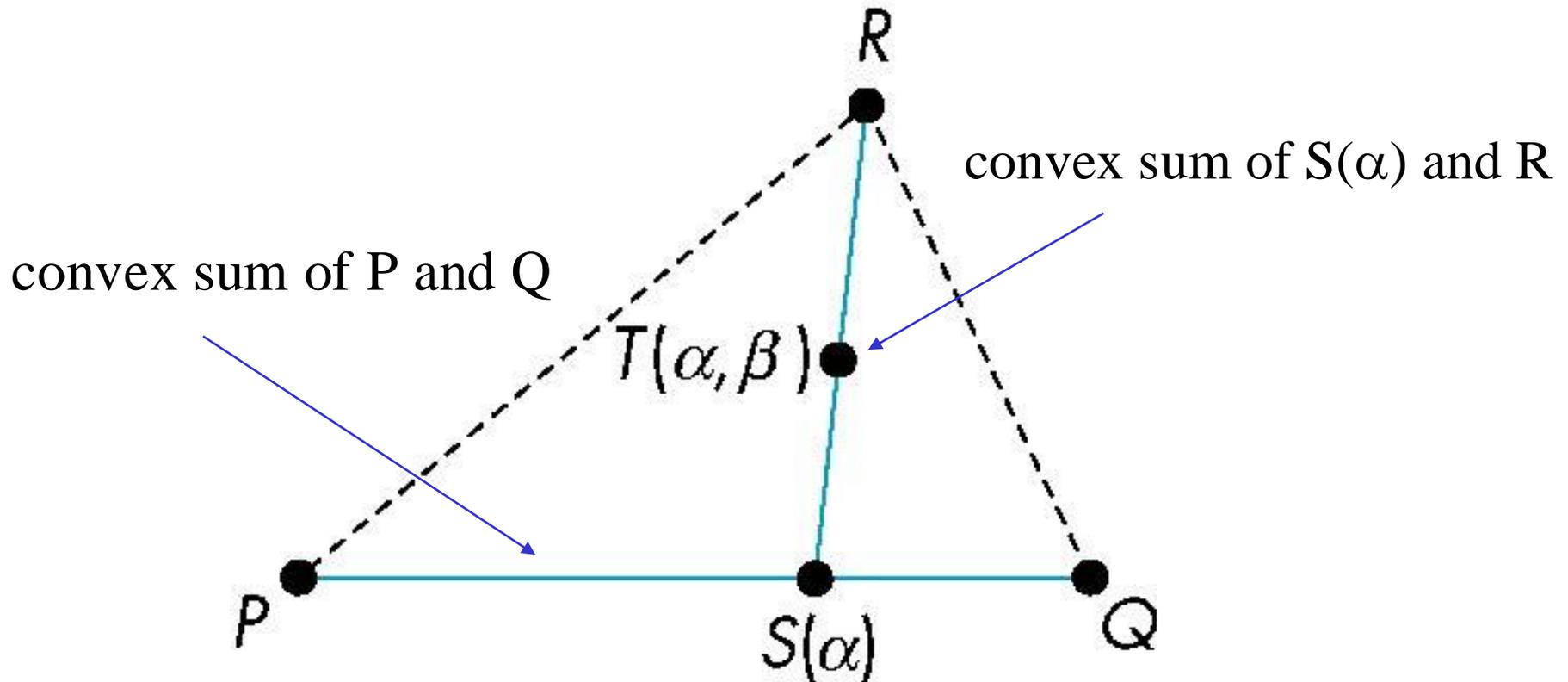


$$P(\alpha, \beta) = R + \alpha u + \beta v$$



$$P(\alpha, \beta) = R + \alpha(Q - R) + \beta(P - R)$$

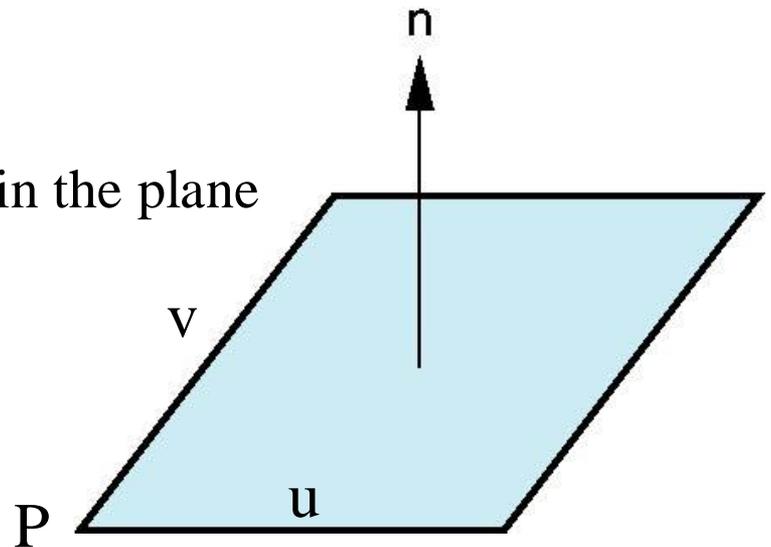
# Triangles



for  $0 \leq \alpha, \beta \leq 1$ , we get all points in triangle

# Normals

- Every plane has a vector  $\mathbf{n}$  normal (perpendicular, orthogonal) to it
- From point/vector form
  - $P(\alpha, \beta) = R + \alpha\mathbf{u} + \beta\mathbf{v}$we know we can use the cross product to find
  - $\mathbf{n} = \mathbf{u} \times \mathbf{v}$
- Plane equation:
  - $\mathbf{n} \cdot \mathbf{x} - d = 0$ ,
  - where  $d = -\mathbf{n} \cdot \mathbf{p}$  and  $\mathbf{p}$  is any point in the plane

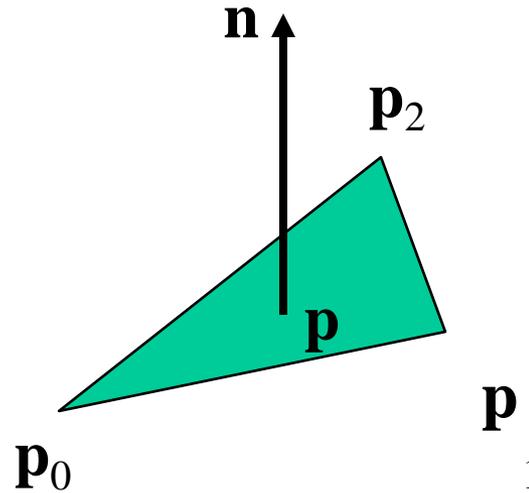


# Normal for Triangle

plane  $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$

$$\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_0) \times (\mathbf{p}_1 - \mathbf{p}_0)$$

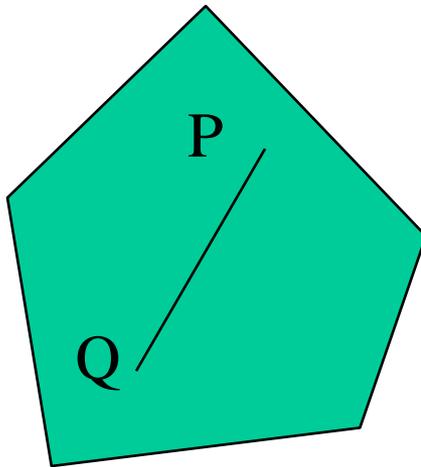
normalize  $\mathbf{n} \leftarrow \mathbf{n} / |\mathbf{n}|$



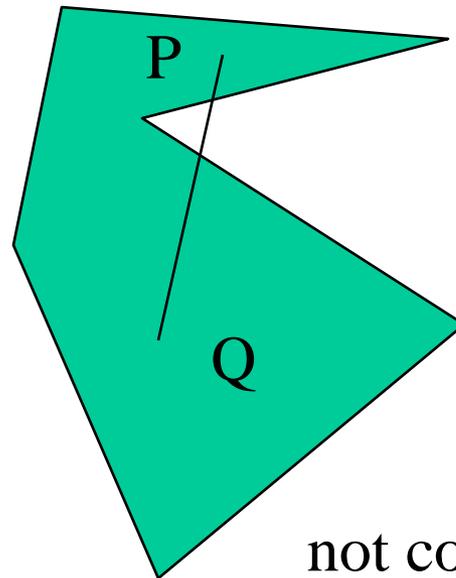
Note that right-hand rule determines outward face

# Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



convex



not convex

# Affine Sums

- Consider the “sum”

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$

Can show by induction that this sum makes sense iff

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$$

in which case we have the *affine sum* of the points  $P_1, P_2, \dots, P_n$

- If, in addition,  $\alpha_i \geq 0$ , we have the *convex hull* of  $P_1, P_2, \dots, P_n$

# Convex Hull

Consider the linear combination

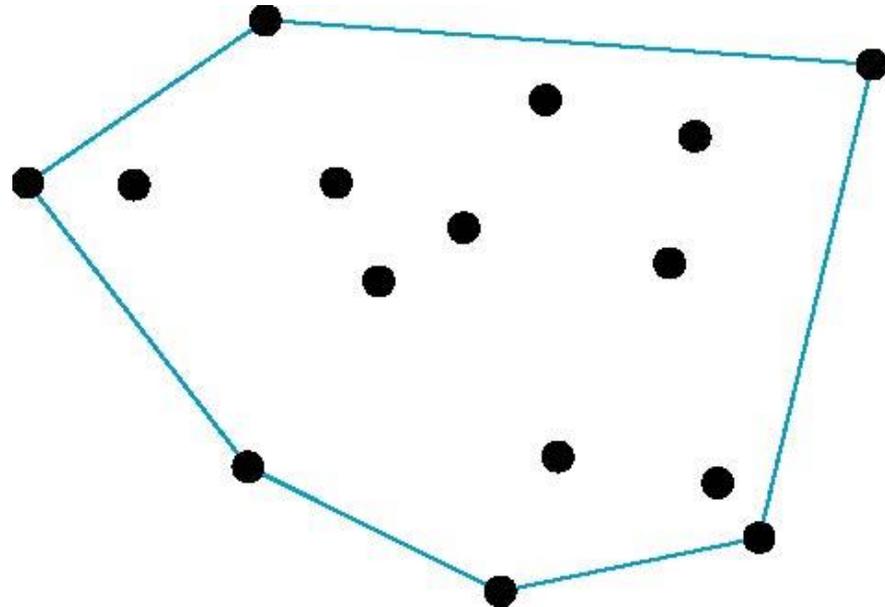
$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$

