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Axiomatic Discrete Geometry

by

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Abstract

The main approaches to image analysis and manipulation, computational geometry, and related fields are based on continuous geometry. This easily leads to trouble with rounding errors and algorithms that return erroneous output, or even fail to terminate gracefully. In view of this we can argue that the proper framework for many algorithms is not continuous, but discrete. Furthermore it is preferable if such a framework is axiomatically defined, so that the essential properties of the system are clearly stated and many models can share the same theory.

In this report we analyse Hübler's axiomatic discrete geometry, one of the few of its kind—perhaps the only one. The system is characterised in terms of torsion free $\mathbb{Z}$-modules satisfying some so-called generator properties. The new axiomatisation obtained is arguably easier to understand than the original one, and the work casts new light on different properties of Hübler's geometries. His system turns out to be too restricted for our purposes, but the results indicate some ways in which to continue this thread of work.

In the process of doing the characterisation of Hübler's system it is shown that all modules over integral domains have a natural (possibly infinite) matroid structure. This was already known, but not well-known, and in this text some of the consequences are examined. Furthermore it is shown how to define a natural oriented matroid structure over all modules over ordered domains, and all torsion free modules over ordered domains are shown to be antimatroids under a certain closure operator. Convexity is examined in relation to both oriented matroids and antimatroids.

Stolfi's oriented projective geometry, which is used in practice, is also treated in the text. The goal is to find a good axiomatisation of oriented projective geometry that has useful discrete models. Two axiomatisations, both in terms of infinite modular oriented matroids, are proposed. It is shown that Stolfi's geometries are models of one of the systems. This part of the report should even more than the rest be seen as a starting point for further investigations.

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A big hug goes to Maria Alsterberg for letting me spend one year away from her, thereby enabling the production of this work.

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Chapter 1

Introduction

1.1 Background

This report is mainly about axiomatic discrete geometry. The main intended application of this subject is image analysis and manipulation, computational geometry, and related fields. Currently the most commonly used approaches to these areas are continuous instead of discrete. Continuous approaches are plagued by the inherent finiteness of computing hardware, though. Disregarding this aspect easily leads to trouble with rounding errors, and hence algorithms who return erroneous output, or even fail to terminate gracefully.

These problems are approached in different ways [Sch00]. One method is to simulate exact real arithmetic with arbitrary precision libraries. This may have performance penalties, especially in cascaded computations where a computed result is the input to another computation. Another technique is to use inexact computation but design the algorithms to be "robust," so that the computed answer is always useful in some way, even though it may not be the correct one. Of course this may not always be appropriate, although there are clearly situations in which a quick, relatively good answer is more important than a correct one. Schirra sums up the situation by writing "Over the past decade much progress has been made on the precision and robust problem, but no satisfactory general-purpose solution has been found" [Sch00].

One of the problems with these approaches are that they more or less ignore that real number comparison, membership of a Euclidean point in a set, etc. are not computable. Using a model which takes these limitations into account, such as the computable solid modelling of Edalat and Lieutier [EL02, ELK01], probably makes it easier to construct provably correct algorithms.

The methods listed above are all based on continuous (such as Euclidean) geometry. In some cases it may be more appropriate to have a discrete framework to work in. Just as Edalat and Lieutier's system takes the inherent limitations of their algorithms' working environment into account, there are situations where the environment is discrete. One example is in image manipulation, where the input consists of discrete pixels. This is the traditional setting for discrete geometry, but perhaps it can also be of use in computational geometry. This would for example be the case if the discrete approach could lead to provable sharp bounds for the time and space behaviour of an algorithm, something which is useful in situations with limited resources, such as computations on embedded hardware. Provable bounds would also be of use to those having a strict finitist viewpoint, in which it is not enough to prove that an algorithm terminates in bounded time (the finitary approach); the computational resources may well be exhausted before the algorithm has terminated [Ben02]. However, our discussions do not concern finite structures, only finitary.

The reason for having an *axiomatic* discrete geometry is that a good axiomatisation picks out just those basic properties that are needed to be able to prove useful results and construct good algorithms. This ensures that for each model sharing these basic properties one can take advantage of the theorems and algorithms developed. An example from computational geometry is Knuth's monograph [Knu92] that presents a small set of axioms which allows the efficient calculation of a planar convex hull. This system has not been generalised to arbitrary dimensions, though. One example of the problems that can arise if a good axiomatisation is not found is given by Kong [Kon01]; in three-dimensional digital topology results have to be proved separately for each "good adjacency relation." A more successful example is given by Herman [Her98]. He describes why fcc grids[1] can be more useful than cubic grids for three-dimensional digital spaces, and his theory works with both kinds of grids. On another note, the usefulness of a geometry is not necessarily limited to a few, low dimensions. Beutelspacher and Rosenbaum give examples of how finite affine and projective geometries of arbitrary dimension can be applied in coding theory and cryptography [BR98]. Hence it can be useful to find an axiomatisation with models of arbitrary dimension.

Before we continue let us point out that discrete geometry is not the same as digital geometry. Digital geometry focuses solely on the "geometric properties of subsets of digital images" [RK02]. Most of the work seems to be geared entirely towards digitisations of Euclidean space, usually in two or three dimensions, and often restricted to the square grid model $\mathbb{Z}^2$ with 4- and 8-connectedness. Of course the treatment of digital images is an important application of discrete geometry, but the hope is that it will be useful for other purposes as well; either for some of the example areas given above or for something else. In other words, discrete geometry is a more general term than digital geometry (at least according to the vague definition employed here).

1.2 Hübler's Geometry, Modules, and Matroids

Given the diverse examples above it is probably time to reveal in more detail what the focus of this report is. We have indicated that discrete geometry may be suitable for use with computers. However, there currently is no established axiomatisation of discrete geometry. The only attempt at such an axiomatisation known to the author is Hübler's work [Hüb89], which is described in relatively high detail in Section 2.6. This axiomatisation is analysed in Chapter 5, where it is completely characterised in terms of torsion free $\mathbb{Z}$-modules with certain so-called generator properties (most of these concepts are introduced in Section 2.5).

[1] Face-centered cubic grids, e.g. the grid consisting of those points of the three-dimensional integer grid which have an even coordinate sum.

This leads to another main topic of the text. To put the modules into a supporting framework it is shown that each module over an integral domain with a certain closure operator is a matroid (possibly infinite). (The basic matroid theory needed for this report is given in Section 2.2.) Matroids capture the essence of independence such as linear independence and affine independence, and matroid theory includes concepts such as bases and dimension. In short, matroid theory provides a good foundation for many branches of geometry in general [FF00], so their usefulness here does not come as a surprise. In Chapter 3 the connection between modules and matroids is examined. It is shown that the structure of the given matroids is closely related to that of the vector space of fractions (see Section 2.5) associated with the module, especially when the module is torsion free.

Even though matroids provide a foundation for geometry there are some important concepts that are missing. Oriented matroids (see Section 2.4) add order and convexity to matroids [BLVS$^+$93, RGZ97]. As an example, Knuth's convexity-related axioms that we mentioned above are equivalent to a certain kind of oriented matroids. Webster makes a case for using oriented matroids as a basis for computational geometry and digital geometry [Web]. In Chapter 4 we show that the matroids obtained from modules over ordered domains can be oriented by defining a half-space structure on the modules.

There is also another approach to abstract convexity, namely using antimatroids (see Section 2.3). In Chapter 4 we show that it is straightforward to define a convex closure operator on a module over an ordered domain such that it satisfies the antimatroid axioms.

Given the theory developed for modules over ordered domains the analysis of Hübler's geometries in Chapter 5 is straightforward. The resulting axiomatisation in terms of modules is arguably easier to understand than Hübler's original one; if nothing else it lends new insight into the geometries.

1.3 Oriented Projective Geometry

Now on to a different but related subject matter. Projective geometry is useful for geometric computations, among other things. There are several reasons for this. (Some of the terms used are explained in Sections 2.2 and 2.7.)

- Many divisions necessary in algorithms based on affine geometry can be avoided by using homogeneous coordinates. (Homogeneous coordinates require somewhat higher memory usage, though.)

- Due to modularity there are less special cases when constructing and implementing algorithms (and proving theoretical results); any two different lines in a plane intersect in a unique point, for instance.

- Duality ensures that every proof is the proof of two theorems; the same applies to algorithms.

These advantages and more are given in Stolfi's book [Sto91]. However, the book also presents some drawbacks to projective geometry. The common theme is the lack of orientability. It is impossible to consistently define which side of a hyperplane a point lies on, what the line segment joining two points should be, what a convex set is, etc., unless we are willing to let go of some important properties of either the projective geometry or the orientation.

Given these drawbacks Stolfi presents his version of *oriented* projective geometry (OPG), briefly introduced in Section 2.7, which resolves the problems. It seems reasonable to do this by adding some more axioms that pinpoint what extra properties a projective geometry needs to be orientable. This is not the way chosen by Stolfi, though. He gives a fixed, canonical model for every dimension, and all other models have to be isomorphic to a canonical one. This is not good seen from the context of this report, since it makes it difficult to construct a discrete OPG. A new theory would be needed, whereas if we had a unifying framework most of the results could probably be reused.

Despite these problems Stolfi's geometries seem to be rather useful. One immediate advantage is that many algorithms already implemented using "real" homogeneous coordinates can be converted to Stolfi's framework just by keeping an extra eye on the order of arguments and coordinates. Such a conversion probably makes it easier to spot bugs in the algorithms and also may make their correctness easier to verify formally.

An example of good use being made of OPG is given by Laveau and Faugeras who use it for hidden surface removal and to reconstruct scenes from multiple camera images. The authors write in their conclusion "We have presented an extension of the usual projective geometric framework which can nicely take into account an information that was previously not used, i.e. the fact that we know that the pixels in an image correspond to points which lie in front of the camera" [LF96]. Another application they consider is the calculation of the three-dimensional convex hull of an object from two images. A previous method constructed by Robert and Faugeras [RF95] did not work in certain cases. By using the full power of OPG this flaw could be fixed.

Since there does not currently seem to exist any good axiomatic foundation of oriented projective geometry this report tries to lay the ground for constructing such a foundation. In Chapter 6 two different possible axiomatisations are proposed. Both are based on oriented matroids that in some sense are projective. It is shown that Stolfi's OPGs are models of one of the systems. The material in Chapter 6 is rather tentative, and should even more than previous chapters be seen as a base for further investigations.

1.4 Limitations

Finally note that there are some aspects of geometry that are not discussed in this report. Some examples are given by metrics, connectedness, and topology. It should be possible to treat these properties within a good discrete geometric framework, but that work is left for others to do. Another area that is missing is the algorithmic aspects. This is somewhat contradictory given the claims above about the usefulness of discrete geometry, but before those claims can be substantiated a solid foundation is needed.

A Note on Previous Presentations of this Work The author has presented a small part of this work at Imperial College as the individual study option Axiomatic Discrete Geometry. Most material reused has been generalised, corrected, or modified in some way, though. Parts of this work have also been presented by the author at the Second Irish Conference on the Mathematical Foundations of Computer Science and Information Technology in Galway, 2002.

Chapter 2

Background Material

This chapter contains background material for the later chapters. Those readers familiar with the subjects treated can skip most of this chapter, but will miss some motivating remarks. Most readers probably will not have met Hübler's geometries, presented in Section 2.6, before. Note that the treatment of matroids, antimatroids, and oriented matroids is somewhat nonstandard, as infinite ground sets are allowed.

Before continuing note that we assume throughout the text that the Axiom of Choice is valid. Otherwise the proof of some important matroid properties, and probably other results as well, would fail.

2.1 Notation and Terminology

A few words on notation and terminology.

When A is defined to be equal to B we sometimes make this explicit by writing $A := B$. We let $A \subseteq_{\text{fin}} B$ be true iff A is a finite subset of B, and $A \subsetneq B$ is equivalent to $A \subseteq B \neq A$. The cardinality of the set A is denoted by $|A|$, the power set by $\wp(A)$, and we let $\wp_{\text{fin}}(A) := \{ B \mid B \subseteq_{\text{fin}} A \}$. When the context allows so the singleton set $\{ x \}$ is often written simply as x. Set difference is always written using $\setminus$, never $-$; this is because $-$ is used for a different purpose. If $f : A \to B$ is a function then, whenever there is no risk of confusion, f also denotes the function $f' : \wp(A) \to \wp(B)$ defined by $f'(P) := \{ f(p) \mid p \in P \}$.

A subset $P \subseteq S$ is *maximal* (*minimal*) with respect to S and some property iff the set has the property and there is not any set $A \subseteq S$ with $P \subsetneq A$ ($A \subsetneq P$) such that A has the property.

The set of naturals (nonnegative integers) is denoted by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, the rational numbers by $\mathbb{Q}$, and the reals by $\mathbb{R}$. The notation X^+, where X is one of the aforementioned sets, stands for all positive elements of X.

2.2 Matroids and Geometries

From the viewpoint of this report matroids capture the essence of independence as found in e.g. linear algebra (linear independence) and affine geometry (affine independence). This general treatment includes concepts such as bases, dimension, etc. Note, however, that the subject of matroids is much larger than this text might indicate.

In this section infinite matroids and certain geometries are briefly introduced. The focus is on the applications used in this report; many aspects of matroid theory are ignored. Any reader who wants more information is referred to a book by Faure and Frölicher [FF00] for information about (infinite) matroids and geometries. Most of the information below is taken from that text. A different perspective on infinite matroids is given in a chapter by Oxley [Oxl92a], and Oxley's book [Oxl92b] is a standard text on finite matroids.

As indicated by the references above, there is a real difference between finite and infinite matroids. There are for instance many equivalent axiomatisations of finite matroids. These axiomatisations do not always give rise to equivalent theories when generalised to infinite matroids. As an example, for finite matroids there is an important concept of orthogonality (not treated here). This concept can be generalised to the infinite case in some of these axiomatisations, but not in the one treated below [Oxl92a].

When the term matroid is used alone, it usually refers to a finite matroid. In this report we are mostly dealing with (possibly) infinite matroids, though, so we do it the other way around; the term matroid, when unqualified, stands for an infinite matroid of the kind defined below.

The definition of matroids we have chosen to use here uses closure operators, and is taken from Faure and Frölicher [FF00]. Coppel gives an equivalent definition [Cop98], but calls a matroid an *exchange alignment*. Oxley presents another equivalent definition [Oxl92a], and he uses the terms *independence space* and *finitary matroid*. The following definition also introduces a number of other concepts which are used throughout the text.

Definition 2.1 (closure operator). *A* closure space *is a set M (the* ground set*) together with a function* (closure operator) $\mathrm{cl} : \wp(M) \to \wp(M)$ *satisfying (for all $A, B \subseteq M$)*

1. (increasing) $A \subseteq \mathrm{cl}(A)$,
2. (monotone) *if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and*
3. (idempotent) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

The following two properties are also important.

4. (The exchange property.) *If $y \in \mathrm{cl}(A \cup x) \setminus \mathrm{cl}(A)$ for some $x \in M$, then $x \in \mathrm{cl}(A \cup y)$.*
5. *If $x \in \mathrm{cl}(A)$, then $x \in \mathrm{cl}(A')$ for some $A' \subseteq_{\mathrm{fin}} A$.*

A closure space satisfying the last property is finitary*. A* matroid *is a finitary closure space satisfying the exchange property. A closure space satisfying*

6. $\mathrm{cl}(\emptyset) = \emptyset$ *and*
7. $\mathrm{cl}(x) = \{ x \}$ *for all $x \in M$*

is simple*. A* geometry *is a simple matroid. A* projective geometry *is a geometry that satisfies the* projective law*:*

8. $\mathrm{cl}(A \cup B) = \bigcup \{ \mathrm{cl}(x, y) \mid x \in \mathrm{cl}(A)\,,\ y \in \mathrm{cl}(B) \}$, *where* $A, B \neq \emptyset$.

A matroid satisfying the projective law is called a projective matroid*.*

The listed axioms/properties are independent. We use the same notation M for the matroid (or closure space/geometry/...) as for its underlying set. Sometimes, when we explicitly want to point out which closure operator is used, we use the notation (M, cl).

The difference between this definition of matroids and the one for finite sets, besides the lack of a requirement of M to be finite, is property 5. If the set is finite then the definitions are equivalent. By dropping finiteness we lose some properties, but for our purposes being finitary is a close enough approximation to being finite.

We usually want the elements of a matroid to be the points of a geometry, or perhaps some kind of function on the points. Typical geometries often have infinite point sets, and hence we cannot restrict ourselves to finite matroids without working only with finite subsets of the actual point set. This may not be what we want, e.g. if we want to examine the properties of an infinite line. In our case we will, among other things, relate a matroidal structure to Hübler's geometries (see Section 2.6). These geometries are infinite, so we have to use infinite matroids.

The closed subsets of a matroid are called *subspaces* or *flats*. A subset $A \subseteq M$ is said to *generate* a subspace E if $\mathrm{cl}(A) = E$, and it is *independent* if it satisfies $x \notin \mathrm{cl}(A \setminus x)$ for all $x \in A$. If A is independent and generates E, then it is a basis of E. A basic result is that every subspace has a basis, and that all bases of a subspace are equipotent. A generalisation of this states that given three sets $A \subseteq D \subseteq E \subseteq M$, where E is a subspace, A is independent and D generates E, there exists a basis B of E with $A \subseteq B \subseteq D$. The *rank* $r(E)$ of a subspace E is the cardinality of any of its bases. The rank function can be extended to arbitrary subsets S by defining $r(S) := r\big(\mathrm{cl}(S)\big)$.

Although the wealth of alternative definitions for finite matroids is not entirely carried over to the infinite case, there are still several equivalent axiomatisations. One alternative uses independent sets, another rank functions. We leave to the reader to find out exactly how these different axiomatisations are related. The first axiomatisation is taken from [Oxl92a], the second from [FF00].

Definition 2.2 (independent sets). *A* matroid *is a set M together with a set $\mathcal{I} \subseteq \wp(M)$ of* independent sets *satisfying*

1. $\mathcal{I} \neq \emptyset$,
2. *if* $B \subseteq A \in \mathcal{I}$ *then* $B \in \mathcal{I}$,
3. *if* $A, B \in \mathcal{I}$ *and* $|A| < |B| < \infty$ *then there is some* $x \in B \setminus A$ *such that* $A \cup x \in \mathcal{I}$, *and*
4. *if* $A \subseteq M$ *and* $B \in \mathcal{I}$ *for every* $B \subseteq_{\mathrm{fin}} A$ *then* $A \in \mathcal{I}$.

Definition 2.3 (rank function). *A* matroid *is a set M together with a* rank function $r : \wp_{\mathrm{fin}}(M) \to \mathbb{N}$ *satisfying (for all $A, B \subseteq_{\mathrm{fin}} M$)*

1. $r(A) \leq |A|$,
2. *if* $A \subseteq B$ *then* $r(A) \leq r(B)$, *and*

3. $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

Note that the rank function which we defined earlier is more general. It maps any subset, finite or infinite, to a possibly infinite cardinality.

Now let E be a subspace of M. Take the quotient set M/E consisting of the equivalence classes of the equivalence relation $\sim$, where $\sim$ is defined by $x \sim y$ iff $\mathrm{cl}(E \cup x) = \mathrm{cl}(E \cup y)$ $(x, y \in M \setminus E)$. Let $\pi : M \setminus E \to M/E$ be the canonical projection. Define the closure operator $\mathrm{cl}_{M/E} : \wp(M/E) \to \wp(M/E)$ by

$$\mathrm{cl}_{M/E}(A) = \pi\left(\mathrm{cl}\big(\pi^{-1}(A) \cup E\big) \setminus E\right). \tag{2.1}$$

Taken together with $\mathrm{cl}_{M/E}$ the quotient set is a geometry, the *quotient geometry*. The quotient geometry $M/\mathrm{cl}(\emptyset)$ is called the *canonical geometry*. The lattice of subspaces (introduced below) of M is isomorphic to that of $M/\mathrm{cl}(\emptyset)$.

