Examination, Models of Computation (DIT310/TDA183/TDA184)

- Date and time: 2018-01-10, 8:30–12:30.
- Author[1]/examiner: Nils Anders Danielsson.
- Responsible during the examination: Daniel Schoepe. Telephone number: 6166. Visits to the examination rooms: ~9:30 and ~11:30.
- Authorised aids (except for aids that are always permitted): None.
- The GU grades Pass (G) and Pass with Distinction (VG) correspond to the Chalmers grades 3 and 5, respectively.
- To get grade n on the exam you have to be awarded grade n or higher on at least n exercises.
- A completely correct solution of one exercise is awarded the grade 5. Solutions with minor mistakes *might* get the grade 5, and solutions with larger mistakes might get lower grades.
- Exercises can contain parts and/or requirements that are only required for a certain grade. To get grade n on such an exercise you have to get grade n or higher on every part marked with grade n or lower (and every unmarked part), and you have to fulfil every requirement marked with grade n or lower (as well as every unmarked requirement).
- Do not hand in solutions for several exercises on the same sheet.
- Write your examination code on each sheet.
- Solutions can be rejected if they are hard to read, unstructured, or poorly motivated.
- After correction the graded exams are available in the student office in room 4482 of the EDIT building. If you want to discuss the grading you can, within three weeks after the result has been reported, contact the examiner and set up a time for a meeting (in which case you should not remove the exam from the student office).

[1]Thanks to Daniel Schoepe for feedback.

1. (a) *For grade 3:* Give an example of a set A for which $\mathbb{N} \to A$ is countable, and give an example of a set B for which $\mathbb{N} \to B$ is not countable. You do not need to provide proofs.

   (b) *For grade 4:* Either prove that the set

   $$\mathit{List}\ \{0,1\} \to \mathit{List}\ \{0,1\}$$

   is countable, or that it is not countable. You can use theorems from the lecture slides without providing proofs for them.

2. Give concrete syntax for the χ expression e for which the standard χ encoding (as presented in the lectures), given using concrete syntax, is

   $$\ulcorner e \urcorner = \mathsf{Case}(\mathsf{Var}(\mathsf{Zero}()), \\ \mathsf{Cons}(\mathsf{Branch}(\mathsf{Suc}(\mathsf{Zero}()), \mathsf{Nil}(), \mathsf{Var}(\mathsf{Zero}())), \mathsf{Nil}())).$$

   Assume that the number 0 corresponds to the variable x, and that the number 1 corresponds to the constructor True.

3. Is the following function χ-decidable?

   $$\begin{aligned} & \textit{is-total} \in \textit{CExp} \to \textit{Bool} \\ & \textit{is-total}\ e = \\ & \quad \textbf{if}\ \forall\, m \in \mathbb{N}.\ \exists\, n \in \mathbb{N}.\ [\![e \ulcorner m \urcorner]\!] = \ulcorner n \urcorner\ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false} \end{aligned}$$

   Here *CExp* is a set containing the abstract syntax of every closed χ expression, and $\ulcorner x \urcorner$ is the standard encoding of the natural number x.

   For grade 3: Motivate your answer.

   For grade 4: Provide a proof. You are allowed to make use of Rice's theorem, and the fact that the halting problem is undecidable, but not other results stating that some function is or is not computable (unless you provide proofs).

   For grade 5: You may not use Rice's theorem (unless you provide a proof).

4. Is the following function χ-decidable?

   $$\begin{aligned} & \textit{is-total} \in \textit{Fun} \to \textit{Bool} \\ & \textit{is-total}\ e = \\ & \quad \textbf{if}\ \forall\, m \in \mathbb{N}.\ \exists\, n \in \mathbb{N}.\ [\![e \ulcorner m \urcorner]\!] = \ulcorner n \urcorner\ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false} \end{aligned}$$

   Here *Fun* is the following set:

   $$\{\, e \in \textit{CExp} \mid f \in \mathbb{N} \to \mathbb{N},\ e \text{ witnesses the } \chi\text{-computability of } f \,\}$$

   The grade criteria of the previous exercise apply to this one as well.

5. Consider the following program in RF_1:

$$p = \mathsf{min}\ (\mathsf{rec}\ (\mathsf{proj}\ 0)\ (\mathsf{comp}\ (\mathsf{min}\ \mathsf{suc})\ \mathsf{nil}))$$

(a) *For grade 3:* What is the value, if any, of $[\![p]\!]\ (\mathsf{nil}, 0)$?