- If  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$

– (in which case we have the *affine sum* of the points  $P_1, P_2, \dots, P_n$ )

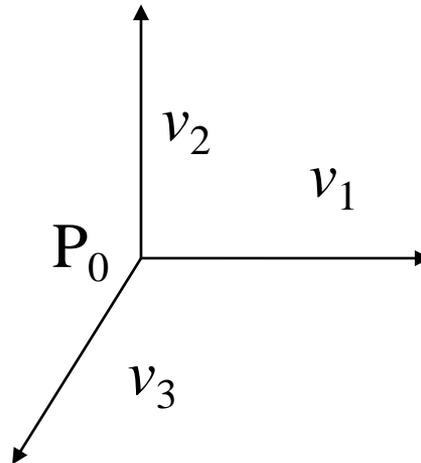
and if  $\alpha_i \geq 0$ , we have the *convex hull* of  $P_1, P_2, \dots, P_n$

- Smallest convex object containing  $P_1, P_2, \dots, P_n$



# Frames

- A coordinate system is insufficient to represent points
- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*



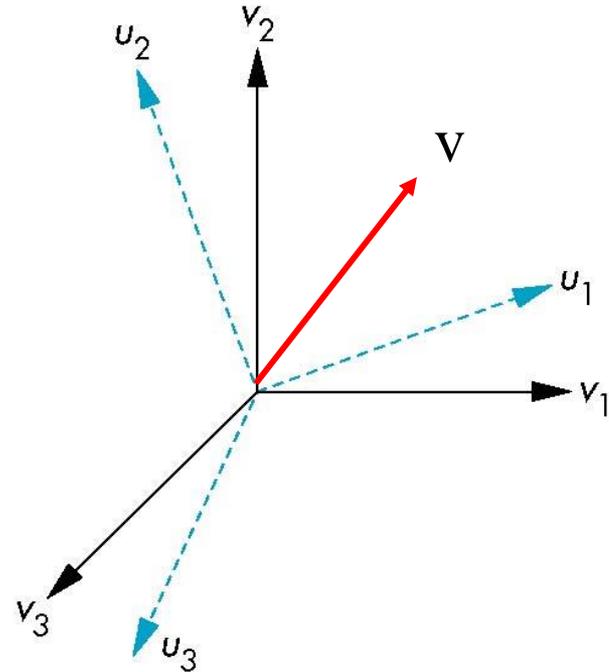
# Representing one basis in terms of another

Each of the basis vectors,  $u_1, u_2, u_3$ , are vectors that can be represented in terms

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$

$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$

$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$



# Matrix Form

The coefficients define a 3 x 3 matrix

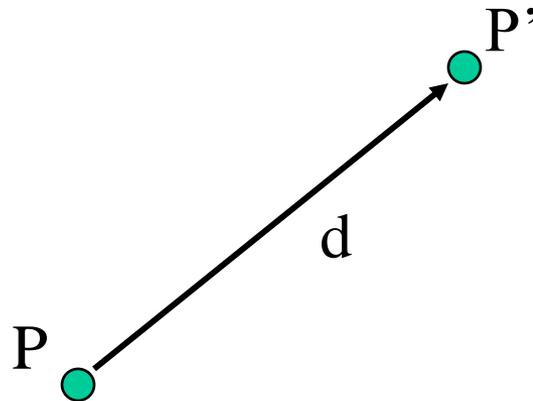
$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

and the bases can be related by

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

# Translation

- Move (translate, displace) a point to a new location



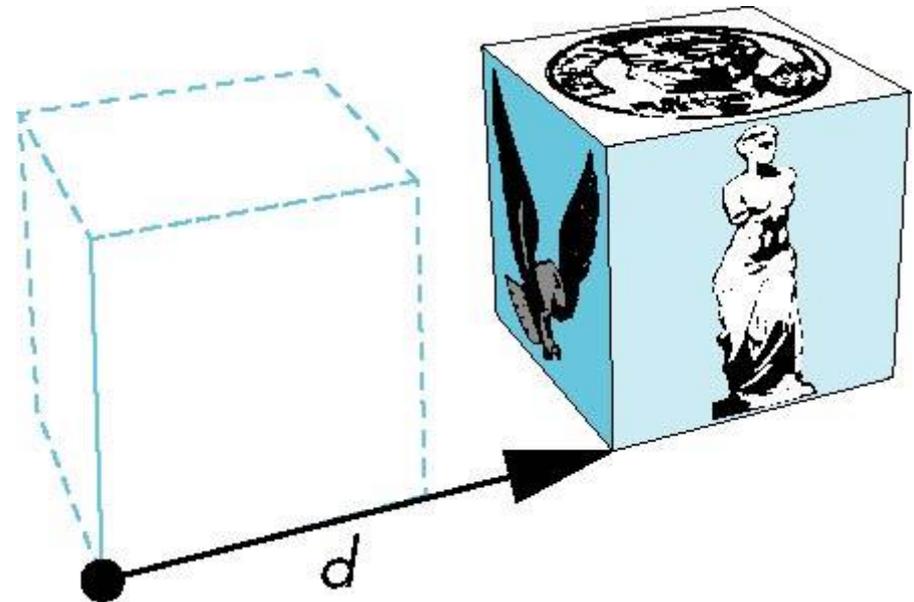
- Displacement determined by a vector  $d$ 
  - Three degrees of freedom
  - $P' = P + d$

# How many ways?

Although we can move a point to a new location in infinite ways, when we move many points there is usually only one way



object



translation: every point displaced  
by same vector

# Translation Using Representations

Using the homogeneous coordinate representation in some frame

$$\mathbf{p} = [x \ y \ z \ 1]^T$$

$$\mathbf{p}' = [x' \ y' \ z' \ 1]^T$$

$$\mathbf{d} = [dx \ dy \ dz \ 0]^T$$

Hence  $\mathbf{p}' = \mathbf{p} + \mathbf{d}$  or

$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$

note that this expression is in four dimensions and expresses point = vector + point

# Translation Matrix

We can also express translation using a 4 x 4 matrix  $\mathbf{T}$  in homogeneous coordinates

$$\mathbf{p}' = \mathbf{T}\mathbf{p} \text{ where } \mathbf{T} = \mathbf{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together

# Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point  $[x \ y \ z]$  is given as

$$\mathbf{p} = [x' \ y' \ z' \ w]^T = [wx \ wy \ wz \ w]^T$$

We return to a three dimensional point (for  $w \neq 0$ ) by

$$x \leftarrow x'/w$$

$$y \leftarrow y'/w$$

$$z \leftarrow z'/w$$

If  $w=0$ , the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For  $w=1$ , the representation of a point is  $[x \ y \ z \ 1]$

# Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using  $4 \times 4$  matrices
  - Hardware pipeline works with 4 dimensional representations
  - For orthographic viewing, we can maintain  $w=0$  for vectors and  $w=1$  for points
  - For perspective we need a *perspective division*

# Rotation about the z axis

- Rotation about z axis in three dimensions leaves all points with the same z
  - Equivalent to rotation in two dimensions in planes of constant z

$$\begin{aligned}x' &= x \cos \theta - y \sin \theta \\y' &= x \sin \theta + y \cos \theta \\z' &= z\end{aligned}$$

–or in homogeneous coordinates

$$\mathbf{p}' = \mathbf{R}_z(\theta)\mathbf{p}$$

# Rotation Matrix

$$\mathbf{R} = \mathbf{R}_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Rotation about x and y axes

- Same argument as for rotation about z axis
  - For rotation about x axis, x is unchanged
  - For rotation about y axis, y is unchanged

$$\mathbf{R} = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Scaling

Expand or contract along each axis (fixed point of origin)