The *corank* of a subspace $E \subseteq M$ is $\overline{r}(E) := r(M/E)$. The corank satisfies $r(E) + \overline{r}(E) = r(M)$. The matroid M itself has corank 0, and a *hyperplane* is defined as a subspace with corank 1.

The subspaces of a matroid, ordered by inclusion, is a lattice. For those into lattice theory we mention that this lattice is geometric and hence complete, atomistic, coatomistic, meet-continuous, algebraic, upper semimodular, and relatively complemented (see [FF00]). The meet of two subspaces E and F is simply $E \wedge F = E \cap F$, while the join is $E \vee F = \mathrm{cl}(E \cup F)$. For any subspaces E, F we have

$$r(E \wedge F) + r(E \vee F) \leq r(E) + r(F), \tag{2.2}$$

and also the related variants

$$\overline{r}(E) + \overline{r}(F) \leq \overline{r}(E \wedge F) + \overline{r}(E \vee F) \tag{2.3}$$

and

$$\overline{r}(E) + r(E \wedge F) \leq \overline{r}(E \vee F) + r(F). \tag{2.4}$$

In some cases these inequalities can be strengthened to equalities.

Definition 2.4. *A matroid M is of* degree n $(n \in \mathbb{N})$ *if it satisfies any of the following equivalent conditions. (Let E and F be subspaces of M.)*

1. If $r(E \wedge F) \geq n$ then $r(E \wedge F) + r(E \vee F) = r(E) + r(F)$.

2. If $r(E) \geq n$ then M/E is a projective geometry.

3. If $r(E) \geq n$ then the lattice $[E, M] := \{\, N \subseteq M \mid E \subseteq N,\ N \text{ is a subspace} \,\}$ is modular.

(There are many more equivalent characterisations of a matroid of degree n.) The inequalities (2.3) and (2.4) hold with equality whenever $r(E \wedge F) \geq n$, where n is the degree of the matroid in question. A lattice L is *modular* if for any $a, b, c \in L$ with $a \leq c$ we have $a \vee (b \wedge c) = (a \vee b) \wedge c$. A matroid of degree 0 is also called modular. A matroid is modular iff it is projective. Furthermore all matroids of rank $n + 2$ are of degree n, so all matroids of rank 2 or less are modular. Trivially any matroid of degree n is of degree m for an arbitrary $m \geq n$.

In a geometry of degree 1 the subspaces of rank 2 are called *lines*, and the subspaces of rank 3 *planes*. Two lines ℓ_1, ℓ_2 are *parallel* ($\ell_1 \,||\, \ell_2$) iff either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$ and $r(\ell_1 \vee \ell_2) = 3$. An *affine geometry* is a geometry M of degree 1 for which for every line $\ell \subseteq M$ and point $p \in M \setminus \ell$ there is a unique line ℓ', parallel to ℓ, with $p \in \ell'$.

To give another view on projective geometries we give some alternative, equivalent definitions as well.

Definition 2.5. *A* projective geometry *is a set G of* points *together with an operator $\star : G \times G \to \wp(G)$ satisfying, for all $a, b, c, d, p \in G$*

1. *$a \star a = \{ a \}$,*
2. *$a \in b \star a$, and*
3. *if $a \in b \star p$, $p \in c \star d$, and $a \neq c$ then $(a \star c) \cap (b \star d) \neq \emptyset$.*

Definition 2.6. *A* projective geometry *is a set G of* points *together with a* collinearity relation *$\ell \subseteq G \times G \times G$ satisfying, for all $a, b, c, d, p, q \in G$*

1. *$\ell(a, b, a)$,*
2. *if $\ell(a, p, q)$, $\ell(b, p, q)$, and $p \neq q$ then $\ell(a, b, p)$, and*
3. *if $\ell(p, a, b)$ and $\ell(p, c, d)$ then there is some $r \in G$ such that $\ell(r, a, c)$ and $\ell(r, b, d)$.*

Given an operator $\star$ we get a valid collinearity relation by defining $\ell(a, b, c)$ iff $b = c$ or $a \in b \star c$. On the other hand, given a collinearity relation we get a valid $\star$ operator by defining

$$a \star b := \begin{cases} \{ c \in G \,|\, \ell(c, a, b) \}, & a \neq b, \\ \{ a \}, & a = b. \end{cases} \tag{2.5}$$

We can also relate this to the closure operator approach. Given a projective geometry with closure operator cl we get a $\star$ operator by defining $a \star b := \mathrm{cl}(a, b)$. Conversely, given a $\star$ operator we get a projective geometry closure operator by defining $\mathrm{cl}(S)$ to be the smallest subspace containing S, where a subspace is a subset of G closed under $\star$.

2.3 Antimatroids

Antimatroids are related to matroids, as the following definition shows. The definition is taken from Coppel's book [Cop98], but Coppel uses the term *anti-exchange alignment* and reserves the term antimatroid for what we would call a finite antimatriod. Most of the theory below is also taken from this book.

Definition 2.7. *An* antimatroid *is a finitary closure space (M, cl) satisfying the* anti-exchange property*: If $x, y \in M$, $x \neq y$, $S \subseteq M$, and $y \in \mathrm{cl}(S \cup x) \setminus \mathrm{cl}(S)$, then $x \notin \mathrm{cl}(S \cup y)$.*

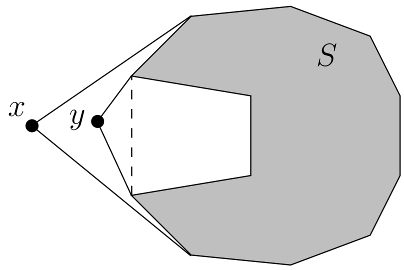

Figure 2.1: Illustration of the anti-exchange property for a planar convex hull operator. The notation used is the same as in Definition 2.7. The point y is not in the convex hull of S, but it is in the convex hull of $S \cup x$. On the other hand, x is not in the convex hull of $S \cup y$. (This illustration is similar to a figure in [Whi92].)

The standard example of an antimatroid is a vector space with the standard convex hull. Figure 2.1 motivates the definition of the anti-exchange property in this context.

Let us finish this section with some simple results about antimatroids, indicating in what way the anti-exchange property is related to convexity. (Note that some results do not depend on the anti-exchange property.) Let S be a subset of the antimatroid M. The point $e \in S$ is an *extreme point of S* if $e \notin \mathrm{cl}(S \setminus e)$. Let $E(S)$ denote the set of all extreme points of S. We have that $E(S) = \bigcap \{ S' \subseteq S \mid \mathrm{cl}(S') = \mathrm{cl}(S) \}$. Independence is defined in the same way as for matroids; S is independent iff $S = E(S)$. We have the following proposition.

Proposition 2.1. *Let (M, cl) be a finitary closure space. Then the following properties are equivalent:*

1. *The anti-exchange property.*

2. *Let $S \subsetneq C \subseteq M$, where C is a closed set. Then S is a maximal closed proper subset of C iff $C \setminus S = \{ e \}$ for some $e \in E(C)$.*

3. *If $S \subseteq M$ then $E(S) = E(\mathrm{cl}(S))$.*

4. *For every closed set $C \subsetneq M$ and point $x \in M \setminus C$ the set $\mathrm{cl}(C \cup x) \setminus x$ is closed.*

5. *If $S \subseteq_{\mathrm{fin}} M$ then $\mathrm{cl}(S) = \mathrm{cl}(E(S))$.*

Let us define bases in the same way as for matroids. We cannot expect the important theorems about bases valid for matroids to be true in this context, since they are based on the exchange property. In fact we have the following result: A closed subset C of an antimatroid has a basis iff $C = \mathrm{cl}(E(C))$. In that case $E(C)$ is the unique basis of C. Given the proposition above we get that all finite closed sets have a unique basis.

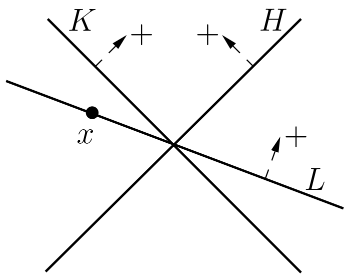

Figure 2.2: The requirement which the hyperplanes and cocircuits of an oriented matroid have to satisfy.

2.4 Oriented Matroids

Oriented matroids add extra structure to the ordinary matroids treated in the last section. From Richter-Gebert and Ziegler [RGZ97] we get the following description: "Roughly speaking, an oriented matroid is a matroid where in addition every basis is equipped with an orientation." As with matroids, the theory of oriented matroids is vast compared to the description given here. The standard reference for oriented matroid theory is the book by Björner et al [BLVS$^+$93]. The previous reference may be easier to digest when all that is needed is a brief introduction, though. The text about finite oriented matroids below is based on these two references.

We define oriented matroids as follows. This definition, which is a straightforward extension of one of the definitions for finite oriented matroids, was suggested by Mike Smyth.

Definition 2.8. *Let M be a matroid where the complement $M \setminus H$ of each hyperplane H is partitioned into two possibly empty sets H^- and H^+, the* negative *and* positive *side of H. The ordered pair (H^-, H^+) is a* cocircuit*. If necessary we can change the* orientation *of the cocircuit, i.e. the* opposite *(H^+, H^-) is also a cocircuit. There are no other cocircuits. The matroid M together with its cocircuits is an* oriented matroid *if the following requirement is satisfied:*

- *Let H and K be two hyperplanes intersecting in a subspace of corank 2 and x a point in $M \setminus (H \cup K)$. If it is possible to choose the orientations of the cocircuits associated with H and K such that $x \in H^+ \cap K^-$ then the hyperplane $L = x \vee (H \wedge K)$ satisfies $L^+ \subseteq H^+ \cup K^+$ and $L^- \subseteq H^- \cup K^-$, given a suitable choice of its orientation.*

The intuition behind the definition can be seen in Figure 2.2.

All the terminology used for ordinary matroids carries over to the oriented case. For instance, we still use the same notation M for both the oriented matroid, its underlying unoriented matroid, and the underlying set of elements. There is a notion of cocircuits for unoriented matroids as well, so the cocircuits above are sometimes called *signed cocircuits*. However, we will only use the term in the signed case, so we use the shorter form.

Given a finite matroid M the definition above gives oriented matroids equivalent to the ones given in Björner et al [BLVS$^+$93, Theorem 3.6.1]. The standard oriented matroid definition assumes the underlying matroid to be finite. As with ordinary matroids, we adopt the convention of omitting the "infinite" qualifier.

As with ordinary matroids there are many equivalent definitions of finite oriented matroids, and some of these do not give rise to equivalent definitions when relaxed to the infinite case. The choice to use the definition above is motivated by the connection between convexity and half-spaces, which we turn to now.

There are two widely used approaches to abstractly/axiomatically characterising convexity. One uses antimatroids as in Section 2.3, the other uses (finite) oriented matroids. Here we try to extend this approach to infinite oriented matroids.

Given a hyperplane H, the sets H^- and H^+ are called *open half-spaces*. The union of an open half-space and the corresponding hyperplane is a *closed half-space*. Let $H_\mathrm{O}(M)$ and $H_\mathrm{C}(M)$ denote all open respectively closed half-spaces in M. We can define a convex hull operator $[\cdot] : \wp(M) \to \wp(M)$ by

$$[S] = \begin{cases} \bigcap \{\, H \in H_\mathrm{C}(M) \mid S \subseteq H \,\}, & S \text{ finite}, \\ \bigcup \{\, [S'] \mid S' \subseteq_\mathrm{fin} S \,\}, & \text{otherwise}. \end{cases} \tag{2.6}$$

Here we use the convention that $\bigcap \emptyset = M$. The reason for having different cases depending on the cardinality of S is that this definition makes $[\cdot]$ finitary.

In the only works about infinite oriented matroids known to the author Buchi and Fenton present a different axiomatisation [BF88, Fen87]. Their system is related to the one (for finite oriented matroids) given by Folkman and Lawrence [FL78], and includes a convex closure operator and an involution (a permutation where all cycles are finite of length at most 2). Its relation to the axiomatisation given above is unknown. To distinguish the two axiomatisations we call these oriented matroids involution-OMs.

Definition 2.9. *An* involution-OM *is a triple* $(M, [\cdot], \star)$ *where* $(M, [\cdot])$ *is a finitary closure space which together with* $\star : M \to M$ *satisfies (for all* $x \in M$*,* $S \subseteq M$*)*

1. $[\emptyset] = \emptyset$,
2. $x^{\star\star} = x$ *and* $x^\star \neq x$,
3. $[S^\star] = [S]^\star$,
4. *if* $x \in [S \cup x^\star]$ *then* $x \in [S]$*, and*
5. *if* $y \in [S \cup x^\star] \setminus [S]$ *and* $y \neq x^\star$ *then* $x \in [(S \cup y^\star) \setminus x]$.

The subsets of M *closed under* $[\cdot]$ *are called* convex. *A convex set closed under* $\star$ *is a* flat.

The two requirements $[\emptyset] = \emptyset$ and $x^\star \neq x$ are not present in the original definitions. All work done in the papers is done under the assumption that both conditions hold, although a claim is made that this assumption is not necessary [BF88, Fen87]. We include the requirements here because it simplifies things and because they are included in [BLVS$^+$93].

The precondition $y \neq x^\star$ in Axiom 5 above is not present in any of the presentations of this axiom system. However, without that requirement there is only one model of the axioms, the one with ground set $\emptyset$. Hence it is natural to believe that this precondition should really be part of the system. All results cited below have been checked to hold also under this modification of the system, except for Carathéodory's theorem (which probably also holds).

Given an involution-OM $(M, [\cdot], \star)$ the *underlying matroid* is defined as (M, cl), where $\mathrm{cl}(S) := [S \cup S^\star]$. By using the result that $y \in [S]$ implies that either $y \in [S \setminus x]$ or $y \in [S \setminus x^\star]$ for any $x, y \in M$, $S \subseteq M$, it is straightforward to show that the underlying matroid is a matroid according to our definition. Note that the subspaces of the matroid are exactly the flats of the involution-OM.

The *rank* $r(S)$ of a subset $S \subseteq M$ is the rank of S in the underlying matroid. The following result, attributed to Carathéodory[1] [Car07], may be of use: Let $x \in M$ and $S \subseteq M$ with $x \in [S]$. Then there is a set $S' \subseteq S$ such that $x \in [S']$ and $|S'| \leq r(S)$.

Since our work using oriented matroids is rather tentative this is all information that is needed to be able to follow the text in later chapters.

2.5 Modules and Rings

This section lists some standard definitions, and a few results, regarding modules and rings. It is assumed that the reader knows the basics about groups, rings, vector spaces, etc. As usual when it comes to this background chapter the text is rather terse. For more information the reader is referred to the book by MacLane and Birkhoff [MB67], from which most of the material is taken. Some material also comes from Taylor [Tay00]; this material may be more easily accessible.

Definition 2.10. *Given a ring $(R, +, \cdot)$ (with multiplicative identity 1) a* (left) R-module *is an abelian group $(G, +)$ together with a scalar multiplication $\times : R \to G \to G$ satisfying, for any $r, r_1, r_2 \in R$ and $g, g_1, g_2 \in G$,*

1. $r(g_1 + g_2) = rg_1 + rg_2$,
2. $(r_1 + r_2)g = r_1 g + r_2 g$,
3. $r_1(r_2 g) = (r_1 r_2)g$, *and*
4. $1g = g$.

Note that the multiplication operators are not written out. Denote the module by the tuple $M = (R, G, \times)$. Below M is often (always) used interchangeably with G. Also note that given an abelian group G we get a $\mathbb{Z}$-module by letting, for $n > 0$, $na = a + \cdots + a$ (n times), $(-n)a = n(-a)$, and $0a = 0$. This is easily seen to be the only $\mathbb{Z}$-module based on G, so abelian groups and $\mathbb{Z}$-modules are essentially the same thing.

A function $f : M_1 \to M_2$ between two R-modules is a *linear transformation* if $f(r_1 m_1 + r_2 m_2) = r_1 f(m_1) + r_2 f(m_2)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M_1$. Two R-modules are *isomorphic* if there exists a bijective linear transformation from one to the other.

We also need the concept of submodules.

[1] Although his article predates matroid theory.

Definition 2.11. *A* submodule *of a module* $(R, G, \times)$ *is a module* $(R, G', \times')$*, where* G' *is an abelian subgroup of* G *and* $\times'$ *is the restriction of* $\times$ *to* G'*.*

Note that the definition above does not guarantee that you get a submodule by taking any abelian subgroup of G; the scalar multiplication has to be closed on G'. A subset of a module is a submodule iff it is nonempty and closed under scalar multiplication and sum.

The modules we are interested in are mainly modules over integral domains and ordered domains.

Definition 2.12. *An* integral domain *is a nontrivial ring* $(R, +, \cdot)$ *(with a multiplicative identity) where the multiplicative semigroup* $(R, \cdot)$ *is commutative and satisfies* $xy \neq 0$ *for all* $x, y \in R \setminus \{ 0 \}$ *(i.e. there are no zero divisors).*

There are two equivalent definitions for ordered rings.

Definition 2.13. *An* ordered ring *is a nontrivial ring* R *with a binary relation* $< \subseteq R \times R$ *satisfying, for all* $a, b, c \in R$*,*

Trichotomy: *exactly one of* $a < b$*,* $a = b$*, and* $a > b$ *holds,*

Transitivity: *if* $a < b$ *and* $b < c$ *then* $a < c$*,*

Additive isotony: *if* $b < c$ *then* $a + b < a + c$*, and*

Multiplicative isotony: *if* $a > 0$ *and* $b < c$ *then* $ab < ac$*.*

Alternative definition: An ordered ring *is a ring* R *with a nonempty subset* $R^+ \subseteq R$ *of* positive elements*, satisfying*

Trichotomy: *for all* $a \in R$ *exactly one of* $a \in R^+$*,* $a = 0$*, and* $-a \in R^+$ *holds, and*

Closure: *if* $a, b \in R^+$*, then* $a + b \in R^+$ *and* $ab \in R^+$*.*

An ordered domain *is an ordered integral domain.*

The definitions are equivalent if we let $a \in R^+$ iff $a > 0$. We use the notation R^+ for the positive elements of an ordered domain R.

Some results are based on the module being torsion free.

Definition 2.14. *An* R*-module* M *is* torsion free *if for any* $r \in R \setminus 0$ *and* $m \in M \setminus 0$ *the product satisfies* $rm \neq 0$*.*

The following three statements are easily seen to be equivalent:

- The R-module M is torsion free,
- $rm = rn$ implies $m = n$ for any $r \in R \setminus \{ 0 \}$ and $m, n \in M$, and
- $rm = sm$ implies $r = s$ for any $r, s \in R$ and $m \in M \setminus \{ 0 \}$.

Hence to say that a module is torsion free is to say that it has a kind of cancellation property. Note that even though all integral domains and groups exhibit different cancellation properties this does not necessarily imply that a module built up from those structures does so. As an example, take the $\mathbb{Z}$-modules over $(\mathbb{Z}_n)^m$, $n \geq 2$, $m \geq 1$. These modules are not torsion free since $n \times (1, 0, \dots, 0) = (0, 0, \dots, 0)$.

