Logical rules for natural deduction

We describe when $\Gamma \vdash \psi$, i.e. ψ is derivable from a finite set $\Gamma = \psi_1, \ldots, \psi_n$ by the following rules.

$$\frac{\psi \in \Gamma}{\Gamma \vdash \psi}$$

$$\frac{\Gamma, \psi \vdash \varphi}{\Gamma \vdash \psi \to \varphi} \qquad \frac{\Gamma \vdash \psi \to \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi}$$

$$\frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \psi} \qquad \frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi \wedge \varphi}$$

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \psi \vee \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi} \qquad \frac{\Gamma \vdash \psi \vee \varphi \qquad \Gamma, \psi \vdash \delta \qquad \Gamma, \varphi \vdash \delta}{\Gamma \vdash \delta}$$

$$\frac{\Gamma, \psi \vdash \bot}{\Gamma \vdash \neg\psi} \qquad \frac{\Gamma \vdash \neg\psi \qquad \Gamma \vdash \psi}{\Gamma \vdash \bot}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \psi(x_0/x)}{\Gamma \vdash \forall x\ \psi} \qquad \frac{\Gamma \vdash \forall x\ \psi}{\Gamma \vdash \psi(t/x)}$$

$$\frac{\Gamma \vdash \psi(t/x)}{\Gamma \vdash \exists x\ \psi} \qquad \frac{\Gamma \vdash \exists x\ \psi \qquad \Gamma, \psi(x_0/x) \vdash \delta}{\Gamma \vdash \delta}$$

In the rule of $\forall$ introduction x_0 should not occur free in the conclusion. This was essentially the rule found by Frege (1879).

In the rule of $\exists$ elimination x_0 should not occur free in Γ and δ and $\exists x\ \psi$.

The rules for equality are.

$$\frac{}{\Gamma \vdash t = t} \qquad \frac{\Gamma \vdash t = u \qquad \Gamma \vdash \psi(t/x)}{\Gamma \vdash \psi(u/x)}$$

We write $\vdash \psi$ for $\Gamma \vdash \psi$ if Γ is empty.

The following is a valid derivation: we have $\vdash x_0 = x_0$ hence $\vdash \exists x\ (x = x)$. It corresponds to the fact that we want to describe the logic of *non empty* universes.