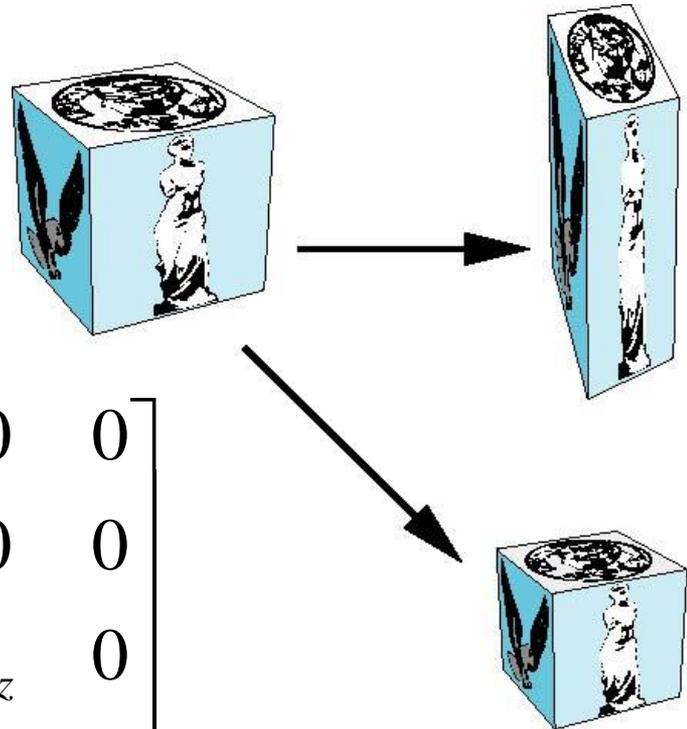
$$x' = s_x x$$

$$y' = s_y x$$

$$z' = s_z x$$

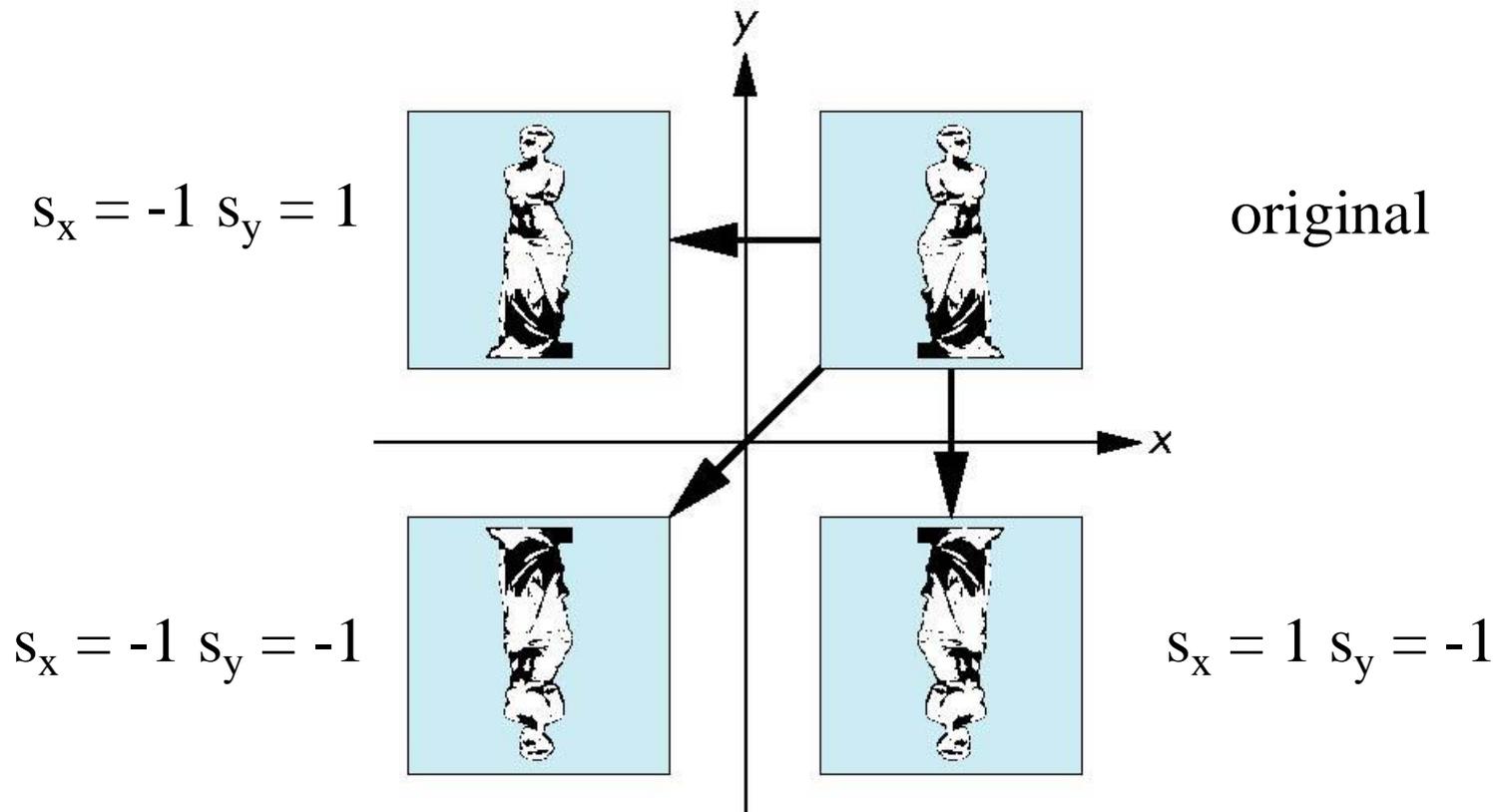
$$\mathbf{p}' = \mathbf{S}\mathbf{p}$$

$$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Reflection

corresponds to negative scale factors



# Inverses

- Although we could compute inverse matrices by general formulas, we can use simple geometric observations

–Translation:  $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$

–Rotation:  $\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$

- Holds for any rotation matrix

- Note that since  $\cos(-\theta) = \cos(\theta)$  and  $\sin(-\theta) = -\sin(\theta)$

$$\mathbf{R}^{-1}(\theta) = \mathbf{R}^T(\theta)$$

–Scaling:  $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$

# Concatenation

- We can form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices
- Because the same transformation is applied to many vertices, the cost of forming a matrix  $\mathbf{M}=\mathbf{ABCD}$  is not significant compared to the cost of computing  $\mathbf{Mp}$  for many vertices  $\mathbf{p}$
- The difficult part is how to form a desired transformation from the specifications in the application

# Order of Transformations

- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

$$\mathbf{p}' = \mathbf{A}\mathbf{B}\mathbf{C}\mathbf{p} = \mathbf{A}(\mathbf{B}(\mathbf{C}\mathbf{p}))$$

- Note many references use column matrices to represent points. In terms of column matrices

$$\mathbf{p}'^T = \mathbf{p}^T \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T$$

# General Rotation About the Origin

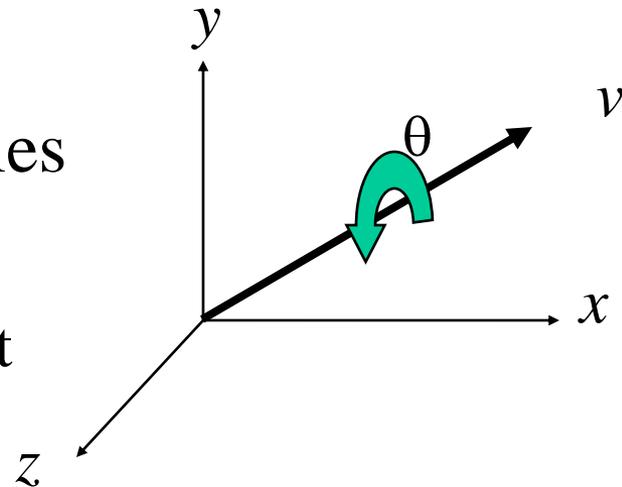
A rotation by  $\theta$  about an arbitrary axis can be decomposed into the concatenation of rotations about the  $x$ ,  $y$ , and  $z$  axes

$$\mathbf{R}(\theta) = \mathbf{R}_z(\theta_z) \mathbf{R}_y(\theta_y) \mathbf{R}_x(\theta_x)$$

$\theta_x$   $\theta_y$   $\theta_z$  are called the Euler angles

Note that rotations do not commute

We can use rotations in another order but with different angles



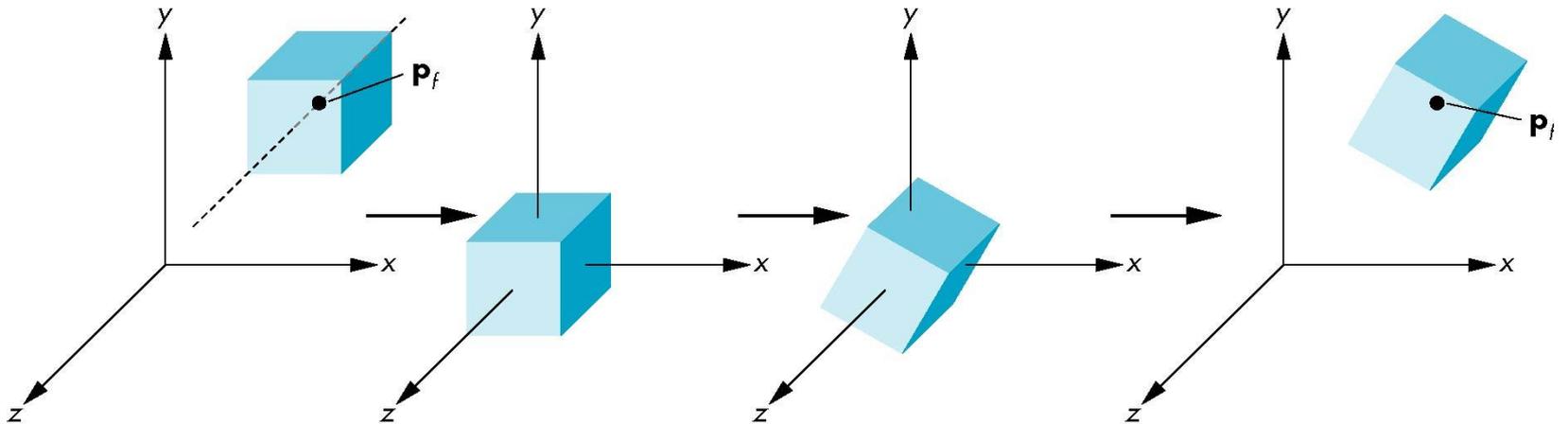
# Rotation About a Fixed Point other than the Origin