Sometimes we will use the notation $\frac{a}{b}$ for some elements a, b of a ring R. This stands for any solutions $r \in R$ of the equation $br = a$. We will also write $\frac{m}{b}$ for some element m of an R-module M. This in turn stands for any solution $n \in M$ of the equation $bn = m$. (In set generators the notation stands for all existing solutions; there may be none.) Due to the ambiguity associated with this notation we will only use it when other forms of notation (such as $\{ n \in M \mid bn = m \}$) are too awkward.

The notation above is related to the field (vector space) of fractions construction, presented here. Given an integral domain[2] R we can define the field of fractions as follows: Define an equivalence relation $\sim$ on all elements in $R \times (R \setminus 0)$ by $(r, s) \sim (r', s')$ iff $rs' = r's$, and treat the equivalence classes as the elements of a new structure. Use the notation $\frac{r}{s}$ for the equivalence class containing (r, s), and identify r with $\frac{r}{1}$ for all $r \in R$. By defining addition in this new structure by $\frac{r}{s} + \frac{r'}{s'} := \frac{rs'+r's}{ss'}$ and multiplication by $\frac{r}{s}\frac{r'}{s'} := \frac{rr'}{ss'}$ we get a field $F(R)$ (field of fractions) with R as a subring. Furthermore $s^{-1} = \frac{1}{s}$ for any $s \in F(R) \setminus 0$.

For a module M over an integral domain R essentially the same procedure works; define an equivalence relation on $M \times (R \setminus 0)$ by $(m, r) \sim (m', r')$ iff there is a ring element $s \in R \setminus 0$ such that $s(r'm - rm') = 0$, and so on. Denote the new $F(R)$-module by $F(M)$ (module, or vector space, of fractions). We do not in general get that M is a submodule of $F(M)$, since M is an R-module. However, for every $F(R)$-submodule $S \subseteq F(M)$ we get that $S \cap M$ is an R-submodule of M, and for every R-submodule $S' \subseteq M$ we get that $F(R)S' := \{ rs \mid r \in F(R),\ s \in S' \}$ is an $F(R)$-submodule of $F(M)$. Furthermore $F(R)S' = F(S')$.

Despite this it may be that M, when treated as a subset of $F(M)$, does not have the same structure as when treated as itself. In fact, $m = m'$ (in M) is equivalent to $m = m'$ (in $F(M)$) iff $F(M)$ is torsion free. If $F(M)$ is not torsion free, then some elements which are different in M become members of the same equivalence class. Hence M retains its own structure in $F(M)$ iff M is torsion free. As an example, the $\mathbb{Z}$-modules over $(\mathbb{Z}_n)^m$ (used in an example above) have an associated vector space of fractions with only one element, 0.

Note that in the vector space $F(M)$ the notation $\frac{m}{r}$ stands for a unique element, but in M the notation is ambiguous. The reason for this is that in the second case the expression stands for any solution $n \in M$ of the equation $rn = m$; this equation may have many solutions, and even if M is torsion free it may have none at all. However, the vector space (or ring) of fractions construction will never be used without comment, so there will not be any great risk of confusion.

For some results we need a different kind of ring.

Definition 2.15. *A* principal ideal domain *(PID) is an integral domain in which every ideal is principal. An* ideal *in a commutative ring is a subset which is closed under addition and multiplication. An ideal I in a ring R is* principal *if there is an element $r \in R$ such that $I = Rr := \{ r'r \mid r' \in R \}$.*

[2]This procedure is possible also with more general rings, but that extra generality is not needed for our purposes.

The ring $\mathbb{Z}$ of integers is a PID, as are all fields.

Let us also introduce some concepts familiar from vector space theory.

Definition 2.16. *Let M be an R-module. A subset $G \subseteq M$* generates *M if $M = \{ \sum_{i=1}^{n} r_i g_i \,|\, n \in \mathbb{N},\ r_i \in R,\ g_i \in G \}$. If M is generated by a finite subset, then M is* finitely generated*. A subset $S \subseteq M$ is* linearly independent *if $\sum_{i=1}^{n} r_i s_i = 0$, $s_i \in S$, $r_i \in R$ implies that $r_i = 0$ for all i, $1 \leq i \leq n$. A linearly independent set that generates M is a* basis *of M. If M has a basis, then it is* free*.*

(The empty set counts as a basis for the module $\{ 0 \}$.)

Definition 2.17. *The* rank *of a module over an integral domain is the dimension of the associated vector space of fractions.*

Let M be a finitely generated, free module over an integral domain. Then every basis of M has the same number of elements, and this number equals the rank of M.

The last few definitions are important since

- every torsion free, finitely generated module over a PID is free,
- every submodule of a free module over a PID is free with rank at most the rank of the module,
- every submodule of a finitely generated module over a PID is finitely generated, and
- all free, finitely generated R-modules are isomorphic to R^n for some $n \in \mathbb{N}$.

Here R^n is the R-module based on the cartesian product of n copies of R, equipped with the same operations as R, applied componentwise.

2.6 Hübler's Discrete Geometry

In this section Hübler's work on discrete geometry [Hüb89] is briefly summarised. Since it is hard to get hold of Hübler's report this section is more detailed than the other sections in this chapter.

2.6.1 Introduction

Hübler's report has three main parts, three approaches to discrete geometry. The first one, totally ignored here, is about so-called digital geometries, and seems to be less abstract than the others. The following two parts, which are summarised here, are about translative neighbourhood graphs and axiomatic discrete geometry.

This text only gives a brief overview of Hübler's report, although it does cover most concepts and important results from the two parts which are treated here. To make the text as compact as possible some parts not considered important for this report are omitted, and at some places a definition or result has been replaced with an equivalent one. The nomenclature is in reasonable correspondence with the one used in Hübler's German report. For exact definitions, proofs etc. Hübler's report has to be consulted.

2.6.2 Discrete Geometry on Translative Neighbourhood Graphs

First Hübler's presentation of neighbourhood graphs is summarised. These structures are not used in other parts of this report (except for some unimportant references), and hence this subsection can be skipped without much loss. However, neighbourhood graphs provide a background setting for Hübler's axiomatic geometries, and furthermore some results in other parts of this report are generalisations of results in this subsection.

Translative Neighbourhood Graphs

A *neighbourhood graph* is a simple, undirected, connected graph with a nonempty node (*point*) set and an edge set with the property that each point has a finite number of *neighbours*. (Two points are each others neighbours if they are connected by an edge.) The *distance* between two points is the length of the shortest path between them.

A *displacement* D is a bijection on the point set of a neighbourhood graph that preserves neighbourhood and has a constant *displacement distance* (i.e. the distance between p and $D(p)$ equals the distance between q and $D(q)$ for any points p and q). The *identity displacement* is denoted by id.

Let $\mathcal{D}$ be the set of all displacements on a neighbourhood graph. The graph is said to be *translative* if

1. $\mathcal{D}$ is closed under composition $\circ$ of displacements,
2. $\circ$ is commutative,
3. for each of a point's neighbours there always exists a displacement that maps the point to the neighbour, and
4. for every displacement D except id, $D^n(p) := (\overbrace{D \circ \ldots \circ D}^{n \text{ times}})(p) \neq p$, for any point p and any $n \in \mathbb{Z}^+$.

The displacements of a translative neighbourhood graph are called translations. Let us use the term t-graph instead of translative neighbourhood graph (these graphs are often denoted by G), and let $\mathcal{P}_G$ be the point set and $\mathcal{T}_G$ the displacement (translation) set of G. The set $\mathcal{T}_G$ is easily seen to be an abelian group under the group operation $\circ$. This is not a sufficient condition for a neighbourhood graph being a t-graph, though. For t-graphs the power notation D^n can be extended to arbitrary $n \in \mathbb{Z}$ in the standard way (i.e., $\mathcal{T}_G$ can be viewed as a $\mathbb{Z}$-module).

A translation with displacement distance 1 is said to be *elementary*. T-graphs have the property that all points have the same number of neighbours (the *neighbourhood degree* of the graph) and that there for each pair of points p, q is exactly one translation that maps p to q. Hence the number of elementary translations is finite. The elementary translations of a t-graph G *generate* $\mathcal{T}_G$, i.e. any translation is equal to a composition of powers of elementary translations. A minimal set of translations that generates G is called a *basis* of G. The *dimension* of a t-graph is the (well-defined) cardinality of its smallest basis. A translation S is *simple* if $S \neq T^n$ for all translations T and all $n \in \mathbb{N} \setminus \{ 1 \}$.

Lines, Parallelity and Convexity

Given a point p and a simple translation S, the associated *line* ℓ is the smallest set that contains p and is closed under S^n for all $n \in \mathbb{Z}$. The translation S is said to be a *generator* of ℓ. The only generators of ℓ are S and S^{-1}, and $\ell = \{ S^n(q) \mid n \in \mathbb{Z} \}$ for all $q \in \ell$. Furthermore, to each pair of distinct points there is exactly one line that contains both points.

Two lines ℓ and ℓ' are said to be *parallel* ($\ell \,||\, \ell'$) if they have the same generators. Two parallel lines either have all or no points in common and two lines are parallel iff there is a translation that maps one of the lines bijectively onto the other. Furthermore, for each line ℓ and point p there is exactly one line ℓ' with $\ell \,||\, \ell'$ and $p \in \ell'$ (compare with the Euclidean parallel axiom).

A betweenness relation B is now introduced: $B(p, q, r)$ holds for three points p, q, and r on a line ℓ if there are positive integers n_1, n_2 with $n_1 < n_2$ and a generator S of ℓ such that $q = S^{n_1}(p)$ and $r = S^{n_2}(p)$. It is easy to check that $B(p, q, r)$ is equivalent to $B(r, q, p)$ and that $B(p, q, r)$ and $B(q, r, s)$ together imply $B(p, q, s)$ and $B(p, r, s)$. A point set P is said to be *convex* if it is closed under B; i.e. if $p, q \in P$ and $B(p, r, q)$ for a point r, then $r \in P$. This convexity definition is in some sense weak, though, for there are convex point sets of t-graphs where the induced subgraph associated with such a set is not connected.

For two-dimensional (2D) t-graphs the convexity definition can be strengthened. For an arbitrary line ℓ with associated generator S in a 2D t-graph G there exists a simple translation T such that $\{ S, T \}$ is a basis of $\mathcal{T}_G$. The sets $H_T^+(\ell) = \{ T^n(p) \mid p \in \ell, n \in \mathbb{N} \}$ and $H_T^-(\ell) = \{ T^{-n}(p) \mid p \in \ell, n \in \mathbb{N} \}$ are called *half planes*. For a given line there are exactly two different half planes, independently of which basis $\{ S, T \}$ is chosen. Furthermore $H_T^+(\ell) \cup H_T^-(\ell) = \mathcal{P}_G$, and $H_T^+(\ell) \cap H_T^-(\ell) = \ell$.

There exist convex sets, contained in 2D t-graphs, which are not expressible as intersections of half planes. A stronger convexity can be defined (in two dimensions): A point set $P \subseteq \mathcal{P}_G$ is *strictly convex* if either it is expressible as an intersection of half planes, or $P = \mathcal{P}_G$. All strictly convex sets are convex. This definition still allows strictly convex sets to have induced subgraphs that are not connected, though.

Isomorphisms

Two t-graphs G_1 and G_2 are said to be *isomorphic* if there exists a bijection φ from $\mathcal{P}_{G_1}$ to $\mathcal{P}_{G_2}$ which is B (betweenness) invariant. (I.e., $B(p, q, r)$ iff $B(\varphi(p), \varphi(q), \varphi(r))$.) The statement that two t-graphs are isomorphic is equivalent to each of the following four statements:

1. There is a bijection φ between points, mapping lines to lines, which is parallelity invariant ($\ell \,||\, \ell'$ iff $\varphi(\ell) \,||\, \varphi(\ell')$).

2. There are a bijection Φ between translations and a bijection φ between points for which $T(p) = q$ iff $\Phi(T)(\varphi(p)) = \varphi(q)$ for all points p and q and all translations T.

3. There is a bijection φ between points which is convexity invariant (P is convex iff $\varphi(P)$ is convex).

4. The two t-graphs have the same dimension.

2.6.3 Axiomatic Discrete Geometry

Let us now turn to Hübler's axiomatisation of discrete geometry.

Basic Axioms and Definitions

Let $\mathcal{P}$ be a set of *points*, and $\mathcal{L} \subseteq \wp(\mathcal{P})$ a nonempty set of *lines*. The first axiom is that for any pair p, q of distinct points there is exactly one line ℓ for which the points *lie on* the line ($p, q \in \ell$). Let $\ell(p, q)$ denote that unique line. The second axiom says that for any line ℓ there exist two different points $p, q \in \ell$ and one point $r \notin \ell$.

The third axiom states that there is an equivalence relation $||$, parallelity, on $\mathcal{L}$ for which for any line ℓ and point p there exists exactly one line ℓ' with $p \in \ell'$ and $\ell \,||\, \ell'$. The corresponding equivalence classes are called *directions*.

A *translation* is defined as a bijection T on $\mathcal{P}$ that either equals the identity bijection id or has the following properties (referred to simply as the first, second, and third translation properties).

1. For all lines ℓ, $T(\ell) := \{\, T(p) \,|\, p \in \ell \,\}$ is a line parallel to ℓ.
2. For all points p, $p \neq T(p)$.
3. The set $\{\, \ell(p, T(p)) \,|\, p \in \mathcal{P} \,\}$ is a direction.

The fourth axiom now states that for any two points p, q there exists a translation T with $T(p) = q$. This translation can be shown to be unique. Another result is that two lines ℓ and ℓ' are parallel iff there exists a translation T such that $T(\ell) = \ell'$. Furthermore, just as with neighbourhood graphs, the set $\mathcal{T}$ of all translations on $\mathcal{P}$ is an abelian group under the group operation composition ($\circ$). Hence $\mathcal{T}$ can be made into a $\mathbb{Z}$-module in the standard way.

Hübler goes on to discuss *cyclic translations*, i.e. translations T for which, for some $n \in \mathbb{Z}^+$, $T^n = \mathsf{id}$. However, the following axioms will have as a result that there are no cyclic translations (except id), so they are not discussed here.

Now let a total[3] order $\leq$ (together with the standard variations $<$, $>$, and $\geq$) be defined on the points of every line. A betweenness relation B is yet again defined; for three different points p, q, and r on a line, $B(p, q, r)$ holds if $p < q < r$ or $r < q < p$. The fifth axiom enforces infinite sets $\mathcal{P}$: For each line ℓ and point $p \in \ell$ there are points $q, r \in \ell$ such that $B(q, p, r)$.

The next, sixth, axiom introduces discreteness: For any two points p and p' there is at most a finite number of points q such that $B(p, q, p')$. This means e.g. that every line is a countably infinite set of points.

The seventh axiom is as follows: Let ℓ_1, ℓ_2, and ℓ_3 be different, parallel lines, and ℓ and ℓ' lines that have points p_i and p'_i, respectively, in common with all the lines ℓ_i, $i \in \{\, 1, 2, 3 \,\}$. Then $B(p_1, p_2, p_3)$ holds iff $B(p'_1, p'_2, p'_3)$. This axiom is the one that rules out cyclic translations.

For each line ℓ there exists a translation G (a *generator*) such that $\ell = \{\, G^n(p) \,|\, n \in \mathbb{Z} \,\}$ for any point $p \in \ell$. For such a triple (ℓ, G, p) the relation $B(G^i(p), G^j(p), G^k(p))$ holds iff $i < j < k$ or $k < j < i$ ($i, j, k \in \mathbb{Z}$). Furthermore, just as for neighbourhood graphs, each line has exactly two generators (G and G^{-1}), and two lines are parallel iff they have the same generators. The

[3] Hübler does not state explicitly that the order is total, but this is probably what he means.

definition of a *simple* translation is just as above (S is simple if $S \neq T^n$ for all translations T and all $n \in \mathbb{N} \setminus \{ 1 \}$), and a result is that a translation is simple iff it generates a line.

Convexity is also defined just as for neighbourhood graphs (a point set is convex if it is closed under B).

Below a notion of betweenness for lines is used. Let ℓ_1, ℓ_2, and ℓ_3 be different, parallel lines. The line ℓ_2 is said to *lie between* ℓ_1 and ℓ_3 ($B(\ell_1, \ell_2, \ell_3)$) if there exist a translation T and $i, j \in \mathbb{Z}^+$ such that $T^i(\ell_1) = \ell_2$ and $T^j(\ell_2) = \ell_3$. Some properties which are easily seen to hold in the point case also hold for lines; at most one of $B(\ell_1, \ell_2, \ell_3)$, $B(\ell_1, \ell_3, \ell_2)$, and $B(\ell_2, \ell_1, \ell_3)$ can hold at once, and if $B(\ell_1, \ell_2, \ell_3)$ and $B(\ell_2, \ell_3, \ell_4)$ hold, then $B(\ell_1, \ell_2, \ell_4)$ and $B(\ell_1, \ell_3, \ell_4)$ also hold.

Planar Grids

Let T_1 and T_2 be translations with different directions, and p an arbitrary point. The set $\mathsf{PG}(p, T_1, T_2) := \{ \big(T_1^i \circ T_2^j\big)(p) \mid i, j \in \mathbb{Z} \}$ is the *planar grid spanned by* p*,* T_1*, and* T_2. A *coordinate system* can be defined for such a planar grid by associating each point with the unique pair (i, j) that "generates" the point. All the axioms presented so far hold for any planar grid, if

1. $\mathcal{P}$ is taken to be the points of the planar grid,
2. $\mathcal{L}$ the lines that have at least two points in the grid,
3. $\mathcal{T}$ the restriction to the grid of those translations which are closed on the grid,
4. for each line in the grid, $<$ is taken to be the restriction to the grid of $<$ for the corresponding original line, and
5. $||$ is restricted to the lines of the grid.

Two planar grids are said to be isomorphic if there exists a bijection between their point sets which is betweenness invariant, just as was the case with t-graphs. A result of this is that all planar grids are isomorphic to each other.

A *planar set* is a point set S

1. whose points do not all belong to one line, and
2. for which for any four, different points $p_i \in S$, $i \in \{ 1, 2, 3, 4 \}$, one has for the lines ℓ_i, $i \in \{ 1, 2, 3 \}$ with $\ell_1 = \ell(p_1, p_2)$, $\ell_2 \,||\, \ell_1$ with $p_3 \in \ell_2$, and $\ell_3 \,||\, \ell_1$ with $p_4 \in \ell_3$ that if the lines are different, then one of the lines lies between the other two lines.

Each planar grid is a planar set. A *plane* is defined to be a Hübler-maximal planar set. (A subset $P \subseteq S$ is *Hübler-maximal* with respect to S and some property iff the set has the property and there is not any $x \in S \setminus P$ such that $P \cup x$ has the property.) For each planar set S there is exactly one plane P with $S \subseteq P$. Furthermore $\ell(p, q) \subseteq P$ holds for any two different points p and q in a plane P, and $P_1 \cap P_2 = \ell(p, q)$ holds for two different planes P_1 and P_2 whose intersection contains at least two different points p and q. Also, by using restrictions to the plane in a way analogous to the one presented for planar grids, one has that all the axioms presented so far hold for a plane.

Let p and q be two points in a plane P with a line ℓ, for which $p, q \notin \ell$. The points are said to *lie on the same side of* ℓ ($p \leftrightarrow_\ell q$) if the lines ℓ_1 and ℓ_2 with $\ell_1 \,||\, \ell$, $p \in \ell_1$ and $\ell_2 \,||\, \ell$, $q \in \ell_2$ satisfy $\ell_1 = \ell_2$, $B(\ell, \ell_1, \ell_2)$, or $B(\ell, \ell_2, \ell_1)$. The relation $\leftrightarrow_\ell$ is an equivalence relation on $P \setminus \ell$ with exactly two associated equivalence classes. These two classes are called the *open half planes induced by* ℓ (and P). The union of an open half plane and the line ℓ is called a *closed half plane*.

