Simply typed lambda-calculus

Types, context, terms and typing rules

We define:

Types $A, B \ ::= \ \mathsf{N} \mid A \to A$
Contexts $\Gamma \ ::= \ () \mid \Gamma.A$
Terms $t \ := \ n \mid t\ t \mid \lambda\ A\ t \mid \mathsf{S}\ t \mid \mathsf{z}$
Typing rules

$$\frac{}{\Gamma.A \vdash\ 0 : A} \qquad \frac{\Gamma \vdash n : A}{\Gamma.B \vdash\ n+1 : A}$$

$$\frac{\Gamma.A \vdash t : B}{\Gamma \vdash \lambda\ A\ t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$$\frac{}{\Gamma \vdash \mathsf{z} : \mathsf{N}} \qquad \frac{\Gamma \vdash t : \mathsf{N}}{\Gamma \vdash \mathsf{S}\ t : \mathsf{N}}$$

Closures and evaluation

We define:

Closures $c \ ::= \ t\rho \mid c\ c \mid \mathsf{z} \mid \mathsf{S}\ c$
Environment $\rho \ ::= \ () \mid \rho, c$
Values $v \ ::= \ \mathsf{z} \mid \mathsf{S}\ v \mid (\lambda\ A\ t)\rho$
Evaluation

$$\frac{}{0(\rho, c) \mapsto c} \qquad \frac{n\rho \mapsto c}{(n+1)(\rho, c') \mapsto c}$$

$$\frac{}{(\lambda\ A\ t)\rho\ c \mapsto t(\rho, c)} \qquad \frac{c_0 \mapsto c_0'}{c_0\ c_1 \mapsto c_0'\ c_1}$$

$$\frac{}{(t_0\ t_1)\rho \mapsto (t_0\rho)\ (t_1\rho)}$$

$$\frac{}{\mathsf{z}\rho \mapsto \mathsf{z}} \qquad \frac{c \mapsto c'}{\mathsf{S}\ c \mapsto \mathsf{S}\ c'} \qquad \frac{}{(\mathsf{S}\ t)\rho \mapsto \mathsf{S}\ (t\rho)}$$

Typing rules for closures and main theorems

$$\frac{}{\mathsf{z} : \mathsf{N}} \qquad \frac{c : \mathsf{N}}{\mathsf{S}\ c : \mathsf{N}}$$

$$\frac{c_0 : A \to B \qquad c_1 : A}{c_0\ c_1 : B} \qquad \frac{\Gamma \vdash t : A \qquad \rho : \Gamma}{t\rho : A}$$

$$\frac{}{() : ()} \qquad \frac{\rho : \Gamma \qquad c : A}{(\rho, c) : \Gamma.A}$$

Theorem 1: *If $c : A$ then c is a value or $\exists c'\ (c \mapsto c')$*

Theorem 2: *If $c : A$ and $c \mapsto c'$ then $c' : A$*

Normalization Theorem

We define $R_A(c)$ by induction on A

$R_{\mathsf{N}}(c)$ is $\exists v\ (c \mapsto^* v)$
$R_{A\to B}(c)$ is $\forall c' : A\ (R_A(c') \to R_B(c\ c'))$

Lemma 1: *If* $c \mapsto c'$ *and* $c : A$ *and* $R_A(c')$ *then* $R_A(c)$

We define $R_\Gamma(\rho)$

$$\frac{}{R_{()}()} \qquad \frac{R_\Gamma(\rho) \qquad c : A}{R_{\Gamma.A}(\rho, c)}$$

Lemma 2: *If* $\Gamma \vdash t : A$ *and* $R_\Gamma(\rho)$ *then* $R_A(t\rho)$. *If* $c : A$ *then* $R_A(c)$.

It follows that we have.

Theorem: *If* $c : \mathsf{N}$ *then* $\exists v\ (c \mapsto^* v)$.

A small term with a large value

We can define $\mathsf{exp}\ A = A \to A$ and the term $\mathsf{twice}\ A : \mathsf{exp}\ (\mathsf{exp}\ A) = \lambda(\mathsf{exp}\ A)\lambda A\ 1\ (1\ 0)$

It is possible then to define $\mathsf{twice}_n = \mathsf{twice}\ (\mathsf{exp}^n\ \mathsf{N})$ and the term

$$t = (((\ldots((\mathsf{twice}_n\ \mathsf{twice}_{n-1})\ \mathsf{twice}_{n-2})\ldots)\ \mathsf{twice}_0)\ \mathsf{S})\ \mathsf{z}$$

is then of type $t : \mathsf{N}$. By the Theorem, there exists v such that $t \mapsto^* v$. However v is of the form $\mathsf{S}^k\ \mathsf{z}$ where k is a tower of n exponentials $k = 2^{2^{2^{\cdots}}}$.