Move fixed point to origin

Rotate

Move fixed point back

$$\mathbf{M} = \mathbf{T}(p_f) \mathbf{R}(\theta) \mathbf{T}(-p_f)$$



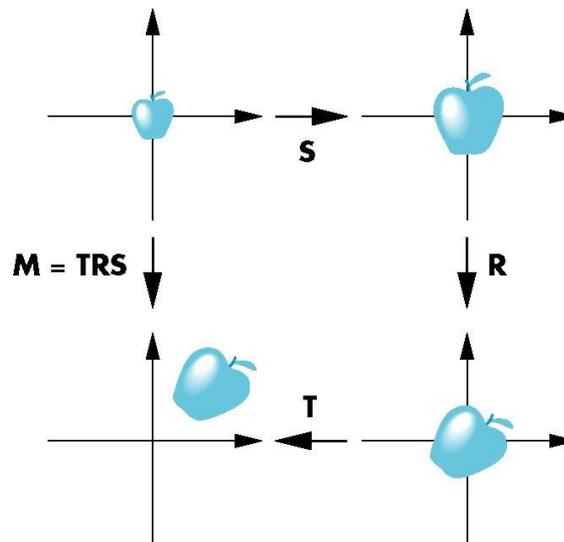
# Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size
- We apply an *instance transformation* to its vertices to

Scale

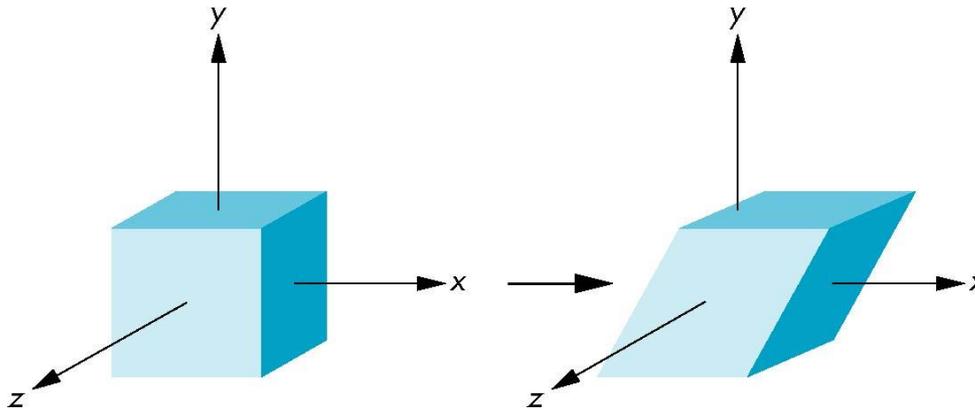
Orient

Locate



# Shear

- Helpful to add one more basic transformation
- Equivalent to pulling faces in opposite directions



# Shear Matrix

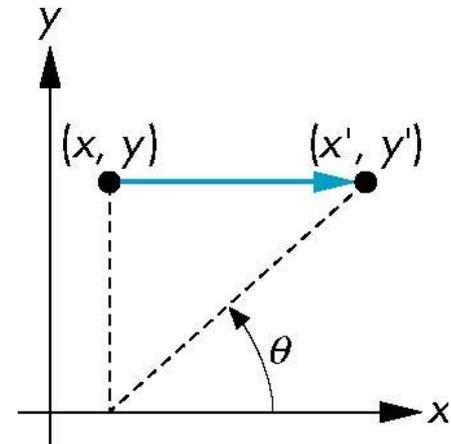
Consider simple shear along  $x$  axis

$$x' = x + y \cot \theta$$

$$y' = y$$

$$z' = z$$

$$\mathbf{H}(\theta) = \begin{bmatrix} 1 & \cot \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Computer Viewing

Ed Angel

Professor of Computer Science,  
Electrical and Computer Engineering,  
and Media Arts

University of New Mexico

# Objectives

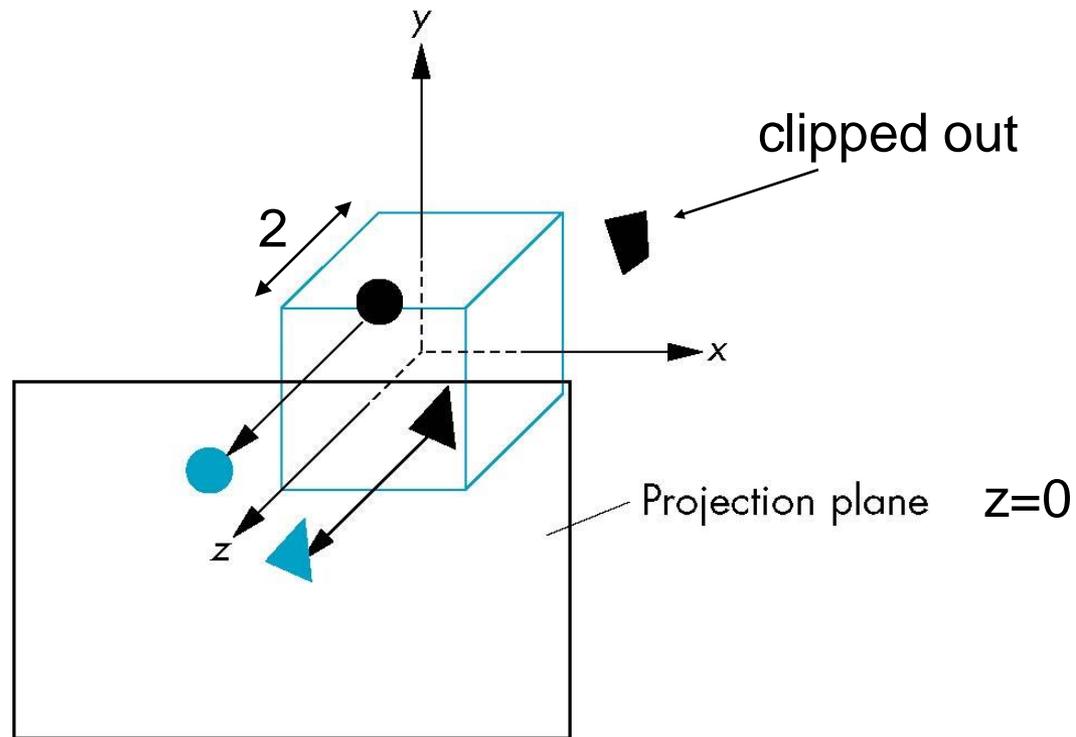
- Introduce the mathematics of projection

# Computer Viewing

- There are three aspects of the viewing process, all of which are implemented in the pipeline,
  - Positioning the camera
    - Setting the model-view matrix
  - Selecting a lens
    - Setting the projection matrix
  - Clipping
    - Setting the view volume
    - (default is unit cube,  $\mathbb{R}^3$ ,  $[-1,1]$ )

# Default Projection

Default projection is orthogonal

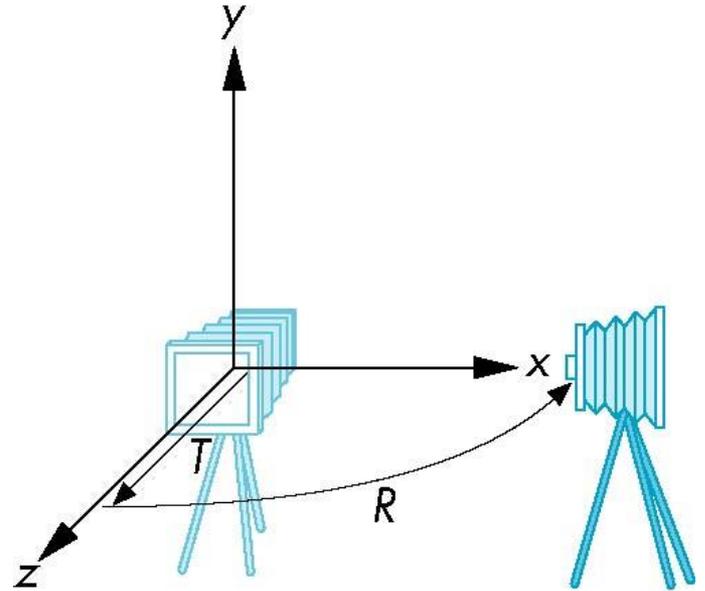


# Moving the Camera Frame

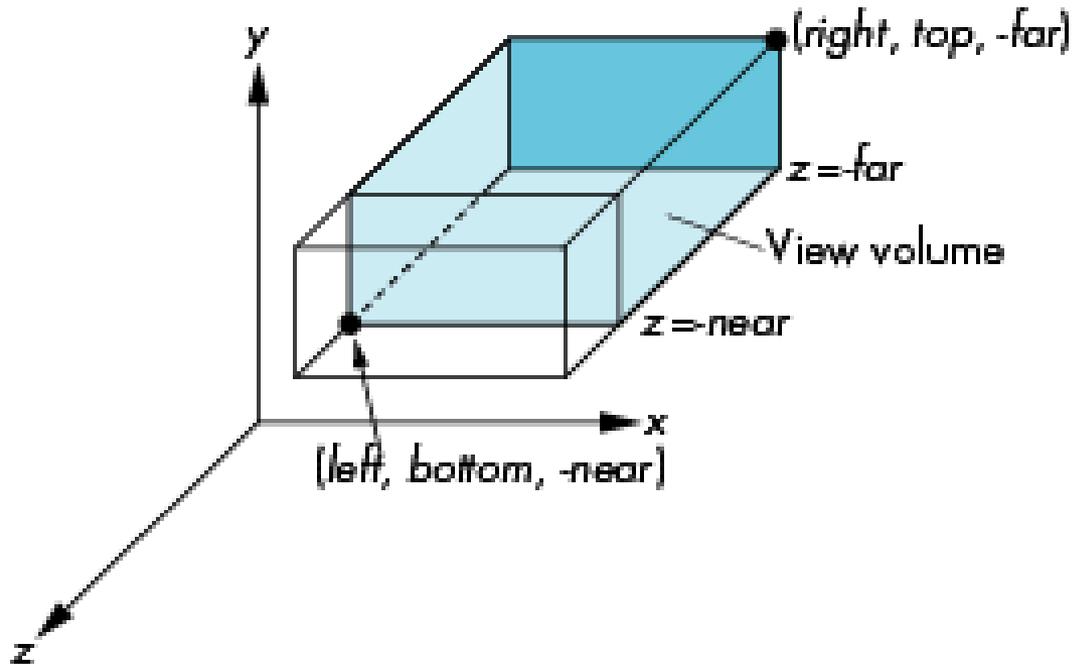
- If we want to visualize object with both positive and negative z values we can either
  - Move the camera in the positive z direction
    - Translate the camera frame
  - Move the objects in the negative z direction
    - Translate the world frame
- Both of these views are equivalent and are determined by the model-view matrix