Note that three non-collinear points define three different half planes (open or closed); the points are contained in exactly one plane, for any pair of points we get a line, the plane and the line induces two half planes, and we can choose the one containing the remaining point. A subset M of a plane P is *bounded* if there are three non-collinear points in P such that M is a subset of the intersection of the three closed half planes defined by the points as above.

All half planes, open or closed, are convex. There are convex sets that are not expressible as intersections of half planes. Hence a stronger form of convexity is yet again defined; a subset S of a plane P is *strictly convex* if $S = P$ or S is expressible as an intersection of half planes. Each strictly convex set is convex.

The Eighth Axiom and a Uniqueness Result

Hübler demonstrates a model of the first seven axioms which is embedded in the real plane; a point set is given and all other concepts are given by the restriction of the corresponding real concept to the point set. Hübler claims that this model has the property that for each point p in the real plane there are points of the model that are arbitrarily close to p (using the standard Euclidean metric). Another property of the model is that for two different, parallel lines ℓ and ℓ' there is an infinite number of lines ℓ'' which satisfy $B(\ell, \ell'', \ell')$.

By closer inspection the model turns out not to be a model (see Appendix A), but it nevertheless explains why Hübler introduced an eighth axiom. This axiom states that there is only a finite number of lines between two parallel lines. The eighth axiom actually makes axiom six, the other discreteness axiom, unnecessary, because it can be deduced from the other axioms. The independence of the remaining axioms does not seem to have been investigated.

The eighth axiom holds for all planar grids, and given the eighth axiom each plane is a planar grid. Hence all planes are isomorphic to each other. Furthermore the axiom ensures that all bounded subsets of a plane are finite.

The axiom system is consistent; there are models satisfying all the axioms. More specifically, the axioms hold for any two-dimensional t-graph. These t-graphs are isomorphic to any plane.

A discrete image geometry is a collection of a point set $\mathcal{P}$, a line set $\mathcal{L}$, a parallelity relation $||$, and a set of total orders $<$ for each line $\ell \in \mathcal{L}$, such that all the axioms are satisfied. Such a geometry is uniquely defined by $\mathcal{P}$ and the set $\mathcal{T}$ of all translations on $\mathcal{P}$.

Since "discrete image geometry" is not a very descriptive term, at least when it comes to distinguishing between different approaches to discrete geometry, we will use the term *Hübler geometry* instead.

2.7 Oriented Projective Geometry

This section presents an extremely brief outline of Stolfi's oriented projective geometry (OPG). The main source of information for the text below is Stolfi's book [Sto91], but for those with less time there is also an extended abstract [Sto87]. Here we only give enough material to be able to put forward the results in later chapters, so those who want to really understand what this is all about are referred to Stolfi's work. Note also that ordinary, axiomatic projective geometry is introduced in Section 2.2.

A standard model for the real projective space of dimension n is, using the notation of Section 2.2, $\mathbb{R}^{n+1}/\mathrm{cl}(\emptyset)$. Here cl is linear, or vector subspace, closure. The difference between these models and Stolfi's OPG is that his spaces are "two-sided" (explained below). Stolfi defines several different isomorphic standard models for each dimension, and all models have to be isomorphic to a standard one. The model presented below is the "analytic" one.

The *analytic model* of dimension n, $n \in \mathbb{N}$, has the quotient set $M := \big(\mathbb{R}^{n+1} \setminus 0\big)/\!\sim$ as point set. Here $\sim$ is the equivalence relation with $x \sim y$ iff there is some $r \in \mathbb{R}^+$ such that $x = ry$. Note that the quotient geometry $M' := \mathbb{R}^{n+1}/\mathrm{cl}(\emptyset)$ (as given above) uses a similar equivalence relation; the difference is that r can be negative as well. Hence where the ordinary projective geometry has one point the OPG has two; this is the two-sidedness alluded to above.

For brevity we call the projective geometries defined here *Stolfi OPGs*. We use the same notation, M, for the points of the geometry as for the geometry itself. We denote the equivalence class including a point $x \in R^{n+1} \setminus 0$ by $[\![x]\!]$. Sometimes we use the notation $[\![0]\!]$ as well. This is not a point of the geometry, but in some formal calculations the value is still needed. The coordinates of x in some basis of R^{n+1} constitute an example of *homogeneous coordinates* for $[\![x]\!]$. Note that these coordinates are determined only up to a positive scalar.

The next step is to define the *subspaces/flats*. Every flat S of M' corresponds to two flats S^+, S^- of M. These two flats have the same set of points but different orientations. Hence a flat in this geometry is not determined solely by the points it contains. The other information is given by an ordered basis of the flat. Two flats are equal if they have the same set of points and their bases are related by a matrix with positive determinant (in the usual vector space sense).

Join and *meet* of flats can be defined similarly to the unoriented case, but since orientation is taken into account the operators are sometimes anticommutative. Furthermore they are undefined in certain degenerate situations. Of course, if the orientation information is discarded from the flats the underlying projective geometry's join and meet can be used.

Projective maps are defined as certain orientation preserving functions between the points of two flats. A Stolfi OPG is uniquely defined by its flats, projective maps, meet, and join. Given this fact *isomorphism* between Stolfi OPGs is defined in a natural way.

Every Stolfi OPG has a *dual*, isomorphic to the original geometry, that is obtained by swapping its meet and join. Hence a *Principle of Duality* holds in Stolfi's framework. This principle states that if a theorem can be proved within the framework, then the dual theorem can also be proved. The dual theorem formulation is obtained by swapping each concept with its dual; meet with join, rank with corank, and so on. This concept can also be applied to algorithms;

one implementation essentially does two different things. There is a principle like this for many unoriented projective geometries as well, see e.g. Csikós [Csi]. We do not treat duality in Section 2.2 since no suitable theory of duality seems to have been worked out for the general projective geometries considered there.

Convexity is defined for Stolfi OPGs in such a way that a subset $S \subseteq M$ is convex iff $[\![x+y]\!] \in S$ for all independent points $[\![x]\!], [\![y]\!] \in S$. Two points $[\![x]\!], [\![y]\!] \in M$ are *independent* if they are not equal or *antipodal*, i.e. if $[\![x]\!] \neq [\![\pm y]\!]$. Every subset S of a Stolfi OPG is contained in a unique minimal convex set. This is the *convex hull* of S.

Chapter 3

Matroids from Modules

This chapter explores some closure operators defined on modules over integral domains and the associated matroids and geometries.

The effect of a closure operator is determined by its closed sets (since every set is mapped to the smallest closed set containing it; this set has to be unique). For a vector space you get a matroid by choosing the vector subspaces as closed sets. Choosing the affine subspaces also yields a matroid, in fact an affine geometry (naturally).

This approach does not in general work for modules, as we show below. The submodules do not always yield a matroid. However, we can simulate vector subspaces by only choosing those submodules which are "closed under existing divisors." In that way we get a matroid. Similarly we can also simulate affine subspaces and get a geometry. We do not in general get an affine geometry, however, since two lines in a plane may cross without intersecting in a discrete setting.

Modules over integral domains are embedded in an associated vector space. The matroids constructed from modules in this chapter turn out to be very similar to the matroids constructed from the corresponding vector spaces. As an example, their respective lattices of subspaces are isomorphic.

3.1 Submodule Closure

Let us first show that the submodules of a module cannot in general make up the subspaces of a matroid.

Lemma 3.1. *Let $M = (R, G, \times)$ be a module and let $\langle\cdot\rangle_{\mathrm{s}} : \wp(M) \to \wp(M)$ take any subset to the smallest submodule containing it. Then $\langle\cdot\rangle_{\mathrm{s}}$ is a well-defined closure operator with the explicit characterisation*

$$\langle S\rangle_{\mathrm{s}} = \left\{ \sum_{i=1}^{n} a_i s_i \,\middle|\, a_i \in R,\ s_i \in S,\ n \in \mathbb{N} \right\}.$$

(The empty sum $\sum_{i=1}^{0} a_i s_i$ is of course interpreted as 0.)

Proof. To show that $\langle\cdot\rangle_{\mathrm{s}}$ is well-defined we have to show that every subset is contained in a unique submodule.

Denote the right hand side of the equation by A. Any submodule containing S has to contain A since all submodules are closed under scalar multiplication and sum and they all contain 0. (The last remark is necessary since we allow $n = 0$.) Furthermore A, by exhibiting the properties just listed, is a submodule and hence the operator is well-defined and $\langle S \rangle_{\mathrm{s}} = A$. By construction the operator satisfies the closure operator axioms. $\square$

Let us now consider the $\mathbb{Z}$-module over $\mathbb{Z}$. Define $n\mathbb{Z} := \{\, nm \mid m \in \mathbb{Z} \,\}$. Observe that $2 \in \langle 10, 3 \rangle_{\mathrm{s}} = \mathbb{Z}$ (since $1 \in \langle 10, 3 \rangle_{\mathrm{s}}$), $2 \notin \langle 10 \rangle_{\mathrm{s}} = 10\mathbb{Z}$, and $3 \notin \langle 10, 2 \rangle_{\mathrm{s}} = \{\, 10m + 2n \mid m, n \in \mathbb{Z} \,\} = 2\,\{\, 5m + n \mid m, n \in \mathbb{Z} \,\} = 2\mathbb{Z}$. Hence the exchange property does not hold, and $\langle \cdot \rangle_{\mathrm{s}}$ is not a matroidal closure operator.

3.2 D-submodule Closure

Given the previous section we know that we cannot (in general) use submodules as subspaces of a matroid. However, by restricting ourselves to d-submodules and modules over integral domains we get a matroid.

Definition 3.1. *A* d-submodule *of a module $M = (R, G, \times)$ is a submodule $S = (R, G', \times')$ with the property that if $rm \in S$ for any $r \in R \setminus \{\, 0 \,\}$ and $m \in M$, then $m \in S$.*

We say that a d-submodule is closed under existing divisors. Thus it is easy to see, intuitively, why this approach works; d-submodules emulate vector subspaces. In Section 3.7 below we formalise this statement.

Theorem 3.2. *Let $M = (R, G, \times)$ be a module over an integral domain and let $\langle \cdot \rangle_{\mathrm{d}} : \wp(M) \to \wp(M)$ take any subset to the smallest d-submodule containing it. Then $\langle \cdot \rangle_{\mathrm{d}}$ is a well-defined matroidal closure operator with the explicit characterisation*

$$\langle S \rangle_{\mathrm{d}} = \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i,\ \ s_i \in S,\ \ a_i, b \in R,\ \ b \neq 0,\ \ n \in \mathbb{N} \right\}.$$

Proof. Compare with the proof of Lemma 3.1. Denote the right hand side of the equation by A. Any d-submodule containing S has to contain A since all d-submodules are closed under scalar multiplication, sum, and existing divisors, and they all contain 0. (The last remark is necessary since we allow $n = 0$.)

Recall that A is a submodule iff it is nonempty and closed under scalar multiplication and sum. If it is also closed under existing divisors then it is a d-submodule.

Nonempty. Because the empty sum is 0 and $b0 = 0$ for any $b \in R$ we have that A is nonempty.

Closed under $\times$. Assume that $m \in A$. Then $bm = \sum_{i=1}^{n} a_i s_i$, $b \neq 0$. By multiplying this expression with $r \in R$, using the commutativity of the integral domain multiplication and the different properties of $\times$, we get $b(rm) = \sum_{i=1}^{n} (ra_i) s_i$. Thus $rm \in A$.

Closed under sum. Let $m, m' \in A$. Then $bm = \sum_{i=1}^{n} a_i s_i$ and $b'm' = \sum_{i=1}^{n'} a'_i s'_i$, $b, b' \neq 0$. By the commutativity of the integral domain and the properties of $\times$ we have $(bb')m = \sum_{i=1}^{n}(b'a_i)s_i$, $(bb')m' = \sum_{i=1}^{n'}(ba'_i)s'_i$, and thus $bb'(m+m') = \sum_{i=1}^{n}(b'a_i)s_i + \sum_{i=1}^{n'}(ba'_i)s'_i$. Because R is an integral domain and $b, b' \neq 0$ we have $bb' \neq 0$, and thus $m + m' \in A$.

Closed under existing divisors. Assume that $rm \in A$, $r \in R \setminus \{0\}$, $m \in M$. Then $b(rm) = \sum_{i=1}^{n} a_i s_i$, $b \neq 0$. By a property of $\times$ we have that $b(rm) = (br)m$, and because $b, r \neq 0$ we have that $br \neq 0$. Thus $m \in A$.

Hence A is a d-submodule, and thus it is the smallest d-submodule containing S, so $\langle S \rangle_{\mathrm{d}} = A$. This means that $\langle \cdot \rangle_{\mathrm{d}}$ is well-defined, and thus by construction all the closure operator axioms hold.

For the exchange property we use the explicit characterisation of $\langle \cdot \rangle_{\mathrm{d}}$. Take any $y \in \langle S \cup x \rangle_{\mathrm{d}} \setminus \langle S \rangle_{\mathrm{d}}$. Then $by = \sum_{i=1}^{n} a_i s_i + ax$ for some $a, b, a_i \in R$, $b \neq 0$, $s_i \in S$, and $n \in \mathbb{N}$. Furthermore $a \neq 0$, because otherwise $y \in \langle S \rangle_{\mathrm{d}}$. Thus we have $ax = \sum_{i=1}^{n}(-a_i)s_i + by$ where $a \neq 0$, and hence $x \in \langle S \cup y \rangle_{\mathrm{d}}$. This means that the fourth axiom is satisfied.

To show that $\langle \cdot \rangle_{\mathrm{d}}$ is finitary, assume that $x \in \langle S \rangle_{\mathrm{d}}$. Then $bx = \sum_{i=1}^{n} a_i s_i$ as usual, and we have that $x \in \langle S' \rangle_{\mathrm{d}}$, where $S' := \{ s_i \mid i \in \mathbb{N},\ 1 \leq i \leq n \}$ is a finite subset of S. $\square$

We note immediately that the matroid obtained from $\langle \cdot \rangle_{\mathrm{d}}$ is not simple, since $\langle \emptyset \rangle_{\mathrm{d}} = \{ 0 \}$. Furthermore all subspaces contain 0, which ensures that they cannot be interpreted as affine lines, planes, etc. Because of this we introduce a-submodules in the next section.

3.3 A-submodule Closure

To get something reminiscent of an affine geometry we define a-submodules. (The term stems from affine submodule, but since the resulting geometry is not in general affine the full name is not used.)

Definition 3.2. *An* a-submodule *A of a module M is a subset of the form $A = D + m$ where $D \subseteq M$ is a d-submodule and $m \in M$ is any element.*

Addition of an element to a set is defined in the obvious way as $D + m := \{ d + m \mid d \in D \}$. Subtraction of an element from a set is defined analogously. Do not confuse this subtraction with set difference, which we always write using $\setminus$.

Lemma 3.3. *Let D be a d-submodule with $m \in D$. Then $D + m = D$.*

Proof. We have $k \in D \Leftrightarrow k - m \in D \Leftrightarrow k \in D + m$. $\square$

Lemma 3.4. *Let A be an a-submodule. Then for any $m \in A$ the set $A - m$ is a d-submodule, and all d-submodules obtained from A in this way are equal.*

Proof. By the definition of a-submodule we know that $A = D + n$ for some d-submodule D and element $n \in M$. Since $D = A - n$ we have $m - n \in D$, and thus, since D is a submodule, also $n - m = -(m - n) \in D$. Hence $D = D + n - m$ or $A - m = D$, so $A - m$ is a d-submodule, and all the obtainable d-submodules are equal to D. $\square$

Corollary 3.5. *A is an a-submodule with $a \in A$ iff $A - a$ is a d-submodule.*

Yet again we define a closure operator, give its explicit representation and prove that the operator is well-defined.

Theorem 3.6. *Let M be an R-module, where R is an integral domain, and let $\langle\cdot\rangle_{\mathrm{a}} : \wp(M) \to \wp(M)$ take any nonempty subset to the smallest a-submodule containing it and $\emptyset$ to $\emptyset$. Then $\langle\cdot\rangle_{\mathrm{a}}$ is a well-defined matroidal closure operator with the explicit characterisation*

$$\langle S\rangle_{\mathrm{a}} = \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i,\ s_i \in S,\ a_i, b \in R,\ b = \sum_{i=1}^{n} a_i \neq 0,\ n \in \mathbb{Z}^+ \right\}.$$

Furthermore, for any $s \in \langle S\rangle_{\mathrm{a}}$, $\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{d}} + s$.

Proof. Compare with the proofs of Lemma 3.1 and Theorem 3.2. Denote the right hand side of the equation by A. When $S = \emptyset$ we have that $\langle S\rangle_{\mathrm{a}} = \emptyset = A$, so assume that $S \neq \emptyset$. Take any $s \in A$. Notice that (leaving out all the side conditions)

$$\begin{aligned} A - s &= \left\{ m - s \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i \right\} \\ &= \left\{ m \;\middle|\; b(m+s) = \sum_{i=1}^{n} a_i s_i \right\} \\ &= \left\{ m \;\middle|\; bm = \sum_{i=1}^{n} a_i s_i - bs \right\} \\ &= \left\{ m \;\middle|\; bm = \sum_{i=1}^{n} a_i (s_i - s) \right\}, \end{aligned} \tag{3.1}$$

where the last step follows since $b = \sum_{i=1}^{n} a_i$. Except for the conditions $n > 0$ and $b = \sum_{i=1}^{n} a_i$ the last expression is equal to $\langle S - s\rangle_{\mathrm{d}}$. The first condition does not play any role since $0 \in A - s$. The second condition can also be dispensed with: Assume that we have $m \in \langle S - s\rangle_{\mathrm{d}}$, i.e. $bm = \sum_{i=1}^{n} a_i(s_i - s)$. Then we also have (assuming that $b's = \sum_{i=1}^{n'} a'_i s'_i$, $b' = \sum_{i=1}^{n'} a'_i \neq 0$)

$$\begin{aligned} b'b(m+0) &= b' \sum_{i=1}^{n} a_i(s_i - s) + b'\left(b - \sum_{i=1}^{n} a_i\right)(s - s) \\ &= b' \sum_{i=1}^{n} a_i(s_i - s) + \left(b - \sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n'} a'_i(s'_i - s)\right), \end{aligned} \tag{3.2}$$

and since $b'b = b'\sum_{i=1}^{n} a_i + (b - \sum_{i=1}^{n} a_i)\sum_{i=1}^{n'} a'_i \neq 0$ we have $m \in A - s$. In other words, $A - s = \langle S - s\rangle_{\mathrm{d}}$. Thus A is an a-submodule. Now take any a-submodule A' containing S. Since $A' - s$ is a d-submodule we know that $\langle S - s\rangle_{\mathrm{d}} \subseteq A' - s$. Hence A is the smallest a-submodule containing S, and $\langle\cdot\rangle_{\mathrm{a}}$ is well-defined. We also get that $\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{d}} + s$.