(b) *For grade 4:* Is the partial function $f \in \mathbb{N} \rightharpoonup \mathbb{N}$ given by

$$f\ n = [\![p]\!]\ (\mathsf{nil}, n)$$

total, i.e. defined for all inputs? Provide a proof.

For reference, here is the abstract syntax of RF:

$$\frac{}{\mathsf{zero} \in RF_0} \qquad \frac{}{\mathsf{suc} \in RF_1} \qquad \frac{0 \le i < n}{\mathsf{proj}\ i \in RF_n}$$

$$\frac{f \in RF_m \qquad gs \in (RF_n)^m}{\mathsf{comp}\ f\ gs \in RF_n} \qquad \frac{f \in RF_n \qquad g \in RF_{2+n}}{\mathsf{rec}\ f\ g \in RF_{1+n}} \qquad \frac{f \in RF_{1+n}}{\mathsf{min}\ f \in RF_n}$$

The operational semantics of RF is also included. The semantics $f\,[\rho] \Downarrow m$ is, for every $n \in \mathbb{N}$, a relation between programs $f \in RF_n$, vectors $\rho \in \mathbb{N}^n$, and natural numbers $m \in \mathbb{N}$, and $fs\,[\rho] \Downarrow^{\star} ms$ is, for all m, $n \in \mathbb{N}$, a relation between vectors $fs \in (RF_m)^n$, vectors $\rho \in \mathbb{N}^m$, and vectors $ms \in \mathbb{N}^n$:

$$\frac{}{\mathsf{zero}\,[\mathsf{nil}] \Downarrow 0} \qquad \frac{}{\mathsf{suc}\,[\mathsf{nil}, n] \Downarrow 1 + n}$$

$$\frac{}{\mathsf{proj}\ i\,[\rho] \Downarrow\ index\ \rho\ i} \qquad \frac{gs\,[\rho] \Downarrow^{\star} \rho' \qquad f\,[\rho'] \Downarrow n}{\mathsf{comp}\ f\ gs\,[\rho] \Downarrow n}$$

$$\frac{f\,[\rho] \Downarrow n}{\mathsf{rec}\ f\ g\,[\rho, \mathsf{zero}] \Downarrow n} \qquad \frac{\mathsf{rec}\ f\ g\,[\rho, m] \Downarrow n \qquad g\,[\rho, n, m] \Downarrow o}{\mathsf{rec}\ f\ g\,[\rho, \mathsf{suc}\ m] \Downarrow o}$$

$$\frac{f\,[\rho, n] \Downarrow 0 \qquad \forall m < n.\ \exists\, k \in \mathbb{N}.\, f\,[\rho, m] \Downarrow 1 + k}{\mathsf{min}\ f\,[\rho] \Downarrow n}$$

$$\frac{}{\mathsf{nil}\,[\rho] \Downarrow^{\star} \mathsf{nil}} \qquad \frac{fs\,[\rho] \Downarrow^{\star} ns \qquad f\,[\rho] \Downarrow n}{fs, f\,[\rho] \Downarrow^{\star} ns, n}$$

The *index* function is defined in the following way (for any set A and natural number n):

$$\begin{aligned}
&index \in A^n \to \{\, i \in \mathbb{N} \mid 0 \le i < n \,\} \to A \\
&index\ (xs, x)\ \mathsf{zero} \quad\ = x \\
&index\ (xs, x)\ (\mathsf{suc}\ i) = index\ xs\ i
\end{aligned}$$

6. Consider the variant of Turing machines that we get if the "move left" instruction (L) is replaced by "stay" (S). The *move* function from the lectures is now defined in the following way:

$$
\begin{aligned}
&\mathit{move} \in \{\mathsf{R}, \mathsf{S}\} \to \mathit{Tape} \to \mathit{Tape} \\
&\mathit{move}\ \mathsf{R}\ (\mathit{ls}, \mathit{rs}) = (\mathit{head}\ \mathit{rs} :: \mathit{ls}, \mathit{tail}\ \mathit{rs}) \\
&\mathit{move}\ \mathsf{S}\ t \qquad\quad\ = t
\end{aligned}
$$

Let us leave all other definitions, including the semantics and the definition of computability, unchanged (except for small, obvious changes to accommodate the switch from L to S).

Prove that there is some Turing-computable partial function $f \in \mathbb{N} \rightharpoonup \mathbb{N}$ that is not computable using this kind of machine.