# Moving the Camera

- We can move the camera to any desired position by a sequence of rotations and translations
- Example: side view
  - Rotate the camera
  - Move it away from origin
  - Model-view matrix  $C = TR$

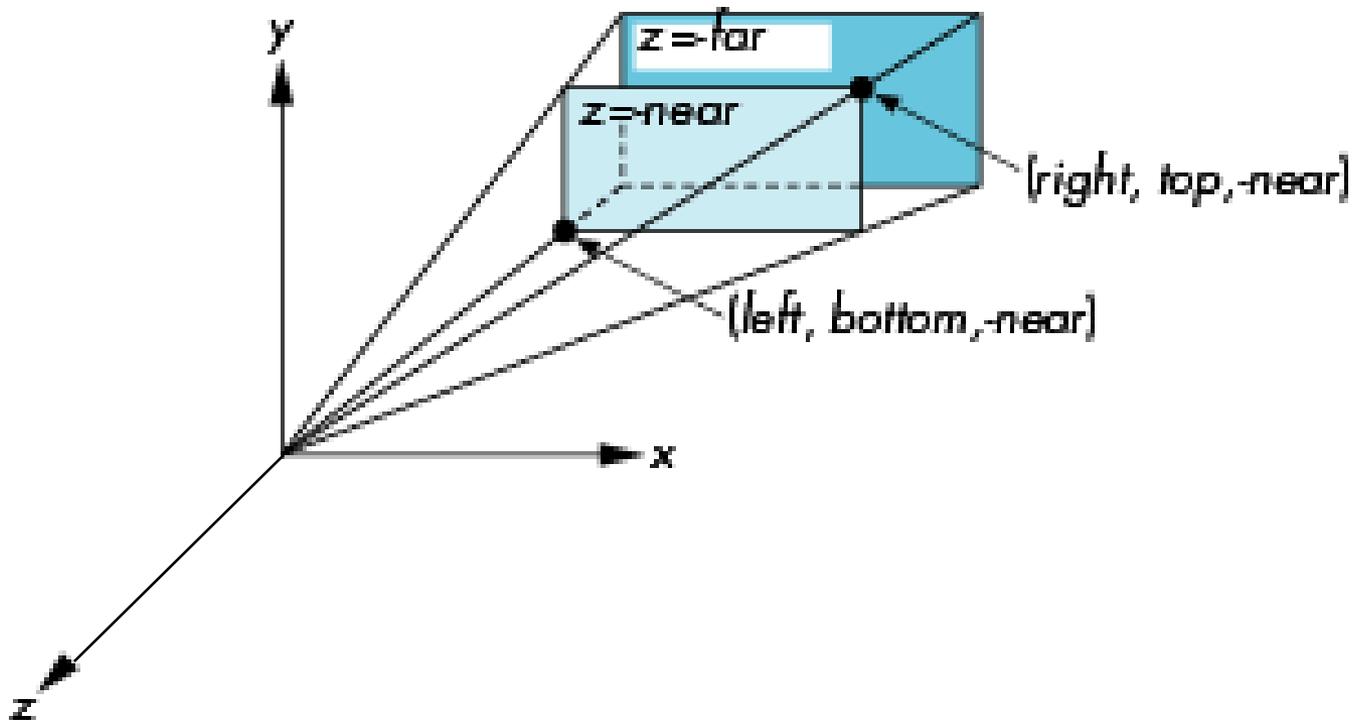


# OpenGL Orthogonal Viewing



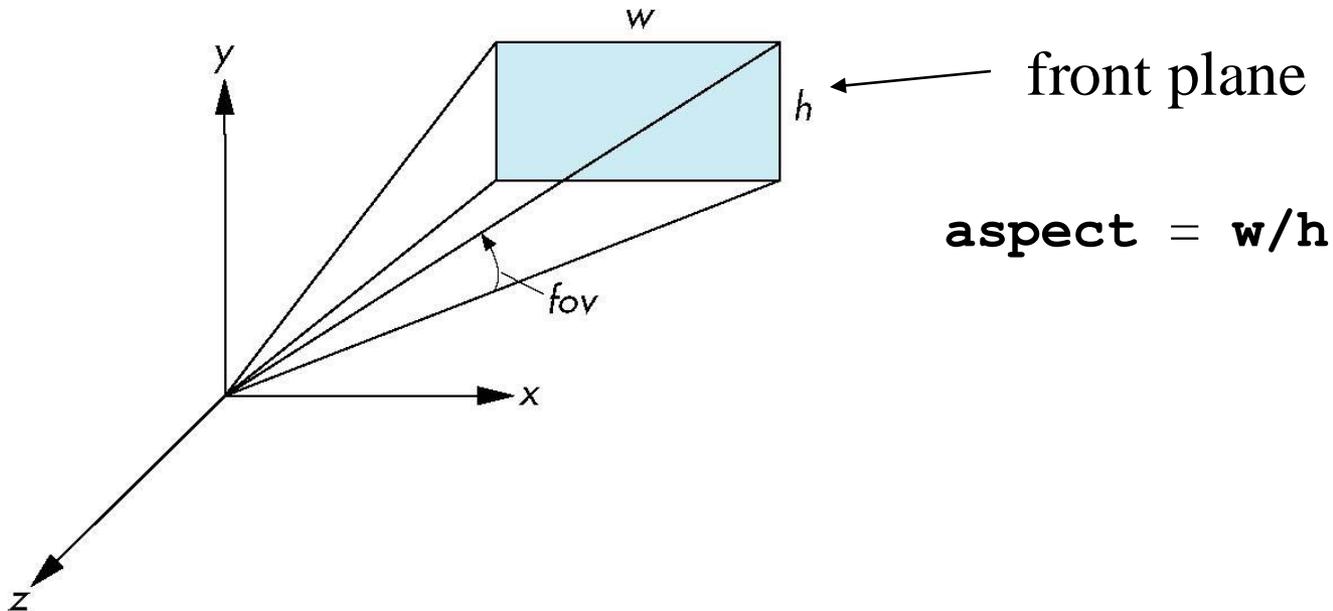
`near` and `far` measured from camera

# OpenGL Perspective



# Using Field of View

- Parameters **fovy**, **aspect**, **near**, **far** often provides a better interface



# Projections explained differently

- Read the following slides about orthogonal and perspective projections by your selves
- They present the same thing, but explained differently

# Projections and Normalization

- The default projection in the eye (camera) frame is orthogonal
- For points within the default view volume

$$x_p = x$$

$$y_p = y$$

$$z_p = 0$$

- Most graphics systems use *view normalization*
  - All other views are converted to the default view by transformations that determine the projection matrix
  - Allows use of the same pipeline for all views

# Homogeneous Coordinate Representation

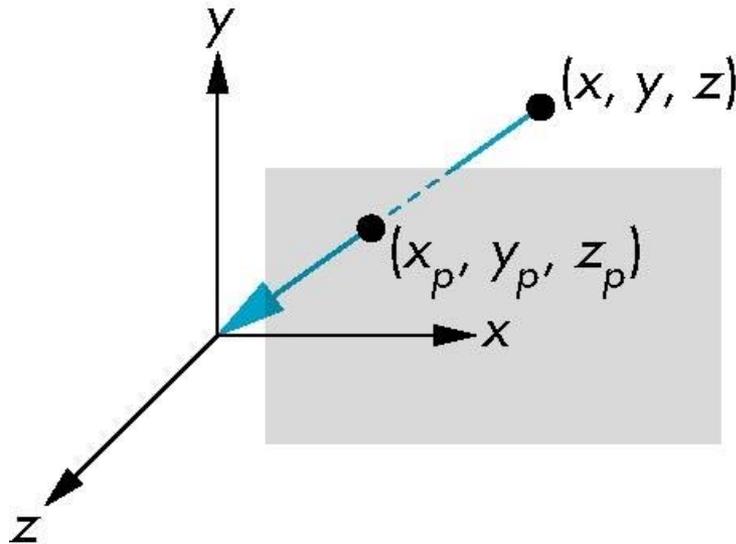
default orthographic projection

$$\begin{aligned}x_p &= x \\y_p &= y \\z_p &= 0 \\w_p &= 1\end{aligned}\quad \mathbf{p}_p = \mathbf{M}\mathbf{p}$$
$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In practice, we can let  $\mathbf{M} = \mathbf{I}$  and set the  $z$  term to zero later