It remains to show that $\langle\cdot\rangle_{\mathrm{a}}$ is a matroidal closure operator. Refer to Definition 2.1. All the axioms are easily seen to hold when $A = \emptyset$, so assume that $A \neq \emptyset$, and pick an element $a \in A$. Now it is easy to see that all the matroid axioms are satisfied by using $\langle A\rangle_{\mathrm{a}} = \langle A - a\rangle_{\mathrm{d}} + a$ and the fact that $\langle\cdot\rangle_{\mathrm{d}}$ satisfies all axioms. ∎

Corollary 3.7. *For any $s \in M$, $\langle S \rangle_{\mathrm{a}} - s = \langle S - s \rangle_{\mathrm{a}}$.*

Proof. Just inspect the explicit representation of $\langle \cdot \rangle_{\mathrm{a}}$. □

Corollary 3.8. *The matroid defined in Theorem 3.6 is a geometry iff the underlying module is torsion free.*

Proof. Since $\langle \emptyset \rangle_{\mathrm{a}} = \emptyset$ we have to check when we have $\langle m \rangle_{\mathrm{a}} = \{ m \}$ for arbitrary $m \in M$. Since $\langle m \rangle_{\mathrm{a}} = \{ n \in M \mid bn = bm,\ b \neq 0 \}$ the corollary follows immediately. □

3.4 Rank

From now on let all modules be modules over integral domains.

Let us distinguish between different kinds of independence and rank. We say that B is d- (a-)independent if it is independent using $\langle \cdot \rangle_{\mathrm{d}}$ ($\langle \cdot \rangle_{\mathrm{a}}$) as the closure operator. Furthermore the rank attained using d- (a-)closure is called d- (a-)rank. This terminology is extended in the obvious way to other concepts, sometimes also using the prefix s- which is associated to the submodule closure of Section 3.1. Note that if we use the term a-geometry we implicitly assume that the underlying module is torsion free; otherwise we do not have a geometry.

Proposition 3.9. *Let D be a d-submodule of the R-module M and $B \subseteq D$ with $p \in B$. Then if B is d-independent, then $B \cup 0$ is a-independent, and if B is a-independent then $(B - p) \setminus 0$ is d-independent. Furthermore $\langle B \rangle_{\mathrm{d}} = \langle B \cup 0 \rangle_{\mathrm{a}}$, and if $\langle B \rangle_{\mathrm{a}} = D$, then $\langle (B - p) \setminus 0 \rangle_{\mathrm{d}} = D$.*

Proof. First assume that B is d-independent. Take any $q \in B \cup 0$. Assume for a contradiction that $q \in \langle (B \cup 0) \setminus q \rangle_{\mathrm{a}}$. If $q \neq 0$, then we have

$$q \in \langle (B \cup 0) \setminus q \rangle_{\mathrm{a}} = \langle (B \setminus q) \cup 0 \rangle_{\mathrm{a}} = \langle ((B \setminus q) \cup 0) - 0 \rangle_{\mathrm{d}} + 0 = \langle B \setminus q \rangle_{\mathrm{d}} \quad (3.3)$$

since $0 \in \langle (B \setminus q) \cup 0 \rangle_{\mathrm{a}}$ and 0 does not play any role in d-closure. This is a contradiction since B is d-independent, so assume that $q = 0$ instead. Then $b0 = \sum_{i=1}^{n} a_i b_i$ for some coefficients $b, a_i \in R$, some $b_i \in B$, and one $n \geq 2$ (since $b = \sum_{i=1}^{n} a_i \neq 0$). We can assume that all the coefficients are nonzero, so we have $a_1 b_1 = \sum_{i=2}^{n} (-a_i) b_i$, and B is not d-independent, which yet again is a contradiction. Hence $B \cup 0$ is a-independent.

Now assume that B is a-independent. Take any $q \in (B - p) \setminus 0$. Assume that $q \in \langle (B - p) \setminus \{ 0, q \} \rangle_{\mathrm{d}}$. Then $bq = \sum_{i=1}^{n} a_i (b_i - p)$ for some $b \neq 0$, $a_i \in R$, $n \in \mathbb{N}$ and $b_i \in B \setminus \{ p, q + p \}$. By adding bp to both sides we get $b(q + p) = \sum_{i=1}^{n} a_i (b_i - p) + bp$, and since the coefficients add up we have $q + p \in \langle B \setminus \{ q + p \} \rangle_{\mathrm{a}}$ (note that $q \neq 0$, i.e. $p \neq q + p$). This is a contradiction, so $(B - p) \setminus 0$ is d-independent.

We immediately have that $\langle B \cup 0 \rangle_{\mathrm{a}} = \langle (B \cup 0) - 0 \rangle_{\mathrm{d}} + 0 = \langle B \rangle_{\mathrm{d}}$ since $0 \in \langle B \cup 0 \rangle_{\mathrm{a}}$. Now assume that $\langle B \rangle_{\mathrm{a}} = D$, which is a d-submodule as well as an a-submodule. We have (omitting the set generator conditions $b \in R \setminus 0$, $a_i \in R$,

$n \in \mathbb{N}$)

$$
\begin{aligned}
&\langle (B-p) \setminus 0 \rangle_{\mathrm{d}} + p \\
&= \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i b_i,\ b_i \in (B \setminus p) - p \right\} + p \\
&= \left\{ m \in M \;\middle|\; b(m-p) = \sum_{i=1}^{n} a_i (b_i - p),\ b_i \in B \setminus p \right\} \\
&= \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i b_i + \left(b - \sum_{i=1}^{n} a_i \right) p,\ b_i \in B \setminus p \right\} \\
&= \left\{ m \in M \;\middle|\; bm = \sum_{i=1}^{n} a_i b_i + a_{n+1} p,\ b = \sum_{i=1}^{n+1} a_i,\ b_i \in B \setminus p \right\} \\
&= \langle B \rangle_{\mathrm{a}} .
\end{aligned}
\tag{3.4}
$$

Since $p \in \langle B \rangle_{\mathrm{a}}$ and $\langle B \rangle_{\mathrm{a}}$ is a d-submodule Lemma 3.3 gives that $\langle B \rangle_{\mathrm{a}} = \langle B \rangle_{\mathrm{a}} - p$, and we are done. ☐

Corollary 3.10. *A module has d-rank n iff it has a-rank $n+1$.*

Proof. Just observe that a basis is an independent generator and that $0 \notin B$ for any d-basis B. ☐

Lemma 3.11. *If B is an a-basis for a d-submodule D, then $B - p$ is also an a-basis for D, for any $p \in \langle B \rangle_{\mathrm{a}}$.*

(If $p \in B$, then this is an immediate corollary of the preceding proposition.)

Proof. Since $\langle B \rangle_{\mathrm{a}}$ is a d-submodule we have, according to Lemma 3.3, that $\langle B \rangle_{\mathrm{a}} = \langle B \rangle_{\mathrm{a}} - p = \langle B - p \rangle_{\mathrm{a}}$ where the last step follows by Corollary 3.7. The independence of $B - p$ follows by the equipotence of all bases. ☐

3.5 Degrees

Let us now determine the degree of d- and a-matroids. First note that a large class of a-matroids are not projective, and hence not of degree 0. (All matroids of rank 2 or less are of degree 0.)

Proposition 3.12. *Let M be an R-module of a-rank at least 3. Then the associated a-matroid does not satisfy the projective law.*

Proof. Let B be an a-basis of M. Due to Lemma 3.11 we can assume that $0 \in B$. Choose two other elements in B and form $B' = \{ 0, x, y \}$ which is an a-basis for a rank 3 subspace. We know that $x + y \in \langle 0, x, y \rangle_{\mathrm{a}}$, and we will show that $x + y \notin \bigcup \{ \langle u, v \rangle_{\mathrm{a}} \mid u \in \langle 0, x \rangle_{\mathrm{a}},\ v \in \langle y \rangle_{\mathrm{a}} \}$, thus showing that the projective law does not hold.

First notice that $\langle 0, x \rangle_{\mathrm{a}} = \left\{ \frac{ax}{b} \,\middle|\, a, b \in R,\ b \neq 0 \right\}$ and $\langle y \rangle_{\mathrm{a}} = \left\{ \frac{cy}{c} \,\middle|\, c \in R \setminus 0 \right\}$. Assume for a contradiction that $x + y \in \left\langle \frac{ax}{b}, \frac{cy}{c} \right\rangle_{\mathrm{a}}$ for some $a, b, c \in R$, $b, c \neq 0$.

This implies that $d(x+y) = e\frac{ax}{b} + f\frac{cy}{c}$ for some $d, e, f \in R$ with $d = e + f \neq 0$. Rewritten this reads $bcd(x+y) = ceax + bfcy$ or

$$(ace - bcd)x = (bcd - bcf)y = bcey. \tag{3.5}$$

We know that $b, c \neq 0$. Furthermore $e = 0$ implies that

$$\begin{aligned} x + y \in \left\{ m \in M \,\middle|\, dm = d\frac{cy}{c},\ c, d \neq 0 \right\} \\ = \{ m \in M \mid cdm = cdy,\ cd \neq 0 \} \subseteq \langle y, 0 \rangle_{\mathrm{a}}, \end{aligned} \tag{3.6}$$

i.e. $x \in \langle y, 0\rangle_{\mathrm{a}} - y = \langle 0, -y\rangle_{\mathrm{a}} = \langle -y\rangle_{\mathrm{d}} = \langle y\rangle_{\mathrm{d}} = \langle 0, y\rangle_{\mathrm{a}}$. This contradicts the independence of B', and hence we have $bce \neq 0$ which shows that $y \in \langle x, 0\rangle_{\mathrm{a}}$. This is another contradiction and we are done. ☐

Now on to showing which degree the matroids do have.

Theorem 3.13. *Let M be an R-module. Then the d-matroid over M is of degree 0, i.e. it satisfies the projective law.*

Proof. We show this by showing that the subspace lattice interval $\left[\langle\emptyset\rangle_{\mathrm{d}}, M\right]$ is modular. Let E, F, and H be arbitrary subspaces. We have to show that $E \subseteq H$ implies that $E \vee (F \wedge H) = (E \vee F) \wedge H$, i.e.

$$\langle E \cup (F \cap H)\rangle_{\mathrm{d}} = \langle E \cup F\rangle_{\mathrm{d}} \cap H. \tag{3.7}$$

We begin by showing that for arbitrary subspaces A, C we have

$$\langle A \cup C\rangle_{\mathrm{d}} = \{ m \in M \mid bm = a + c,\ b \in R \setminus \{0\},\ a \in A,\ c \in C \}. \tag{3.8}$$

The $\supseteq$ inclusion is trivial. Assume that $m \in \langle A \cup C\rangle_{\mathrm{d}}$, i.e. $bm = \sum_{i=1}^{n_1} b_{1i}a_i + \sum_{i=1}^{n_2} b_{2i}c_i$ for some $b \in R \setminus \{0\}$, $b_{ji} \in R$, $a_i \in A$, $c_i \in C$, and $n_j \in \mathbb{N}$. Then $\sum_{i=1}^{n_1} b_{1i}a_i \in A$ and $\sum_{i=1}^{n_2} b_{2i}c_i \in C$ (recall that A and C are d-submodules), so the other inclusion also holds.

In the light of (3.8) we can rewrite the two sides of (3.7) as

$$P := \{ m \in M \mid bm = e + \tilde{f},\ b \in R \setminus \{0\},\ e \in E,\ \tilde{f} \in F \cap H \} \tag{3.9}$$

and

$$Q := \{ \tilde{m} \in M \mid b\tilde{m} = e + f,\ b \in R \setminus \{0\},\ e \in E,\ f \in F \} \cap H. \tag{3.10}$$

Take $m \in P$, i.e. $bm = e + \tilde{f}$. Since H is a d-submodule, $E \subseteq H$, and $\tilde{f} \in H$ we have that $m \in H$. Hence $m \in Q$ and $P \subseteq Q$. Now take $\tilde{m} \in Q \subseteq H$ with $b\tilde{m} = e + f$. Since $f = b\tilde{m} - e \in H$ we get that $\tilde{m} \in P$ and $Q \subseteq P$. Thus $P = Q$ and we are done. ☐

Corollary 3.14. *Let M be an R-module. Then the a-matroid over M is of degree 1.*

Proof. Take any point $a \in M$. We have to show that the a-lattice $\left[\langle a\rangle_{\mathrm{a}}, M\right]$ is modular. We have

$$\begin{aligned} \left[\langle a\rangle_{\mathrm{a}}, M\right] - a &= \{ A - a \mid \langle a\rangle_{\mathrm{a}} \subseteq A \subseteq M,\ A \text{ is an a-submodule} \} \\ &= \{ A \mid \langle a\rangle_{\mathrm{a}} - a \subseteq A \subseteq M - a,\ A \text{ is a d-submodule} \} \\ &= \left[\langle\emptyset\rangle_{\mathrm{d}}, M\right], \end{aligned} \tag{3.11}$$

where we have used $\langle a \rangle_{\mathrm{a}} - a = \langle 0 \rangle_{\mathrm{a}} = \langle \emptyset \rangle_{\mathrm{d}}$, $M - a = M$, and Corollary 3.5. This shows that the a-lattice $\left[\langle a \rangle_{\mathrm{a}}, M\right]$ is isomorphic to the d-lattice $\left[\langle \emptyset \rangle_{\mathrm{d}}, M\right]$ which is modular. □

3.6 Representations

This section lists some results about the representation of an element in a basis.

Theorem 3.15 (The Representation Theorem). *Let B be an a-basis for the R-module M. Assume that $p \in M$ has the representation*

$$cp = \sum_{i=1}^{n} a_i b_i,\ n \in \mathbb{Z}^+,\ c, a_i \in R \setminus 0,\ c = \sum_{i=1}^{n} a_i,\ b_i \in B,\ b_i \neq b_j \text{ if } i \neq j$$

in this basis. Then the only other representations of p in this basis are

$$dp = \sum_{i=1}^{n} \frac{da_i}{c} b_i,$$

where $d \in R \setminus 0$ and all $\frac{da_i}{c}$ are assumed to be well-defined.

Of course, if the module is not torsion free, then any particular representation does not necessarily stand for a unique module element.

Proof. Suppose that we have another representation $\left(n', c', \{a'_i\}_{i=1}^{n'}, \{b'_i\}_{i=1}^{n'}\right)$ of p in B. First assume that $n \neq n'$ or $n = n'$ but $\{\, b_i \,|\, 1 \leq i \leq n \,\} \neq \{\, b'_i \,|\, 1 \leq i \leq n' \,\}$. Then there is one basis element, say b_1, for which

$$c' a_1 b_1 = \sum_{i=2}^{n} (-c' a_i) b_i + \sum_{i=1}^{n'} c a'_i b'_i, \tag{3.12}$$

where b_1 does not occur in the right hand side of the equation. Since

$$\sum_{i=2}^{n} (-c' a_i) + \sum_{i=1}^{n'} c a'_i = -(c'c - c' a_1) + cc' = c' a_1 \neq 0 \tag{3.13}$$

we get that B is not independent, which is a contradiction. Hence $n = n'$ and $\{\, b_i \,|\, 1 \leq i \leq n \,\} = \{\, b'_i \,|\, 1 \leq i \leq n' \,\}$.

For simplicity let us reorder the basis elements such that $b_i = b'_i$ for all i, $1 \leq i \leq n$. If $c' a_i \neq c a'_i$ for some i then we get a contradiction as above, so $c' a_i = c a'_i$ for all i, $1 \leq i \leq n$. Hence $a'_i = \frac{c' a_i}{c}$, and by noticing that $d = c'$ we are almost done. The only thing remaining is to point out that every choice of $d \neq 0$ such that $\frac{da_i}{c}$ is defined for all i gives a correct representation. (Since R is an integral domain we have $\frac{dc}{c} = d$.) □

Corollary 3.16. *The Representation Theorem also holds for d-representations (where $c = \sum_{i=1}^{n} a_i$ does not necessarily hold).*

Proof. Apply Proposition 3.9. If B is a d-basis for M, then $B \cup 0$ is an a-basis for M. A consequence of this is that any d-representation in B of a point $p \in M$ is also an a-representation in $B \cup 0$ of p (using 0 to make the coefficients add up) and vice versa (removing 0). The corollary follows. □

Corollary 3.17. *If R is well-ordered then we get a* canonical representation *of p by choosing the smallest possible positive d, and if R is a field then we can choose $d = 1$ (using the notation of the preceding theorem).*

3.7 Embedding

All torsion free modules over integral domains are naturally embedded in a vector space, as pointed out in Section 2.5. Here we explore some of the connections with the matroids defined above.

In fact, the connections also work out for modules that are not torsion free. Let M be a module over an integral domain R, and define the canonical map $\pi : M \to F(M)$ by

$$\pi(m) := \frac{m}{1}. \tag{3.14}$$

Denote the preimage of π by π^{-1}, i.e.

$$\pi^{-1}\left(\frac{m}{r}\right) = \left\{ m' \in M \,\middle|\, \pi(m') = \frac{m}{r} \right\}. \tag{3.15}$$

For notational convenience let us stray slightly from our convention for functions applied to sets. Let $\pi^{-1}(S) := \bigcup \left\{ \pi^{-1}(s) \,\middle|\, s \in S \right\}$ for any subset $S \subseteq F(M)$. If M is torsion free then it can be seen as a subset of $F(M)$, and we get that π is essentially just the identity, while π^{-1} for subsets $S \subseteq F(M)$ can be expressed as $\pi^{-1}(S) = S \cap M$.

Let us also define another function, $\mu : F(M) \to M$, which maps 0 to 0 and an element $m' \in F(M) \setminus 0$ to any element $m \in M \setminus 0$ with the property that $m = rm'$ for some $r \in R$. Since $m' = \frac{m''}{r''}$ for some $m'' \in M \setminus 0$ and $r'' \in R \setminus 0$ this is always possible by choosing $r = r''$. (Just note that this choice usually is not unique).

In a vector space, the d-submodule closure $\langle\cdot\rangle_{\mathrm{d}}$ equals the vector subspace closure. Given this closure operator independence, bases, etc. match the matroid definitions exactly.

Given the preliminaries above, let us now show in what way the matroid structure carries over to the vector space.

Theorem 3.18. *Let M be an R-module, and let $F(M)$ be the module of fractions associated with M. Denote the d-submodule closure in $F(M)$ by $\langle\cdot\rangle_{\mathrm{D}}$. Then we have the following properties.*

1. *For any subset $S \subseteq M$ the equality $\langle S\rangle_{\mathrm{d}} = \pi^{-1}\big(\langle\pi(S)\rangle_{\mathrm{D}}\big)$ holds.*
2. *Let D be a d-submodule of M with d-basis B. Then $F(R)\pi(D) = F(D)$ is a vector subspace with basis $\pi(B)$, and the d-rank of D equals the dimension of $F(D)$.*
3. *Let S be a vector subspace of $F(M)$ with basis B. Then $\pi^{-1}(S)$ is a d-submodule of M with d-basis $\mu(B)$, and the dimension of S equals the d-rank of $\pi^{-1}(S)$.*

Remember that the expressions involving π and π^{-1} can be simplified whenever M is torsion free.

Proof. 1. We have

$$\langle S\rangle_{\mathrm{d}} = \left\{ m \in M \,\middle|\, bm = \sum_{i=1}^{n} a_i s_i,\ b, a_i \in R \setminus 0,\ s_i \in S,\ n \in \mathbb{N} \right\} \tag{3.16}$$

and

$$\tilde{S} := \pi^{-1}\left(\langle \pi(S)\rangle_{\mathrm{D}}\right) = \left\{ m \in M \,\middle|\, \pi(m) = \sum_{i=1}^{n} a_i \pi(s_i),\ a_i \in F(R) \setminus 0,\ s_i \in S,\ n \in \mathbb{N} \right\}. \tag{3.17}$$

It is obvious that $\langle S\rangle_{\mathrm{d}} \subseteq \tilde{S}$. Now assume $m \in \tilde{S}$, i.e. $\pi(m) = \sum_{i=1}^{n} a_i \pi(s_i)$. Assume that $a_i = \frac{b_i}{c_i}$ for all i, $1 \le i \le n$. We get

$$\left(\prod_{i=1}^{n} c_i\right) \pi(m) = \sum_{i=1}^{n} b_i \left(\prod_{j \ne i} c_j\right) \pi(s_i). \tag{3.18}$$

Hence, by the definition of $F(M)$, we get that for some $s \in R \setminus 0$ the equality

$$s\left(\left(\prod_{i=1}^{n} c_i\right) m - \sum_{i=1}^{n} b_i \left(\prod_{j \ne i} c_j\right) s_i\right) = 0 \tag{3.19}$$

holds in M. Since $s\prod_{i=1}^{n} c_i \ne 0$ it follows that $m \in \langle S\rangle_{\mathrm{d}}$.

