Logical rules for natural deduction

We describe when $\Gamma \vdash \psi$, i.e. ψ is derivable from a finite set $\Gamma = \psi_1, \ldots, \psi_n$ by the following rules. We write $\vdash \psi$ for $\Gamma \vdash \psi$ if Γ is empty.

$$\frac{\psi \in \Gamma}{\Gamma \vdash \psi}$$

$$\frac{\Gamma, \psi \vdash \varphi}{\Gamma \vdash \psi \to \varphi} \qquad \frac{\Gamma \vdash \psi \to \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi}$$

$$\frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \psi} \qquad \frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi \wedge \varphi}$$

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \psi \vee \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi} \qquad \frac{\Gamma \vdash \psi \vee \varphi \qquad \Gamma, \psi \vdash \delta \qquad \Gamma, \varphi \vdash \delta}{\Gamma \vdash \delta}$$

$$\frac{\Gamma, \psi \vdash \bot}{\Gamma \vdash \neg\psi} \qquad \frac{\Gamma \vdash \neg\psi \qquad \Gamma \vdash \psi}{\Gamma \vdash \bot}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \psi[x_0/x]}{\Gamma \vdash \forall x\ \psi} \qquad \frac{\Gamma \vdash \forall x\ \psi}{\Gamma \vdash \psi[t/x]}$$

$$\frac{\Gamma \vdash \psi[t/x]}{\Gamma \vdash \exists x\ \psi} \qquad \frac{\Gamma \vdash \exists x\ \psi \qquad \Gamma, \psi[x_0/x] \vdash \delta}{\Gamma \vdash \delta}$$

In the rule of $\forall$ introduction x_0 should not occur free in the conclusion. This was essentially the rule found by Frege (1879).

In the rule of $\exists$ elimination x_0 should not occur free in Γ and δ and $\exists x\ \psi$.

The following example illustrates well the use of Frege's rule for $\forall$ introduction

$$\forall x\ (P(x) \to Q(x)),\ \forall x\ P(x) \vdash \forall x\ Q(x)$$

Russell, who was the one of the first to understand the importance of Frege's discovery, talks about the difference between *all* and *any*. In order to prove $\forall x\ Q(x)$ we prove that $Q(x_0)$ holds for *any* x_0

$$\forall x\ (P(x) \to Q(x)),\ \forall x\ P(x) \vdash Q(x_0)$$

and this we can prove since we have from the hypotheses $P(x_0) \to Q(x_0)$ and $P(x_0)$ and we can use modus-ponens.

0.1 Classical logic

The rule for classical logic (how to prove something true by assuming something false) is

$$\frac{\Gamma, \neg\psi \vdash \psi}{\Gamma \vdash \psi}(1)$$

or, alternatively

$$\frac{\Gamma, \neg\psi \vdash \bot}{\Gamma \vdash \psi}(2)$$

It is a good exercise to show that these rules (1) and (2) are equivalent. The formulation (1) is due to Peirce (1885), who even had a (apparently more general) equivalent formulation

$$\frac{\Gamma, \psi \to \varphi \vdash \psi}{\Gamma \vdash \psi}(3)$$

It is remarkable that it corresponds to the type of *continuation operators* in programming languages.

The formulations (1) and (3) are interesting since they illustrate how to use classical logic: in order to prove ψ from some hypotheses, we can always add $\neg\psi$, or any formula $\psi \to \delta$ in the hypotheses. For instance, we can show p from $\Gamma = (p \to q) \to r, r \to p$ since we can show r, and hence p, from $\Gamma, p \to q$.

0.2 Soundness Theorem

All these rules are valid for the relation $\Gamma \vDash \psi$. For instance if both $\psi \to \varphi$ and ψ are valid in a model, then so is φ.

Since $\Gamma \vdash \psi$ is (by definition) the *least* relation satisfying these rules, it follows that we have

$$\Gamma \vdash \psi \quad \Rightarrow \quad \Gamma \vDash \psi$$

which is precisely the *soundness* Theorem.

0.3 Equality

The rules for equality are.

$$\frac{}{\Gamma \vdash t = t} \qquad \frac{\Gamma \vdash t = u \qquad \Gamma \vdash \psi[t/x]}{\Gamma \vdash \psi(u/x)}$$

This implies symmetry and transitivity of equality.

This implies that we have $t = v,\ u = v \vdash t = u$: the relation of equality is *euclidean*, two objects which are "equal to the same are equal to each other".

Equality reasoning can be really powerful.

Here is an example: if we know $f(a, x) = x$ and $f(x, g(x)) = a$ then we deduce $g(a) = a$.

This is because, if we consider the substitution $[a/x]$, we both get $f(a, g(a)) = g(a)$ and $f(a, g(a)) = a$ and hence $g(a) = a$.

(This is connected to the *Knuth-Bendix algorithm*, which is a general technique to deduce interesting equational consequences from a set of equations.)

0.4 Non empty domain

The following is a valid derivation: we have $\vdash x_0 = x_0$ hence $\vdash \exists x\ (x = x)$. It corresponds to the fact that we want to describe the logic of *non empty* universes.

Similarly we can show $\forall x\ \psi \vdash \exists x\ \psi$.

0.5 Examples

We show $\forall x\ \neg P(x)$ from $\Gamma = \neg(\exists x\ P(x))$.

This is because $\Gamma, P(x_0)$ is contradictory.

We show $\psi = \exists x\ \neg P(x)$ from $\Gamma = \neg(\forall x\ P(x))$. This is because $\Gamma' = \Gamma, \neg\psi$ is contradictory, which is because we can show $P(x_0)$ $\forall x\ P(x)$ from Γ'. In turn this is because $\Gamma', \neg P(x_0)$ is contradictory.