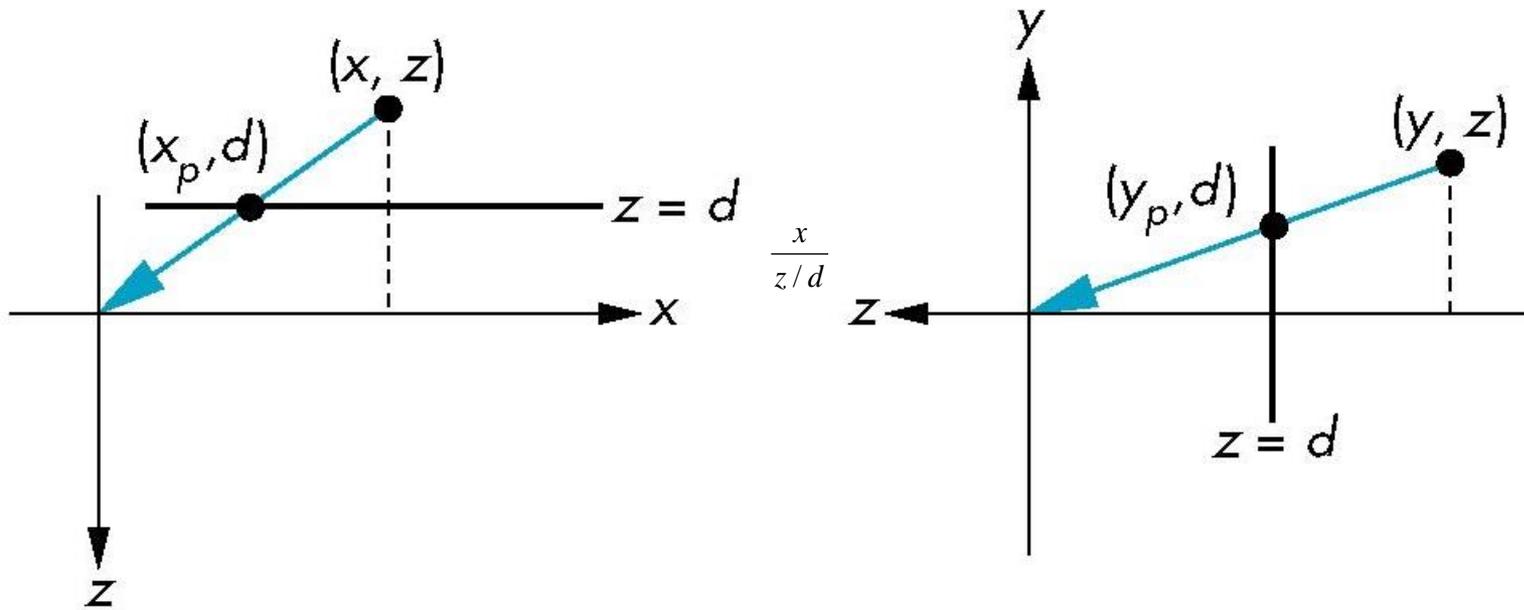
# Simple Perspective

- Center of projection at the origin
- Projection plane  $z = d$ ,  $d < 0$



# Perspective Equations

Consider top and side views



$$x_p = \frac{x}{z/d}$$

$$y_p = \frac{y}{z/d}$$

$$z_p = d$$

# Homogeneous Coordinate Form

consider  $\mathbf{q} = \mathbf{M}\mathbf{p}$  where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$

# Perspective Division

- However  $w \neq 1$ , so we must divide by  $w$  to return from homogeneous coordinates
- This *perspective division* yields

$$x_p = \frac{x}{z/d} \quad y_p = \frac{y}{z/d} \quad z_p = d$$

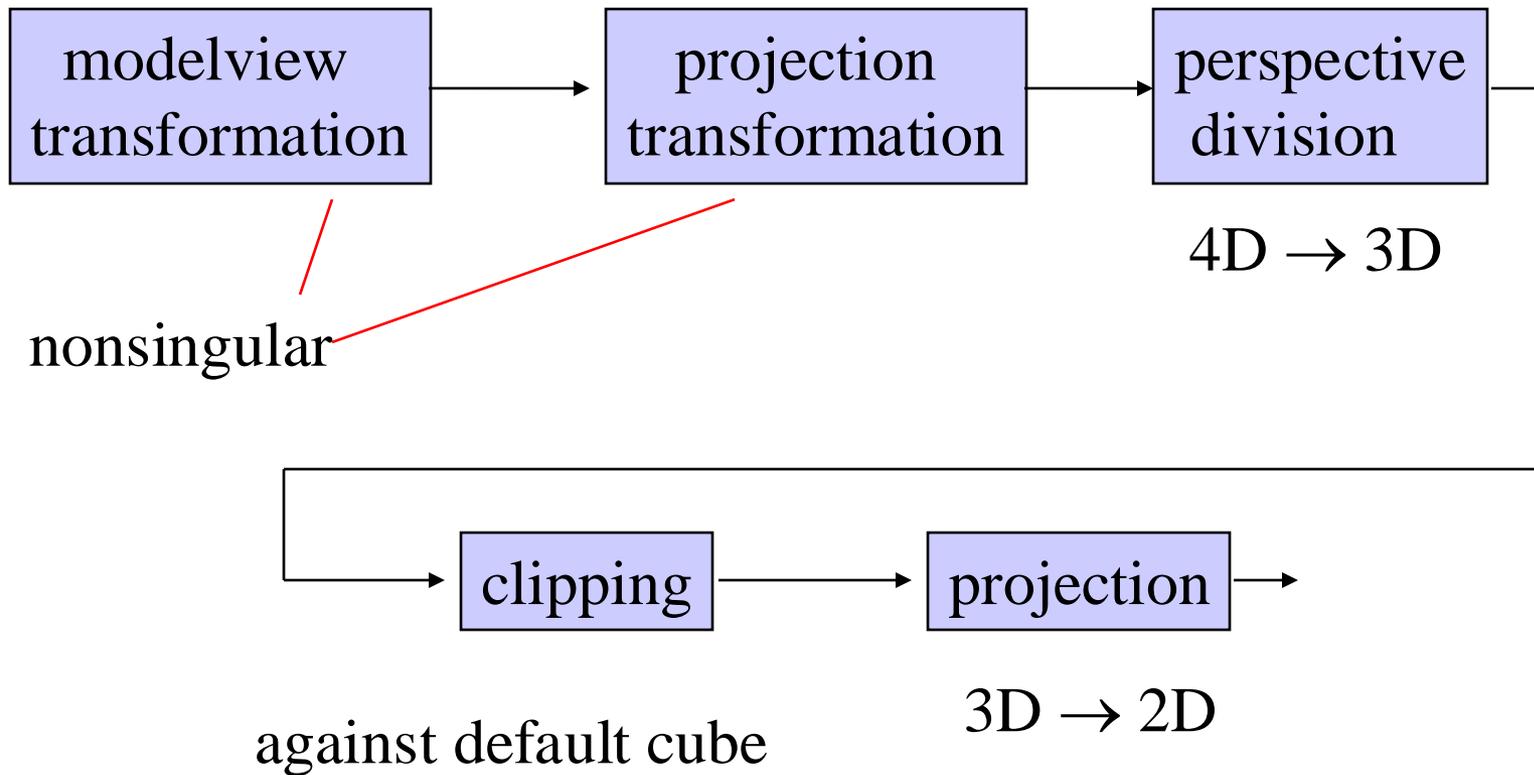
the desired perspective equations

- We will consider the corresponding clipping volume with the OpenGL functions

# Normalization

- Rather than derive a different projection matrix for each type of projection, we can convert all projections to orthogonal projections with the default view volume
- This strategy allows us to use standard transformations in the pipeline and makes for efficient clipping

# Pipeline View

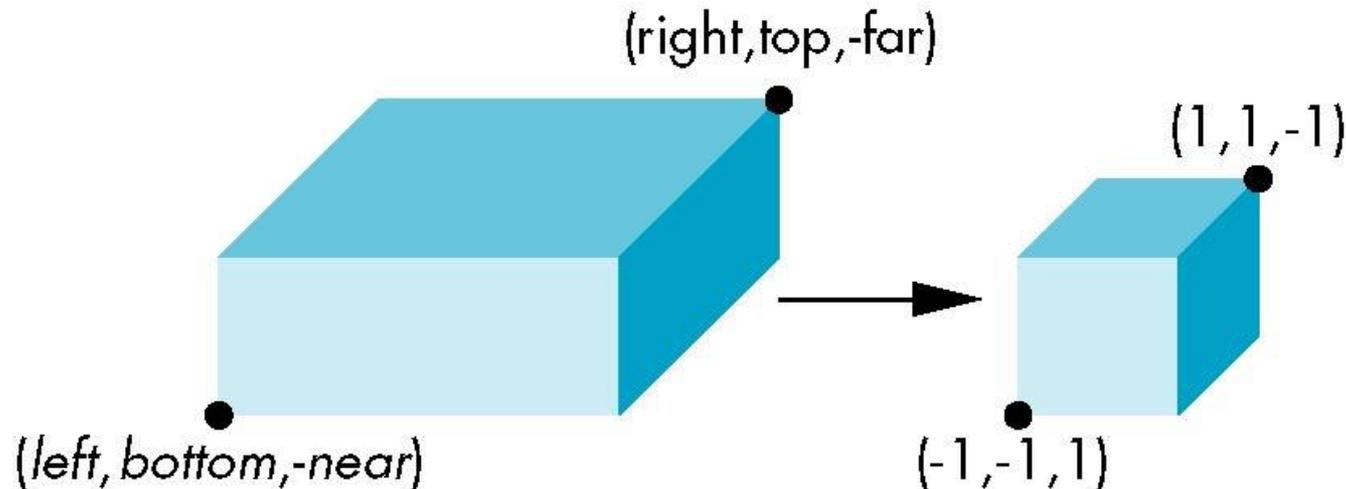


# Notes

- We stay in four-dimensional homogeneous coordinates through both the modelview and projection transformations
  - Both these transformations are nonsingular
  - Default to identity matrices (orthogonal view)
- Normalization lets us clip against simple cube regardless of type of projection
- Delay final projection until end
  - Important for hidden-surface removal to retain depth information as long as possible

# Orthogonal Normalization

normalization  $\Rightarrow$  find transformation to convert specified clipping volume to default



# Orthogonal Matrix

- Two steps

- Move center to origin

- $T(-(\text{left}+\text{right})/2, -(\text{bottom}+\text{top})/2, (\text{near}+\text{far})/2))$