2. We already know that $F(D)$ is a vector subspace. Since we have

$$\sum_i a_i \pi(b_i) = \sum_i a_i \frac{b_i}{1} = \frac{\sum_i a_i b_i}{1} = \pi\left(\sum_i a_i b_i\right) \tag{3.20}$$

($a_i \in R$, $b_i \in M$) it is easy to check that $\pi(B)$ generates $F(D) = F(R)\pi(D)$.

By showing that $b, b' \in B$, $b \ne b'$ implies $\pi(b) \ne \pi(b')$ we get that $|B| = |\pi(B)|$. To see this, note that if $\pi(b) = \pi(b')$ then $s(b - b') = 0$ for some $s \in R \setminus 0$. We get that $b \in \langle b'\rangle_{\mathrm{d}}$, i.e. B is not independent, a contradiction.

We also need to check that $\pi(B)$ is independent. Assume that $\pi(b) \in \langle \pi(B) \setminus \pi(b)\rangle_{\mathrm{D}}$ for some $b \in B$. Then we have $b \in \pi^{-1}\big(\langle \pi(B \setminus b)\rangle_{\mathrm{D}}\big) = \langle B \setminus b\rangle_{\mathrm{d}}$, another contradiction, and we are done.

3. We already know that $\pi^{-1}(S)$ is a submodule. It is easy to check that it is also closed under existing divisors, so it is a d-submodule. We have that $\pi(\mu(B))$ is a basis of S, so $\langle \pi(\mu(B))\rangle_{\mathrm{D}} = S$. Hence $\langle \mu(B)\rangle_{\mathrm{d}} = \pi^{-1}\big(\langle \pi(\mu(B))\rangle_{\mathrm{D}}\big) = \pi^{-1}(S)$. Furthermore we trivially have $|B| = |\mu(B)|$.

It remains to show that $\mu(B)$ is independent. Assume $b \in \langle \mu(B) \setminus b\rangle_{\mathrm{d}} = \pi^{-1}\big(\langle \pi(\mu(B) \setminus b)\rangle_{\mathrm{D}}\big)$ for some $b \in \mu(B)$. Since $\pi\big(\mu(B) \setminus b\big) = \pi(\mu(B)) \setminus \pi(b)$ we get

$$\pi(b) \in \left(\pi \circ \pi^{-1}\right)\left(\langle \pi(\mu(B)) \setminus \pi(b)\rangle_{\mathrm{D}}\right), \tag{3.21}$$

i.e. $\pi(b) \in \langle \pi(\mu(B)) \setminus \pi(b)\rangle_{\mathrm{D}}$, and by the independence of $\pi(\mu(B))$ we are done. ∎

Corollary 3.19. *There is a bijective correspondence between the d-submodules of M and the vector subspaces of $F(M)$.*

Proof. Just note that $\pi^{-1}(F(R)\pi(D)) = D$ for any d-submodule $D \subseteq M$, and $F(R)\pi(\pi^{-1}(S)) = S$ for any vector subspace $S \subseteq F(M)$. This is easiest to see by observing how the bases are related. □

Corollary 3.20. *The lattice of subspaces of M is isomorphic to that of $F(M)$.*

Corollary 3.21. *The d-rank and the module rank of a module (over an integral domain) coincides.*

Since a-matroids have subspaces which are just translations of d-subspaces all the results above can be transformed to an a-matroid context. For instance, for any subset $S \subseteq M$ and any $s \in \langle S\rangle_{\mathrm{a}}$ we have

$$\langle S\rangle_{\mathrm{a}} = \langle S - s\rangle_{\mathrm{d}} + s = \pi^{-1}\big(\langle \pi(S - s)\rangle_{\mathrm{D}}\big) + s. \tag{3.22}$$

It is easy to verify that this can be rewritten as

$$\langle S\rangle_{\mathrm{a}} = \pi^{-1}\big(\langle \pi(S) - \pi(s)\rangle_{\mathrm{D}} + \pi(s)\big) = \pi^{-1}\big(\langle \pi(S)\rangle_{\mathrm{A}}\big), \tag{3.23}$$

where $\langle\cdot\rangle_{\mathrm{A}}$ is a-submodule closure in $F(M)$. By using similar techniques and adapting some pieces from the proof above we can proceed to conclude that the lattice of subspaces of any a-matroid M is isomorphic to that of the a-matroid over $F(M)$.

Note that some of the results in previous sections could have been proved by reduction to the corresponding proofs for vector spaces. The most glaring examples are probably those results in Section 3.5 which only refer to the lattice of subspaces. The reason for still having those "unnecessary" proofs in this text is to make the text more accessible. Furthermore, reduction to another proof may be quicker, but it sometimes obscures the reasoning behind the results. A good example illustrating how a reduction can hide some interesting details is given in Section 4.2.

3.8 Affine Geometry

Given that all a-matroids are of degree 1, and that all a-matroids over modules that are torsion free are geometries, is an a-geometry an affine geometry? Not necessarily, as we will show.

Take the $\mathbb{Z}$-module over $\mathbb{Z}^2$. This module is torsion free and is hence an a-geometry of degree 1. Consider the line $\ell = \langle (0,0),(2,1)\rangle_{\mathrm{a}}$ and the point $p = (1,0)$. (All subsets of cardinality two are independent, and hence bases, since this is a geometry.) Both the lines $\ell_1 = \langle (1,0),(1,1)\rangle_{\mathrm{a}}$ and $\ell_2 = \langle (1,0),(-1,1)\rangle_{\mathrm{a}}$ are parallel to ℓ, so this geometry is not affine. See Figure 3.1 for an indication of the situation.

The $\mathbb{Z}$-module over $\mathbb{Z}^2$ clearly has an affine feel to it. The reason why it is not affine is that two lines which are non-parallel in the associated vector space can be disjoint, and hence parallel; the problem lies in the discreteness of the structure. The following proposition shows that it is easy to define a notion of parallelity which at least satisfies some of the usual requirements of

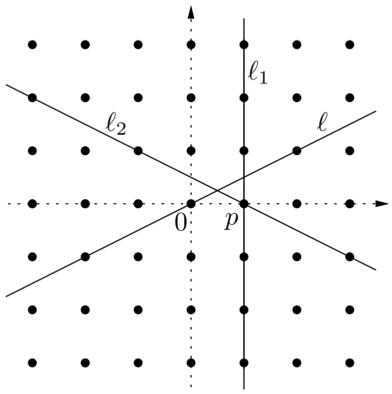

Figure 3.1: An example demonstrating why the $\mathbb{Z}$-module $\mathbb{Z}^2$ is not affine. The lines ℓ_1 and ℓ_2 are both parallel to ℓ. The notation is the same as in the text.

a parallelity relation. Of course this definition is influenced by the fact that the same approach gives the proper parallelity relation in an a-geometry over a vector space.

Proposition 3.22. *Let M be a torsion free R-module. Define a binary relation $|||$ on the lines of M by $\ell \,|||\, \ell'$ iff there is some $p \in M$ such that $\ell + p = \ell'$. Then $|||$ is an equivalence relation, and for any point $p \in M$ and line $\ell \subseteq M$ there is a unique line ℓ' such that $p \in \ell'$ and $\ell \,|||\, \ell'$.*

Proof. The relation is easily seen to be reflexive, symmetric, and transitive. Now consider a point $p \in M$ and a line $\ell \subseteq M$. Assume $q \in \ell$. The set $\ell' = \ell + (p - q)$ is then a line with $\ell \,|||\, \ell'$. Assume that ℓ'' is another such line with $\ell'' = \ell + r$ and $p \in \ell''$, $r \in M$. Notice that $\ell - q$ is a d-submodule, and hence a subgroup. (A d-submodule contains 0 and is closed under inverse and addition.) Thus $\ell' = (\ell - q) + p$ and $\ell'' = (\ell - q) + (q + r)$ are non-disjoint cosets of the same subgroup, and hence equal. $\square$

Note that the proposition does not really require the module to be torsion free, if we replace "line" with "a-rank 2 subspace." This also applies to the following proposition, relating $||$ and $|||$, if we weaken the definition of $||$ to allow matroids that are not geometries.

Proposition 3.23. *Let ℓ and ℓ' be two lines of a torsion free R-module M. Then $\ell \,|||\, \ell'$ implies $\ell \,||\, \ell'$.*

Proof. Assume that $\ell \,|||\, \ell'$, i.e. $\ell + p = \ell'$ for some $p \in M$. If $\ell = \ell'$ then $\ell \,||\, \ell'$, so assume $\ell \neq \ell'$. This implies, by Proposition 3.22, that $\ell \cap \ell' = \emptyset$. Assume $\ell = \langle q, r\rangle_{\mathrm{a}}$, and let $B = \{ q, r, q + p \}$. Since $q + p \in \ell'$ we have $q + p \notin \langle q, r\rangle_{\mathrm{a}}$. Furthermore $\langle q, q + p\rangle_{\mathrm{a}} \neq \ell$, and hence $\langle q, q + p\rangle_{\mathrm{a}} \cap \ell = \{ q \}$ (two points determine a line uniquely), whereby $r \notin \langle q, q + p\rangle_{\mathrm{a}}$. Analogously $q \notin \langle r, q + p\rangle_{\mathrm{a}}$, so B is a-independent. We obviously have $\ell \subseteq \langle B\rangle_{\mathrm{a}}$, but also $\ell' = \langle q, r\rangle_{\mathrm{a}} + p = \langle q + p, r + p\rangle_{\mathrm{a}} \subseteq \langle q + p, q, r\rangle_{\mathrm{a}} = \langle B\rangle_{\mathrm{a}}$ since $br = a_1(q+p) + a_2(r+p)$ implies $br = a_1(q + p) + a_2(r + (q + p) - q)$. Hence $\ell \vee \ell' = \langle \ell \cup \ell'\rangle_{\mathrm{a}} \subseteq \langle B\rangle_{\mathrm{a}}$. The union of two disjoint lines cannot have a-rank 2, and hence the inclusion is an equality. Thus $r(\ell \vee \ell') = 3$ and $\ell \,||\, \ell'$. $\square$

The question about what would make a suitable definition of a discrete affine geometry remains open. However, we can at least motivate why $|||$ seems to be a valid parallelity relation (although, in general, it is not). Note that in an a-geometry over a vector space we have $||| = ||$. Let us weaken the definition of $|||$ to apply to modules that are not torsion free as well. Then we get the following result.

Proposition 3.24. *Let M be an R-module, and let $\ell_1, \ell_2 \subseteq M$ be two a-rank 2 subspaces. Then $\ell_1 \,|||\, \ell_2$ iff $\pi(\ell_1) \,|||\, \pi(\ell_2)$, i.e. iff $\pi(\ell_1) \,||\, \pi(\ell_2)$, where π is the canonical map into the associated vector space as defined in* (3.14).

Proof. First note that $\pi(\ell)$ is a line if ℓ is an a-rank 2 subspace (Theorem 3.18).

Assume that $\ell_1 \,|||\, \ell_2$. Then $\ell_1 + p = \ell_2$ for some $p \in M$. Hence $\pi(\ell_1 + p) = \pi(\ell_1) + \pi(p) = \pi(\ell_2)$, and thus $\pi(\ell_1) \,|||\, \pi(\ell_2)$.

Assume instead that $\pi(\ell_1) \,|||\, \pi(\ell_2)$. Then $\pi(\ell_1) + x = \pi(\ell_2)$ for some $x \in F(M)$. Take any two points $p \in \ell_1$, $q \in \ell_2$, and set $y = \pi(q) - \pi(p)$. We get that $y - x$ is a vector parallel to ℓ_1, and hence $\pi(\ell_2) = \pi(\ell_1) + x + (y - x) = \pi(\ell_1 + q - p)$. For any two points $r_1, r_2 \in M$ we have that $\pi(r_1) = \pi(r_2)$ implies $s(r_1 - r_2) = 0$ for some $s \in R \setminus 0$, and hence $r_1 \in \langle r_2 \rangle_{\mathrm{a}}$. It follows that $\ell_2 = \ell_1 + (q - p)$, since all sets involved are a-rank 2 subspaces (in either M or $F(M)$). In other words $\ell_1 \,|||\, \ell_2$, and we are done. □

3.9 Generators and Isomorphism

The properties defined in this section can perhaps serve as an indication of whether a geometry is discrete or not. They are based on a generalisation of Hübler's generators.

Definition 3.3. *Let M be a d-matroid over an R-module with d-rank at least n. This matroid has the* rank n generator property *if all d-submodules of d-rank n are s-generated by a set of points of cardinality n. The elements of this set are called* generators.

If a d-submodule with rank n is s-generated by a set of cardinality n, then by matroid arguments this set has to be d-independent, and thus it is easily seen to be linearly independent. Hence the property ensures that all d-submodules of rank n are free.

Note that the rank n generator property implies that all a-submodules of rank $n + 1$ are generated by a set of cardinality n (plus the usual translation). Hence, when $n = 1$ we use the term *line generator property*, or more often just the generator property. For $n = 2$ we use the term *plane generator property.*

Let us show that in some cases the properties for different n are not independent. In fact, while we are indulging in the theory of finitely generated, torsion free modules over principal ideal domains we might as well throw in a result about isomorphism as well. Note that the term finitely generated stands for finitely s-generated. Obviously any a-geometry of d-rank n satisfying the rank n generator property is finitely generated.

Theorem 3.25. *All finitely generated a-geometries over a principal ideal domain R with d-rank n are isomorphic to the R-module over R^n, and they satisfy the rank m generator property for any $m \leq n$.*

Proof. All finitely generated, torsion free modules over R are free, and all finitely generated, free modules with rank n are isomorphic to the R-module over R^n. Furthermore all d-submodules of a finitely generated, free module are free and finitely generated with module bases of the same size as the d-rank (by Corollary 3.21), so we are done. □

Corollary 3.26. *Let M be an a-geometry over an R-module satisfying the rank n generator property, where R is a principal ideal domain and n is finite. Then the rank m generator property holds for any $m \leq n$.*

Proof. Let D be a d-rank m subspace of M. Let P be any d-rank n subspace containing D (known to exist since M has d-rank at least n). Since P by the rank n generator property is finitely generated we can treat it as a module with D as a d-submodule, apply the preceding Theorem, and we are done. □

As a final exercise of this section we show that the rank 1 generator property is equivalent to the irreducible element property.

Definition 3.4. *Let M be an R-module. An element $m \in M \setminus \langle\emptyset\rangle_{\mathrm{d}}$ is* irreducible *if $bm = am'$ for some $a, b \in R$, $a \neq 0$, $m' \in M$ implies $a|b$. The module has the* irreducible element property *if, given any element $m \in M$, there is an irreducible element $m' \in M$ such that $m = r'm'$ for some $r' \in R$.*

The term irreducible is taken from [Eck01], where irreducible translation means the same as simple translation means here (see Section 2.6). The term irreducible is arguably more descriptive than simple, and the definition above is not identical to the one for simple translations, motivating the change in terminology. (Note, though, that by the following proposition and Theorem 5.13 the irreducible elements of a Hübler geometry are exactly the simple translations.)

Proposition 3.27. *Let M be an R-module of d-rank at least 1. Then the rank 1 generator property is equivalent to the irreducible element property. Furthermore the irreducible elements are exactly those elements which are generators for some d-rank 1 subspace.*

Proof. First assume that M has the rank 1 generator property. Take any element $m \in M$. We know that $\langle m\rangle_{\mathrm{d}}$ has a generator, say $g \in M$. (Unless $\langle m\rangle_{\mathrm{d}} = \langle\emptyset\rangle_{\mathrm{d}}$, in which case we can choose a generator from any d-rank 1 subspace.) It follows that $m = rg$ for some $r \in R$. We will now show that g is irreducible. Assume that $bg = am'$ for some $a, b \in R$, $a \neq 0$, $m' \in M$. It follows that $m' \in \langle g\rangle_{\mathrm{d}}$. Hence $m' = cg$ for some $c \in R$. By the Representation Theorem 3.15 we get that $c = \frac{b}{a}$, i.e. $a|b$.

Now let M have the irreducible element property instead. Take any d-rank 1 subspace $\langle m\rangle_{\mathrm{d}}$, $m \in M \setminus 0$. Let $g \in M$ be an irreducible element with $m = rg$, $r \in R$. Since g is irreducible it follows that $\langle m\rangle_{\mathrm{d}} = \langle g\rangle_{\mathrm{d}} = \langle g\rangle_{\mathrm{s}}$, whereby g is a generator.

The procedure above also proves the second statement of the proposition, and we are done. □

3.10 Acknowledgements

The idea to use d-submodules was suggested by Mike Smyth. He also came up with the explicit characterisation of the d-closure. After trying several

approaches to "find a matroid" in Hübler's geometries the author asked him whether modules are matroids (the same thought had also struck him). No information was found in the existing literature, but d-submodules did the trick. Later it was discovered that exactly the same construction (for left Ore domains) is mentioned in Faure and Fröhlicher's book [FF00, Exercise 3.7.4.5].

The ideas behind a-closure and some of the associated results also come from [FF00]—indeed, several of the proofs in this chapter are relatively straightforward adaptations of results in that text. However, [FF00] only treats the case of vector spaces over division rings, not modules over integral domains (except for some specific examples).

3.11 Conclusions and Future Work

One of the standard examples of a matroid is a vector space with its linear (vector) subspaces. Modules are hardly ever mentioned. This chapter clearly shows that there is no reason to restrict the attention to vector spaces, modules over integral domains work equally well. In fact, some may say, they work too well. At least those that are torsion free; they are naturally embedded in a unique vector space, and hence they can be treated within the framework of vector space theory. (Of course vector spaces are more well-known than modules, and hence more appropriate for introductory examples.)

Given the context of this report another viewpoint is also possible. Some modules over integral domains are better suited for discrete geometry than vector spaces, and even though they are often embedded in a vector space this work shows that there is no reason to take the detour via vector space theory. Furthermore the module approach gives a simple characterisation of Hübler's geometries, as we will show in Chapter 5. This characterisation would be more awkward if we had to go via subsets of vector spaces.

On the other hand, there is no claim made that modules over integral domains (with some discreteness assumption added) should provide a suitable framework for discrete geometry in general. On the contrary, a good axiomatisation of discrete geometry should have many models of different kinds. One example which is difficult to treat within this framework is the geometry of a cylinder; the closure of a singleton is likely to include all points in a circle going around the cylinder. Finite geometries also seem hard to model. The standard example of a finite model used here, the $\mathbb{Z}_n$-module over $(\mathbb{Z}_n)^m$, has a strange geometry where lines "wrap around" in a fashion that may not always be wanted.

The modules that are not torsion free are not as naturally embedded in their associated vector spaces, and hence cannot (easily) be treated within vector space theory. However, all interesting modules considered by the author are torsion free when viewed in a proper way. For instance, the $\mathbb{Z}$-module over $(\mathbb{Z}_2)^2$ is not torsion free, but the $\mathbb{Z}_2$-module is, and the second example is arguably more natural. Similarly, the $\mathbb{Z}$-module over $\mathbb{Z} \times \mathbb{Z}_2$ is not torsion free, but it has the same associated vector space and essentially the same structure as the $\mathbb{Z}$-module over $\mathbb{Z}$, which is torsion free. It is an open question whether there are any "interesting" modules over integral domains that are not torsion free.

Chapter 4

Modules with Order

In this chapter we assume that all modules are modules over ordered domains. These modules, together with a natural notion of half-spaces, are shown to be oriented matroids.

First we show that the points of a line can be totally ordered. This actually shows that every line is an oriented matroid, and the results are used as a special case when showing that a-matroids are oriented matroids. These line orders are also used when characterising Hübler geometries in Chapter 5.

Convexity is introduced in two ways. First we explicitly define a convex closure operator. This operator yields an antimatroid whenever the module is torsion free, thereby motivating the use of the term convex. The closed sets are shown to be betweenness closed, using the line orders previously defined. The second way in which convexity is introduced is via oriented matroid theory. The relation between the two closure operators is partly investigated, but more work needs to be done in that area.

4.1 Ordered Lines

By using the order of ordered domains we can easily induce orders on the points of every line.

Theorem 4.1. *Let M be an R-module and $\ell = \langle p, q\rangle_{\mathrm{a}} \subseteq M$ an a-rank 2 subspace. Take any two a-rank 1 subspaces $r_1, r_2 \subseteq M$ with elements $r_1' \in r_1$, $r_2' \in r_2$ given by $b_i r_i' = a_{i1}p + a_{i2}q$, $b_i, a_{ij} \in R$, $b_i = a_{i1} + a_{i2} > 0$, $i, j \in \{1, 2\}$. These subspaces satisfy $r_1 = r_2$ iff $a_{12}a_{21} = a_{11}a_{22}$, independently of which members r_1', r_2' were chosen.*

Define $r_1 < r_2$ iff $a_{12}a_{21} < a_{11}a_{22}$. This is well-defined, and the relation $\leq := < \bigcup =$ is a total order on the a-rank 1 subspaces of ℓ. By swapping p and q we get the opposite order ($\geq$), and these two orders are invariant when going to another basis of ℓ.

Note that in the case of torsion free modules this means that the points of every line can be totally ordered. For convenience we define $r_1' < r_2'$ ($r_1' \leq r_2'$) iff $r_1' \in r_1$, $r_2' \in r_2$ and $r_1 < r_2$ ($r_1 \leq r_2$). Beware that when the module is not torsion free these new relations do not have all their usual properties, though.

Before we prove this proposition let us establish a simple equivalence.

Lemma 4.2. *Given two points r_1, r_2 as in Theorem 4.1 above the following statements are equivalent:*

1. $a_{12}a_{21} < a_{11}a_{22}$.
2. $b_1a_{21} < b_2a_{11}$.
3. $b_2a_{12} < b_1a_{22}$.

The same applies to the corresponding equalities, i.e. when $=$ is substituted for $<$.

Proof. Since $b_i = a_{i1}+a_{i2}$, $i \in \{1,2\}$ we get the equivalence by additive isotony of $<$; for $1 \Leftrightarrow 2$ add or subtract $a_{11}a_{21}$, and for $1 \Leftrightarrow 3$ add or subtract $a_{12}a_{22}$. The same method works for the equalities. □

Proof of Theorem 4.1. Take any two members $r_1'' \in r_1$, $r_2'' \in r_2$ with $b_i'r_i'' = a_{i1}'p + a_{i2}'q$, $b_i', a_{ij}' \in R$, $b_i' = a_{i1}' + a_{i2}' > 0$, $i,j \in \{1,2\}$. These elements could be different from the one chosen above, or the same elements, perhaps with different representations. We know that $c_1r_1'' = c_1r_1'$ and $c_2r_2'' = c_2r_2'$ for some $c_1, c_2 \in R^+$. Hence by the Representation Theorem 3.15 and multiplicative isotony (and $b_i, c_i > 0$, $i \in \{1,2\}$) we get $a_{12}a_{21} < a_{11}a_{22}$ iff $a_{12}'a_{21}' < a_{11}'a_{22}'$. This shows that $<$ is well-defined.

Similarly we get $a_{12}a_{21} = a_{11}a_{22}$ iff $a_{12}'a_{21}' = a_{11}'a_{22}'$. Furthermore $a_{12}a_{21} = a_{11}a_{22} \Leftrightarrow r_1'$ and r_2' have the same representations (compare the Representation Theorem) $\Leftrightarrow r_1'$ and r_2' are members of the same a-rank 1 subspace $\Leftrightarrow r_1 = r_2$. Hence we know that $r_1 = r_2$ iff $a_{12}a_{21} = a_{11}a_{22}$, independently of the choice of r_1', r_2'.

Now let us check if $\leq$ is a total order. It is obviously reflexive, and the results above imply that it is antisymmetric and total. Assume that $r_1 \leq r_2 \leq r_3$, i.e. (using Lemma 4.2) $b_1a_{21} \leq b_2a_{11}$ and $b_2a_{31} \leq b_3a_{21}$ ($r_3' \in r_3$, $b_3r_3' = a_{31}p+a_{32}q$, $b_3 = a_{31} + a_{32} > 0$). Since $b_i > 0$, $i \in \{1,2,3\}$ we can pass to the field of fractions associated with R. We get $\frac{a_{11}}{b_1} \geq \frac{a_{21}}{b_2} \geq \frac{a_{31}}{b_3}$, and hence $b_1a_{31} \leq b_3a_{11}$, $r_1 \leq r_3$, and $\leq$ is transitive. Thus $\leq$ is a total order.

It is obvious that we get the converse relation by swapping p and q in the definition. It remains to show that the orders are invariant when going to another basis of the line. It is enough to show that this holds when q is exchanged for another point $q' \neq p$; if both points are exchanged then this can be accommodated by two of these exchanges (plus some swapping of the order of the basis elements). Assume $dq' = d_1p + d_2q$ with $d = d_1 + d_2 > 0$. (Here $d_2 \neq 0$ since $q' \neq p$.) Then

$$d_2b_1r_1' = (d_2a_{11} - d_1a_{12})p + a_{12}dq' \tag{4.1}$$

and

$$d_2b_2r_2' = (d_2a_{21} - d_1a_{22})p + a_{22}dq'. \tag{4.2}$$

Now assume that $r_1 < r_2$, i.e. $a_{12}a_{21} < a_{11}a_{22}$. If $d_2 > 0$ we get $dd_2a_{21}a_{12} < dd_2a_{11}a_{22}$. By subtracting $dd_1a_{12}a_{22}$ from both sides we get

$$a_{12}d(d_2a_{21} - d_1a_{22}) < (d_2a_{11} - d_1a_{12})a_{22}d, \tag{4.3}$$

i.e. $r_1 < r_2$ also in the order based on p and q'. If $d_2 < 0$ we get

$$a_{12}d(d_2a_{21} - d_1a_{22}) > (d_2a_{11} - d_1a_{12})a_{22}d, \tag{4.4}$$

i.e. the opposite order. (The coefficients d_2b_1 and d_2b_2 are negative, so all coefficients have to be negated. However, since this applies to both points it does not make any difference.) $\square$

4.2 Convexity

We will now treat convexity. We depart from the procedure in the sections introducing the d- and a-closure. Instead of defining the closed sets first, and then deducing the closure operator, we just define the closure operator. Note that this convex hull operator is related to the standard vector space convex hull operator (as given in [FF00]),

$$[S]_{\mathrm{V}} := \left\{ \sum_{i=1}^{n} a_i s_i \,\middle|\, s_i \in S,\ a_i \in R,\ \sum_{i=1}^{n} a_i = 1,\ 0 \le a_i \le 1,\ n \in \mathbb{Z}^+ \right\}, \tag{4.5}$$

in the same way as the d- and a-closures are related to the corresponding vector space closures. Of course the operators are chosen to agree in the case where the module is a vector space.

Theorem 4.3. *Let M be an R-module. Define the* convex hull operator $[\cdot] : \wp(M) \to \wp(M)$ *by*

$$[S] := \left\{ m \in M \,\middle|\, \begin{array}{c} bm = \sum_{i=1}^{n} a_i s_i,\ s_i \in S,\ b, a_i \in R, \\ b = \sum_{i=1}^{n} a_i > 0,\ 0 \le a_i \le b,\ n \in \mathbb{Z}^+ \end{array} \right\}.$$

Then $(M, [\cdot])$ is a finitary closure space satisfying $[\emptyset] = \emptyset$. The subspaces are called convex. *Furthermore the following properties are equivalent:*

1. *M is torsion free,*
2. *$(M, [\cdot])$ is an antimatroid, and*
3. *$(M, [\cdot])$ is simple.*

Maybe it is inappropriate to use the term convex if the anti-exchange property does not hold, but that is a minor issue. Note, by the way, that the requirement $a_i \le b$ is redundant; it is included mostly for clarity.

Before we prove this result, let us introduce some more terminology.

Definition 4.1. *Let M be an R-module, and let $p, q \in M$. If $\{ p, q \}$ is a-independent then the* line segment $\langle p, q \rangle_\ell$ *is defined by*

$$\langle p, q \rangle_\ell := \{ r \in \langle p, q \rangle_{\mathrm{a}} \mid p \le r \le q \text{ or } p \ge r \ge q \}.$$

Here $\le$ and $\ge$ are the two point orders for $\langle p, q \rangle_{\mathrm{a}}$ as given in Theorem 4.1. If $\{ p, q \}$ is not a-independent then $\langle p, q \rangle_\ell := \langle p, q \rangle_{\mathrm{a}}$.

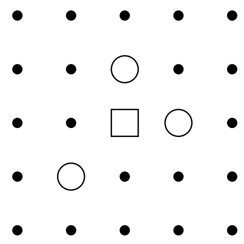

Figure 4.1: The set consisting of the three circles is betweenness closed, but the convex hull of these points also includes the square. This standard example in the $\mathbb{Z}$-module over $\mathbb{Z}^2$ is presented in e.g. Hübler's report [Hüb89].

Note that the word "line" is not really correct when M is not torsion free, but the alternative "a-rank 2 subspace segment" is too unwieldy.

The following result is easy to verify.

Lemma 4.4. *The line segment $\langle p, q\rangle_\ell$ is equal to $[p, q]$.*

Proof. First note that for any single point $r \in M$ we have $[r] = \langle r\rangle_{\mathrm{a}}$. If $\{ p, q \}$ is not a-independent then $\langle p, q\rangle_{\mathrm{a}}$ has a-rank 1, so $\langle p, q\rangle_\ell = \langle p, q\rangle_{\mathrm{a}} = [p, q]$. Hence we can assume that $\{ p, q \}$ is a-independent.

Now assume that $r \in \langle p, q\rangle_\ell$. We get that $br = a_1p+a_2q$, where $b = a_1+a_2 > 0$ and either $0 \le a_1, a_2$ or $0 \ge a_1, a_2$. The second case is clearly impossible (since we chose $b > 0$) so we have $r \in [p, q]$.

Assume conversely that $r \in [p, q]$. Then $br = a_1p+a_2q$ where $b = a_1+a_2 > 0$ and $a_1, a_2 \ge 0$. We obviously have $0a_2 \le 1a_1$ and $0a_1 \le 1a_2$, so $p \le r \le q$ (or $q \le r \le p$), and we are done. □

Corollary 4.5. *A convex set is betweenness closed; i.e., if $p, q \in [S]$, then $\langle p, q\rangle_\ell \subseteq [S]$.*

The converse is not true, a betweenness closed set may not be convex. See Figure 4.1.

We also need the following lemma.

Lemma 4.6. *Let M be a torsion free R-module with $p, q, r, s \in M$. If $q \ne r$, $q \in \langle p, r\rangle_\ell$, and $r \in \langle q, s\rangle_\ell$ then $q, r \in \langle p, s\rangle_\ell$.*

If $q = r$ then all we know is that the two lines $\langle p, r\rangle_{\mathrm{a}}$ and $\langle q, s\rangle_{\mathrm{a}}$ intersect in $p = q$, so there is no reason to suspect that the two lines should be equal.

Proof. Since $q \ne r$ we know that $p \ne r$ and $q \ne s$. Hence $b_1q = a_{11}p + a_{12}r$ and $b_2r = a_{21}q + a_{22}s$, $b_i = a_{i1} + a_{i2} > 0$, $a_{ij} \ge 0$. We get that $b_2b_1q = b_2a_{11}p + a_{12}(a_{21}q + a_{22}s)$, i.e. $(b_2b_1 - a_{12}a_{21})q = b_2a_{11}p + a_{12}a_{22}s$. Since $q \ne r$ we know that $a_{12} < b_1$ and $a_{21} < b_2$. Hence $b_2b_1 - a_{12}a_{21} > 0$. Furthermore $b_2b_1 - a_{12}a_{21} = b_2(a_{11} + a_{12}) - a_{12}(b_2 - a_{22}) = b_2a_{11} + a_{12}a_{22}$. Hence $q \in \langle p, s\rangle_\ell$. The other part is shown analogously. □

Now on to the proof we postponed.

Proof of Theorem 4.3. By construction $[\cdot]$ is increasing, monotone, and finitary, and $\mathrm{cl}(\emptyset) = \emptyset$ holds. It is easy to see that $[\cdot]$ is simple iff M is torsion free.

The idempotence is proved as follows: Assume that $m \in [[S]]$ for some $S \subseteq M$. It follows that $bm = \sum_{i=1}^{n} a_i s_i$ for some $a_i \in R$, $s_i \in [S]$, and $n \in \mathbb{Z}^+$, and furthermore for each i that $b_i s_i = \sum_{j=1}^{n_i} a_{ij} s_{ij}$ for some $a_{ij} \in R$, $s_{ij} \in S$, and $n_i \in \mathbb{Z}^+$. All these coefficients also satisfy the restrictions given in the definition of $[\cdot]$. We get that

$$\left(b \prod_{i=1}^{n} b_i\right) m = \sum_{i=1}^{n} a_i \left(\prod_{j \neq i} b_j\right) \sum_{j=1}^{n_i} a_{ij} s_{ij}. \tag{4.6}$$

We also have

$$0 < b \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i \left(\prod_{j \neq i} b_j\right) \sum_{j=1}^{n_i} a_{ij}. \tag{4.7}$$

Since furthermore all coefficients are nonnegative this means that $b \in [S]$, and we are done with the idempotence.

Suppose now that M is not torsion free. Then there are two elements, $b \in R \setminus 0$ and $m \in M \setminus 0$, such that $bm = 0$. It follows that $0, m \in [m]$. Hence $0 \in [\emptyset \cup m] \setminus m$, but $m \notin [\emptyset \cup m] \setminus m$. Since $m \in [0]$ this means that $[\emptyset \cup m] \setminus m$ is not convex, and because $\emptyset$ is convex this violates one of the properties of Proposition 2.1. Hence the anti-exchange property is not satisfied and $(M, [\cdot])$ is not an antimatroid.

From now on M is assumed to be torsion free. It remains to prove that in this case the anti-exchange property does hold. Assume for a contradiction that $x, y \in M$, $x \neq y$, $S \subseteq M$, and both $y \in [S \cup x] \setminus [S]$ and $x \in [S \cup y]$. Since this implies that $x \notin [S]$ we get that $b_1 y = \sum_{i=1}^{n_1} a_{i1} s_{i1} + a_1 x$ and $b_2 x = \sum_{i=1}^{n_2} a_{i2} s_{i2} + a_2 y$, where $a_1, a_2 \neq 0$, $s_{ij} \in S$ and the coefficients satisfy the usual restrictions.

In the case of vector spaces we can go on to prove that x and y are both on the same line, in between members of $[S]$, which is absurd. This does not work in some discrete cases, though. The intuition behind this is given in Figure 4.2. Fortunately we can make the procedure work by scaling the problem. For this we need the results of the following paragraph.

Notice that if $n \in \mathbb{Z}^+$, $a_i \in R$, $a_i \geq 0$, $\sum_{i=1}^{n} a_i > 0$, and $s_i \in S$, then $\sum_{i=1}^{n} a_i s_i \in [(\sum_{i=1}^{n} a_i) S]$. (Here $aS = \{ as \mid s \in S \}$.) This follows since

$$\left(\sum_{j=1}^{n} a_j\right) \sum_{i=1}^{n} a_i s_i = \sum_{i=1}^{n} a_i \left(\left(\sum_{j=1}^{n} a_j\right) s_i\right). \tag{4.8}$$

Furthermore for any $a \in R \setminus 0$ we have $m \in [S]$ iff $am \in [aS]$ (since M is torsion free).

Now let $c_1 = \sum_{i=1}^{n_1} a_{i1}$ and $c_2 = \sum_{i=1}^{n_2} a_{i2}$, and define $c = c_1 c_2$. Since $x \neq y$ we know that $c_1, c_2 > 0$, so we get that $z_1 := \sum_{i=1}^{n_1} a_{i1} s_{i1} \in [c_1 S]$ and

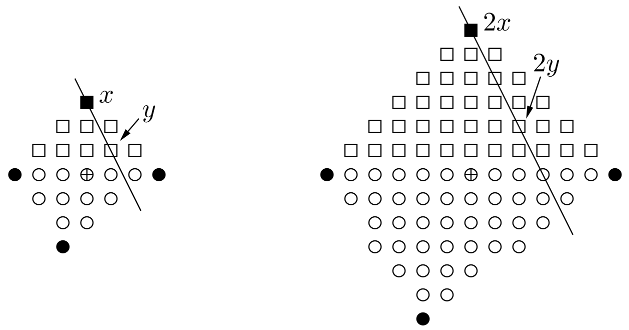

Figure 4.2: This is an illustration of why, intuitively, a method for proving the anti-exchange property of the convex hull operator $[\cdot]$ which works for vector spaces may fail in a discrete setting; see the proof of Theorem 4.3 for details about this method. The particular discrete setting is the $\mathbb{Z}$-module over $\mathbb{Z}^2$. Start with the left figure. The round points constitute a convex set (S) with the black points as extreme points. The convex hull of these points together with x consists of all the points in the figure. We can see that the line through x and y "passes through" S without intersecting. In a vector space this could not have happened. Fortunately, by "enlarging" the setup, scaling all extreme points, as in the figure to the right, we can make the line intersect S. The scaling is done by a factor of 2 around the origin, the point with the crosshairs.

$z_2 := \sum_{i=1}^{n_2} a_{i2}s_{i2} \in [c_2S]$. Thus we also have $c_2z_1, c_1z_2 \in [cS]$. Furthermore we know that $b_1(cy) = c_1(c_2z_1) + a_1(cx)$ and $b_2(cx) = c_2(c_1z_2) + a_2(cy)$. Since $c_i = b_i - a_i$, $i \in \{ 1, 2 \}$ this implies that $cx \in \langle c_1z_2, cy\rangle_\ell$ and $cy \in \langle c_2z_1, cx\rangle_\ell$. In other words (due to Lemma 4.6 and $cx \neq cy$) $cx, cy \in \langle c_2z_1, c_1z_2\rangle_\ell$. Since convex sets are betweenness closed this implies that $cx, cy \in [cS]$, and hence $x, y \in [S]$. This is a contradiction, and we are done. □

Let us now show how we could have proved that $(M, [\cdot])$ is an antimatroid using the associated vector space, assuming that we know that $(F(M), [\cdot]_\mathrm{V})$ is an antimatroid.