- Scale to have sides of length 2

- $S(2/(\text{left}-\text{right}), 2/(\text{top}-\text{bottom}), 2/(\text{near}-\text{far}))$

$$\mathbf{P} = \mathbf{ST} = \begin{bmatrix} \frac{2}{\text{right} - \text{left}} & 0 & 0 & -\frac{\text{right} - \text{left}}{\text{right} - \text{left}} \\ 0 & \frac{2}{\text{top} - \text{bottom}} & 0 & -\frac{\text{top} + \text{bottom}}{\text{top} - \text{bottom}} \\ 0 & 0 & \frac{2}{\text{near} - \text{far}} & \frac{\text{far} + \text{near}}{\text{far} - \text{near}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Final Projection

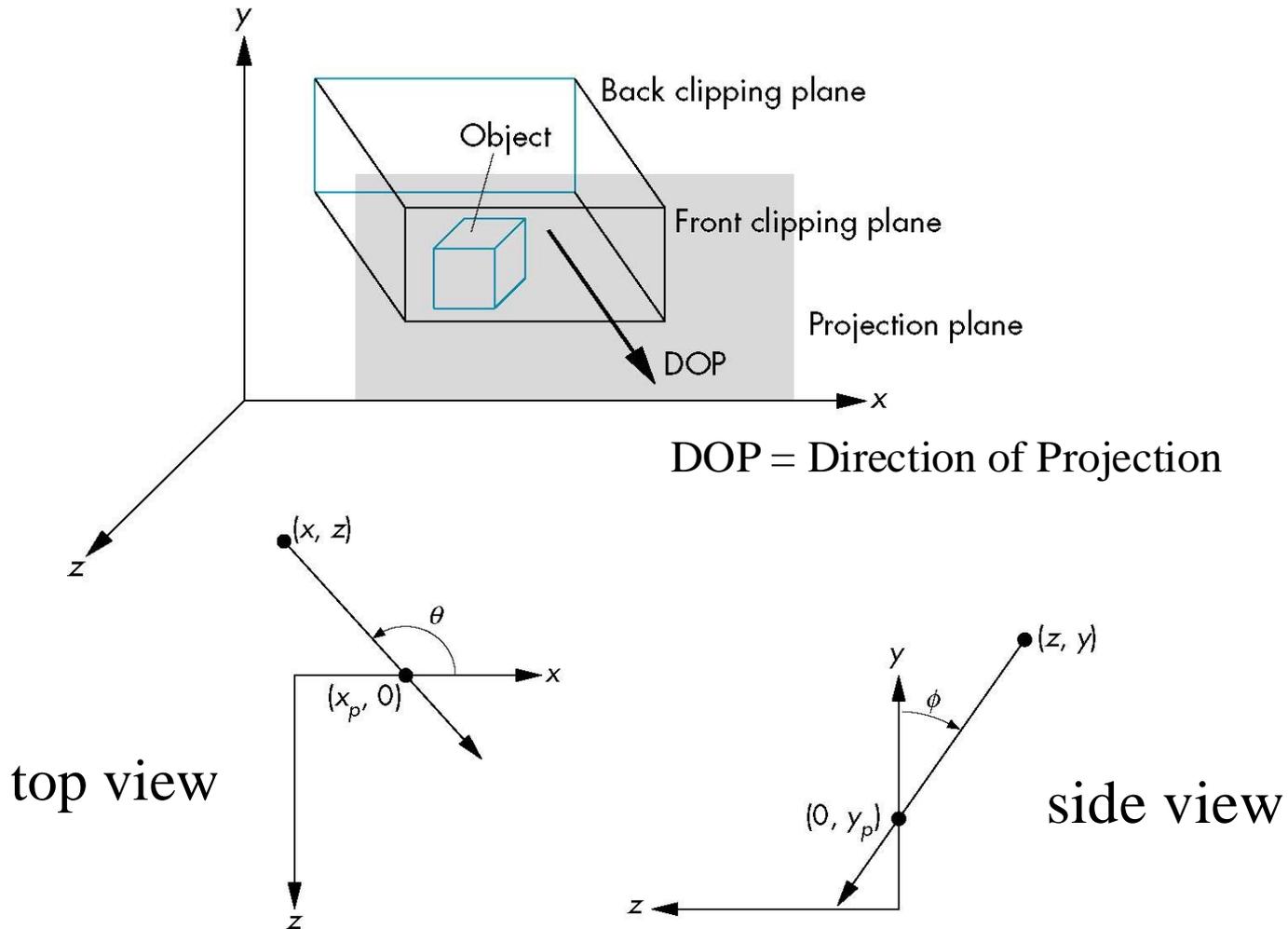
- Set  $z = 0$
- Equivalent to the homogeneous coordinate transformation

$$\mathbf{M}_{\text{orth}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Hence, general orthogonal projection in 4D is

$$\mathbf{P} = \mathbf{M}_{\text{orth}} \mathbf{S} \mathbf{T}$$

# General Shear



# Shear Matrix

$xy$  shear ( $z$  values unchanged)

$$\mathbf{H}(\theta, \phi) = \begin{bmatrix} 1 & 0 & -\cot \theta & 0 \\ 0 & 1 & -\cot \phi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

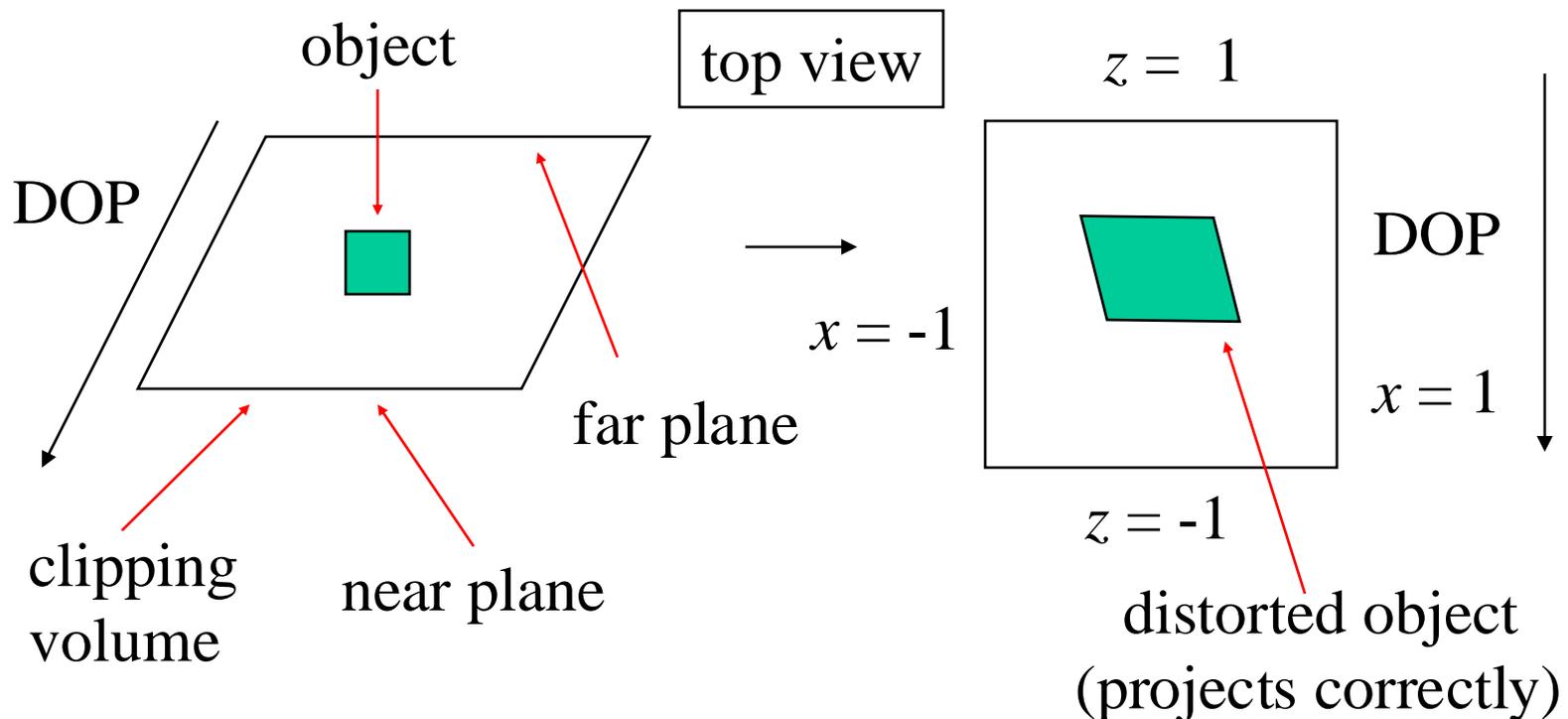
Projection matrix

$$\mathbf{P} = \mathbf{M}_{\text{orth}} \mathbf{H}(\theta, \phi)$$

General case:  $\mathbf{P} = \mathbf{M}_{\text{orth}} \mathbf{STH}(\theta, \phi)$

# Effect on Clipping

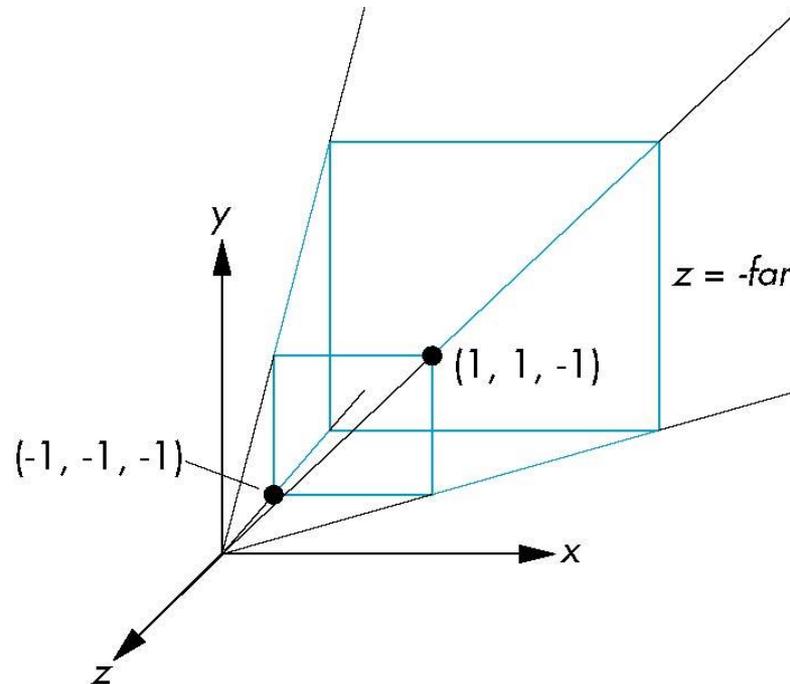
- The projection matrix  $\mathbf{P} = \mathbf{STH}$  transforms the original clipping volume to the default clipping volume



# Simple Perspective

Consider a simple perspective with the COP (=center of projection) at the origin, the near clipping plane at  $z = -1$ , and a 90 degree field of view determined by the planes

$$x = \pm z, y = \pm z$$



# Perspective Matrices

Simple projection matrix in homogeneous coordinates

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Note that this matrix is independent of the far clipping plane

# Generalization

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

after perspective division, the point  $(x, y, z, 1)$  goes to

$$x'' = x/z$$

$$y'' = y/z$$

$$Z'' = -(\alpha + \beta/z)$$

which projects orthogonally to the desired point  
regardless of  $\alpha$  and  $\beta$

# Picking $\alpha$ and $\beta$

If we pick

$$\alpha = \frac{\text{near} + \text{far}}{\text{far} - \text{near}}$$

$$\beta = \frac{2\text{near} * \text{far}}{\text{near} - \text{far}}$$

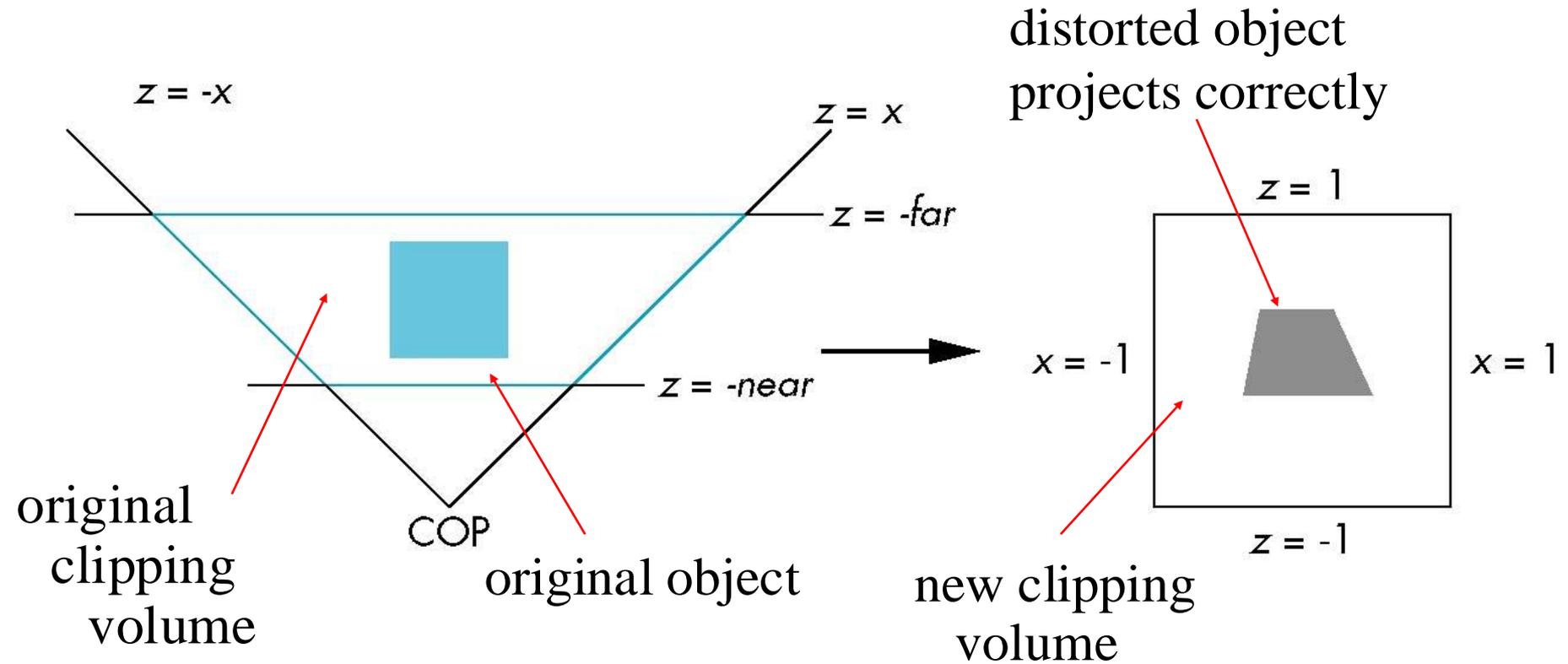
the near plane is mapped to  $z = -1$

the far plane is mapped to  $z = 1$

and the sides are mapped to  $x = \pm 1, y = \pm 1$

Hence the new clipping volume is the default clipping volume

# Normalization Transformation

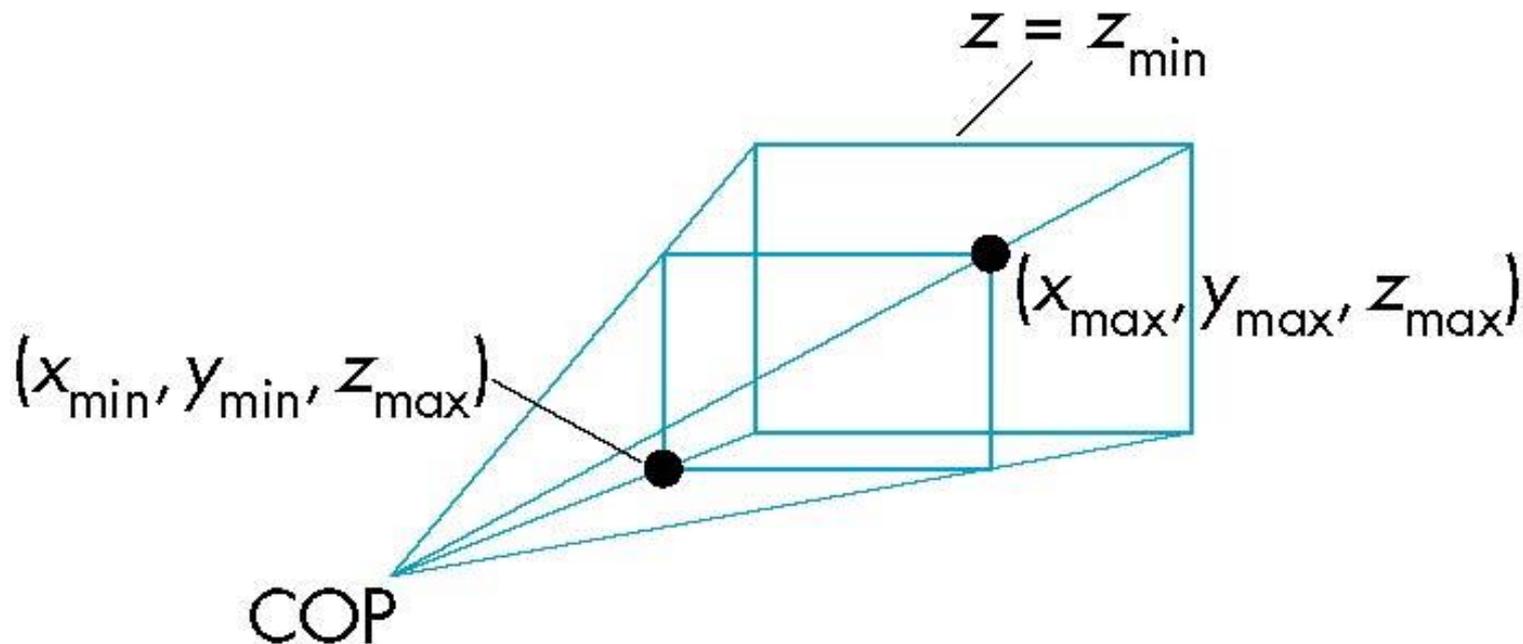


# Normalization and Hidden-Surface Removal

- Although our selection of the form of the perspective matrices may appear somewhat arbitrary, it was chosen so that if  $z_1 > z_2$  in the original clipping volume then the for the transformed points  $z_1' > z_2'$
- Thus hidden surface removal works if we first apply the normalization transformation
- However, the formula  $z'' = -(\alpha + \beta/z)$  implies that the distances are distorted by the normalization which can cause numerical problems especially if the near distance is small

# OpenGL Perspective

- Unsymmetric viewing frustum possible:



# OpenGL Perspective Matrix

- The normalization by a perspective projection requires an initial shear to form a right viewing pyramid, followed by a scaling to get the normalized perspective volume. Finally, the perspective matrix results in needing only a final orthogonal transformation

$$\mathbf{P} = \mathbf{NSH}$$

our previously defined  
perspective matrix

shear and scale

# Why do we do it this way?

- Normalization allows for a single pipeline for both perspective and orthogonal viewing
- We stay in four dimensional homogeneous coordinates as long as possible to retain three-dimensional information needed for hidden-surface removal and shading
- We simplify clipping