Proposition 4.7. *Let M be an R-module with convex hull operator $[\cdot]$, and let $[\cdot]_\mathrm{V}$ be the convex hull operator of $F(M)$. Then*

$$[S] = \pi^{-1}\big([\pi(S)]_\mathrm{V}\big),$$

where π and π^{-1} are the functions defined in Section 3.7.

Recall that for a torsion free module this means that $[S] = [S]_\mathrm{V} \cap M$.

Proof. This proof is almost identical to the proof of the corresponding statement for d-closure in Theorem 3.18. □

Corollary 4.8. *The pair $(M, [\cdot])$ is an antimatroid whenever M is torsion free.*

Proof. Everything except for idempotence follows immediately since $[S] = [S]_{\mathrm{V}} \cap M$ and $(F(M), [\cdot]_{\mathrm{V}})$ is an antimatroid. The idempotence is not as straightforward to prove in this way since $[[A]] = [[A]_{\mathrm{V}} \cap M]_{\mathrm{V}} \cap M$, so we cannot immediately take advantage of the idempotence of $[\cdot]_{\mathrm{V}}$. Of course the method we used in the proof of Theorem 4.3 still works, though. □

It is obvious from this example that the method of reusing knowledge about vector spaces can lead to shorter proofs. However, compared to the proof above the proof of Theorem 4.3 shows more clearly *why* the anti-exchange property holds in the discrete case. This motivates keeping that proof in the text.

4.3 Oriented Matroids from Modules

We will now take another approach to convexity by using oriented matroid theory. First we have to show in what way we get an oriented matroid from a module. The following definition was suggested by Mike Smyth.

Definition 4.2. *Let M be an R-module, $H \subseteq M$ a d-hyperplane, and $x \in M \backslash H$. Define the* open half-spaces *H^+ and H^- by*

$$H^+ := \{\, m \in M \mid bm = h + ax,\ a, b \in R,\ h \in H,\ ab > 0 \,\}$$

and

$$H^- := \{\, m \in M \mid bm = h + ax,\ a, b \in R,\ h \in H,\ ab < 0 \,\}\,.$$

For ease of reference let us denote x as the positive point *of H.*

Let us now show that this definition is consistent with oriented matroid terminology, and that it gives rise to an oriented matroid structure on M.

Lemma 4.9. *The open half-spaces of Definition 4.2 are both non-empty, and they satisfy $H^+ \cup H^- = M \setminus H$ and $H^+ \cap H^- = \emptyset$. In other words (H^-, H^+) is a cocircuit, as is the opposite (H^+, H^-). Furthermore the half-spaces are independent of the choice of positive point $x \in M \setminus H$; given another $x' \in H^+$ we get the same cocircuit, and given $x'' \in H^-$ we get the opposite cocircuit.*

Proof. The sets are both non-empty since $ab = 1$ and $ab = -1$ is always possible.

Let B be a basis of H. Then $B \cup x$ is a d-basis of $\langle B \cup x \rangle_{\mathrm{d}} = M$. By the Representation Theorem 3.15 we get that $H^+ \cap H^- = \emptyset$. Furthermore it follows that $\{\, m \in M \mid bm = h + ax,\ a, b \in R,\ h \in H,\ b \neq 0 \,\} = M$, and since $m \in H$ iff $a = 0$ in this set generator we get that $H^+ \cup H^- = M \setminus H$.

Given $x' \in H^+$ we know that $bx' = h + ax$ for some $h \in H$, $a, b \in R$, $ab > 0$. Denote the open half-spaces constructed from x' instead of x by H'^+ and H'^-. Assume that $m' \in H'^+$. We get that $b'm' = h' + a'x'$ for some $h' \in H$, $a', b' \in R$, $a'b' > 0$. Hence $bb'm' = (bh' + a'h) + a'ax$. Since $bh' + a'h \in H$ and $aa'bb' > 0$ this implies that $m' \in H^+$. The other direction is analogous, $H'^+ = H^+$, and hence also $H'^- = H^-$. The proof for a point $x'' \in H^-$ is almost identical. □

Lemma 4.10. *Let M be an R-module with two different d-hyperplanes H and K intersecting in a subspace L of d-corank 2. Let x be any point in $M \setminus (H \cup K)$. Choose the orientation of the cocircuits associated with H and K such that $x \in H^+ \cap K^-$. Then the hyperplane $L = x \vee (H \wedge K)$ satisfies $L^+ \subseteq H^+ \cup K^+$ and $L^- \subseteq H^- \cup K^-$, given a suitable choice of its orientation.*

Proof. Let us choose the orientations by letting x be the positive point for H, $-x$ be the positive point for K, and any point $y \in H^+ \cap K$ be the positive point for L. (We know that $H^+ \cap K \neq \emptyset$ because $K \neq H$, $H^\pm \neq \emptyset$, and K since it is a d-submodule is closed under negation and hence intersects both H^+ and H^-. Furthermore $H^+ \cap K \cap L = \emptyset$ since $L \cap K = L \wedge K = H \wedge K$, so $y \in M \setminus L$.)

Take any point $p \in L^+$. This point satisfies $dp = \ell + cy$ for some $\ell \in L$, $c, d \in R$, $cd > 0$. Now take any basis B of $H \wedge K$. We get that $B \cup x$ is a basis of L. It follows that we have $b\ell = g + ax$ for some $a, b \in R$, $b \neq 0$, $g = \sum_{i=1}^{n} a_i b_i$, $a_i \in R$, $b_i \in B$, $n \in \mathbb{N}$. Hence $bdp = g + ax + bcy$.

We have three cases, depending on the sign of ab.

$ab = 0$ Then $\ell \in H \wedge K$, so since $y \in H^+$ and $cd > 0$ we get that $p \in H^+$. (Remember that any point in H^+ can be regarded as the positive point of H^+.)

$ab < 0$ Then $\ell \in H^- \cap K^+$. Here we have two subcases depending on the sign of d. If $d > 0$ then $abd < 0$ which implies that $p \in K^+$ since $g + bcy \in K$. If on the other hand $d < 0$ then we have to use that $\kappa y = h + \lambda x$ for some $h \in H$, $\kappa, \lambda \in R$, $\kappa\lambda > 0$. It follows that $\kappa bdp = \kappa(g + ax) + bc(h + \lambda x) = \kappa g + bch + (\kappa a + bc\lambda)x$. Since $\kappa g + bch \in H$ and $\kappa bd(\kappa a + bc\lambda) = \kappa^2(ab)d + (\lambda\kappa)b^2(cd) > 0$ we get that $p \in H^+$.

$ab > 0$ Here $\ell \in H^+ \cap K^-$. This case is entirely analogous to the previous one.

Thus we know that $p \in H^+ \cup K^+$. The proof of $L^- \subseteq H^- \cup K^-$ is symmetrical to this one. □

We can sum up the preceding results in a theorem.

Theorem 4.11. *Every module over an ordered domain has an oriented matroid structure given by its d-matroid structure and the open half-spaces as defined in Definition 4.2.*

Note that for some modules, e.g. our ubiquitous example, the $\mathbb{Z}$-module over $(\mathbb{Z}_n)^m$, the oriented matroid structure obtained is trivial. This follows since the underlying matroid is trivial and does not have any hyperplanes.

The half-spaces also satisfy the following property regarding the "quadrants" induced by two hyperplanes.

Proposition 4.12. *Let M be an R-module with two different d-hyperplanes H and K intersecting in a subspace Λ of d-corank 2. Assume $h \in H \setminus K$ and $k \in K \setminus H$. Then*

$$M \setminus (H \cup K) = \{\, m \in M \mid bm = \lambda + a_1 h + a_2 k,\ \lambda \in \Lambda,\ a_1, a_2, b \in R \setminus 0 \,\},$$

and this set is partitioned into four sets characterised by the signs of $a_1 b$ and $a_2 b$.

Proof. Since $\pm 1 \in R$ all the four sets are nonempty.

Let B be a basis of Λ. Then $B \cup h \cup k$ is a basis of M, whereby the Representation Theorem 3.15 implies that the four sets are disjoint. Furthermore $\{\, m \in M \mid bm = \lambda + a_1 h + a_2 k,\ \lambda \in \Lambda,\ a_1, a_2, b \in R,\ b \neq 0 \,\} = M$, and in this set generator $m \in H \cup K$ iff $a_1 a_2 = 0$. □

It is easy to extend the results above to a-matroids. Given a hyperplane H of an a-matroid M with $h \in H$ and $x \in M \setminus H$ we get that $H - h$ is a d-hyperplane, and by letting $x - h$ be the positive point we get the cocircuit $((H-h)^-, (H-h)^+)$. Hence we can define the cocircuit associated with H with x as positive point to be $(H^-, H^+) := ((H-h)^- + h, (H-h)^+ + h)$. It is easy to check that this is a proper cocircuit, and that it is independent of the choice of $h \in H$.

Furthermore, if the a-rank of the module is at least 3 the proof of the oriented matroid axiom can be reduced to the proof given for d-matroids above. This is done by observing that the three hyperplanes all intersect in a common point, and this point can be translated to the origin. This does not work when the a-rank is 2, though, since then the hyperplanes intersect in a subspace of rank 0, i.e. they are disjoint. In this case we reduce the problem to another result instead. Theorem 4.1 shows that the a-rank 1 subspaces in an a-rank 2 subspace can be totally ordered. It is straightforward to check that given an a-rank 1 hyperplane $H \subseteq M$ we have (given suitable orientations) $H^+ = \{\, r \in M \mid \langle r \rangle_{\mathrm{a}} > H \,\}$ and $H^- = \{\, r \in M \mid \langle r \rangle_{\mathrm{a}} < H \,\}$. Now it is easy to verify that the oriented matroid axiom is satisfied.

Denote the convex closure obtained from a-matroids as above by $[\cdot]_{\mathrm{OM}}$. It is easy to show that $[S] \subseteq [S]_{\mathrm{OM}}$ using some simple algebraic manipulations. It is unknown whether the converse holds though, although it seems likely.

4.4 Conclusions and Future Work

Geometry without order and convexity is not very useful. Oriented matroids and antimatroids provide different frameworks for treating convexity, and in this chapter we have shown that modules over ordered domains can be fitted into both frameworks.

This chapter also provides more arguments for why modules over ordered domains do not provide a good framework for discrete geometry in general; compare the conclusion of Chapter 3. Since finite rings cannot be ordered, if convexity is an issue then finite geometries are even harder to model in this framework than we have already pointed out.

There is more work to be done here, e.g. properly relating the two different convex closures. This should be done in the general case, not just for modules. The relation is not fully explored even in the finite case, but it is known that so-called "simple acyclic" finite oriented matroids satisfy the anti-exchange property [Ede82, BLVS+93]. Since the convex closure we have defined for an oriented matroid is finitary, and in the acyclic finite case corresponds exactly to the standard oriented matroid convex closure [BLVS+93], it is probably straightforward to extend that result to the infinite case. It is probably useful to also include the convexity related to involution-OMs when the different convexity approaches are compared.

These examples are part of a more extensive task, namely to develop the infinite oriented matroid theory further. Probably many results valid in the finite case can be straightforwardly extended to the infinite case. Exploring the different possible axiomatisations and the relations between them is also important. What this chapter does in this respect is showing that the axiomatisation used has infinite models.

Chapter 5

Hübler Geometries and Modules

This chapter characterises Hübler geometries exactly in terms of the geometries investigated in Chapters 3 and 4. The correspondence is really simple, only a few properties to ensure discreteness are needed. This characterisation gives a new view of Hübler geometries which is arguably easier to understand, at least for those who are used to abelian groups and matroids.

We also define isomorphism for Hübler geometries, something which Hübler does not do, and explore a few results. All geometries are completely characterised by their translation groups, so isomorphism is naturally defined as group isomorphism.

Finally an exemplifying group of models is briefly treated, the square grid geometries. This example shows that there are Hübler geometries of any "dimension" larger than or equal to 2, including infinite dimensions.

Remember that, unless otherwise stated, all modules are assumed to be modules over integral domains.

5.1 Characterisation of Hübler Geometries

By using the results from Chapter 3 we can show that any Hübler geometry is a geometry.

Lemma 5.1. *The translation group $\mathcal{T}$ of a Hübler geometry, when viewed as a $\mathbb{Z}$-module, is torsion free.*

Proof. Let $S, T \in \mathcal{T}$, and suppose $S^k = T^k$ for some $k \in \mathbb{Z} \setminus \{ 0 \}$. Since $S^k = T^k$ we have $\left(S \circ T^{-1}\right)^k = \mathsf{id}$. There are no cyclic translations except for id, and because $k \neq 0$ this implies that $S \circ T^{-1} = \mathsf{id}$. Hence $S = T$. $\square$

Corollary 5.2. *The translation group $\mathcal{T}$ of a Hübler geometry is an a-geometry.*

Let us now determine what kind of correspondences we can draw between the geometrical constructions of an a-geometry and a Hübler geometry. First note that there is a superficially significant difference between those two kinds of geometries in that Hübler geometries have both points and translations, while the a-geometries only have points. However, by choosing an origin we get that each Hübler point corresponds to a unique translation (by the first four axioms). From now on we will assume that an origin has been chosen, thereby avoiding the differences by identifying points and translations.

Furthermore, from now on we will adopt the more geometrical notation from Chapter 3 and treat $\mathcal{T}$ as an additive group. Thus we will use $+$ instead of $\circ$, 0 instead of id, and multiplication instead of exponentiation. The function composition notation used in Section 2.6 is what Hübler used [Hüb89], and is suitable when you view translations as functions and separate the points from the translations.

Let us begin by characterising Hübler lines and parallelity in the context of module geometries. We begin with a simple lemma.

Lemma 5.3. *If s is a simple translation and $ms = nt$ for some $t \in \mathcal{T}$ and $m, n \in \mathbb{Z}$, then $n|m$.*

Proof. We know that there is some simple translation s' such that $t = ks'$ for some $k \in \mathbb{N}$ (just take one of the generators of the line $\ell(0, t)$). The lemma follows immediately if $m = 0$, so assume $m \neq 0$. We have $ms = kns'$, which since $m \neq 0$ implies that s and s' generate the same line. (Given two different points, e.g. 0 and $ms \neq 0$, Axiom 1 says that there is exactly one line containing those points.) Hence $s = s'$ or $s = -s'$, so since $\mathcal{T}$ is torsion free $m = \pm kn$ and the lemma follows. □

Proposition 5.4. *A set of points is a Hübler line iff it is an a-rank 2 subspace (an a-geometry line).*

Proof. Let $\{ p, q \}$ be the basis of an a-rank 2 subspace. Take the simple translation g with $kg = q - p$. Let $\ell = \{ p + ng \mid n \in \mathbb{Z} \}$ (this is a Hübler line). For any $r = p + ng \in \ell$ we have $1r = p + ng = p + n(q - p) = (1 - n)p + nq$ and we get $\ell \subseteq \langle p, q \rangle_{\mathrm{a}}$. Now take any $r \in \langle p, q \rangle_{\mathrm{a}}$, i.e. $br = a_1 p + a_2 q$, $b = a_1 + a_2 \neq 0$. We can rewrite this as $br = (a_1 + a_2)p + a_2 g$, i.e. $b(r - p) = a_2 g$. Since g is simple we get by Lemma 5.3 that $a_2|b$, i.e. $r = p + \frac{a_2}{b} g$ (remember that the d-submodule $\langle g \rangle_{\mathrm{d}}$ is closed under existing divisors). Hence $\langle p, q \rangle_{\mathrm{a}} \subseteq \ell$, and thus $\langle p, q \rangle_{\mathrm{a}} = \ell$. This means that all a-rank 2 subspaces are Hübler lines.

Now take a Hübler line ℓ with generator g and a point $p \in \ell$. Notice that since g is simple we have (using Lemma 5.3)

$$\ell = p + \{ ng \mid n \in \mathbb{Z} \} = p + \langle g \rangle_{\mathrm{d}} = \langle p, p + g \rangle_{\mathrm{a}}, \tag{5.1}$$

and the proof is finished. □

Proposition 5.5. *The Hübler parallelity $||$ and the relation $|||$ (as defined in Proposition 3.22) over the corresponding a-geometry coincide.*

Proof. Two lines ℓ and ℓ' are Hübler parallel iff $t + \ell = \ell'$ for some translation $t \in \mathcal{T}$. This obviously coincides with $|||$. □

We will show below (Theorem 5.13) that the $\mathbb{Z}$-module over $\mathbb{Z}^2$ is a Hübler geometry. Hence the example of Section 3.8 demonstrating that the "standard" a-geometry parallelity and $|||$ do not coincide is applicable to (some) Hübler geometries as well.

Let us now characterise Hübler orders. Generators and the generator property are defined in Section 3.9.

Lemma 5.6. *Let M be an a-geometry over an R-module, where R is an ordered domain. Let $\ell \subseteq M$ be a line with a generator g and a point $p \in \ell$, and let an order on the points of the line be defined as in Theorem 4.1. Then, depending on which of the opposite orders is chosen, we either have $p + r_1 g < p + r_2 g$ iff $r_1 < r_2$, or $p + r_1 g < p + r_2 g$ iff $r_1 > r_2$ (for arbitrary $r_1, r_2 \in R$).*

Proof. Note that $\{ p, p + g \}$ is a basis of ℓ (since $g \neq 0$). For $i \in \{ 1, 2 \}$ let $q_i = p + r_i g$ and assume that $b_i q_i = a_{i1} p + a_{i2}(p + g)$ for some coefficients in R with $b_i = a_{i1} + a_{i2} > 0$. By using both relationships for q_i we get $b_i(p + r_i g) = b_i q_i = (a_{i1} + a_{i2})p + a_{i2}(p + g - p) = b_i p + a_{i2} g$. By subtracting $b_i p$ and using that M is torsion free we get that $b_i r_i = a_{i2}$, $i \in \{ 1, 2 \}$. We have (by Lemma 4.2) that $q_1 < q_2$ iff $b_2 a_{12} < b_1 a_{22}$, which by the equalities above is equivalent to $b_1 b_2 r_1 < b_1 b_2 r_2$. Since $b_1 b_2 > 0$ this is in turn equivalent to $r_1 < r_2$. If we had swapped p and $p + g$ then we would have gotten the opposite order, $r_1 > r_2$, and we are done. □

Corollary 5.7. *A point order $<$ of a Hübler line is identical to one of the point orders $<$ and $>$ of the corresponding a-geometry line (as defined in Theorem 4.1).*

Proof. Just note that all Hübler geometries satisfy the generator property. □

Let us also find a sufficient condition for when each line has exactly two generators, as in Hübler geometries.

Corollary 5.8. *If g is a generator of a line ℓ in an a-geometry M over a $\mathbb{Z}$-module, then $-g \neq g$ is the only other generator of ℓ.*

Proof. Note that if g is a generator, then $-g$ is also a generator, but we may have $g = -g$.

Let us first show that $g \neq -g$. If $g = -g$ then $2g = 0$, which since $g \neq 0$ leads to a contradiction; M is torsion free.

Now assume that g' generates ℓ. Let $p \in \ell$. Then for every $n \in \mathbb{Z}$ there is some $n' \in \mathbb{Z}$ such that $p + ng = p + n'g'$, i.e. $ng = n'g'$. By choosing $n = 1$ we get $g = n'g'$. Since by the preceding lemma all points in the set $\{ p + mg' \mid 0 < m < |n'| \}$ lie between p and $p + g$ (or $p - g$ if $n' < 0$) we get that $|n'| \leq 1$, and since $n' \neq 0$ we are done. □

We now turn to planes and planar sets. Remember that a set of points S is of a-rank n iff $r\big(\langle S \rangle_{\mathrm{a}}\big) = n$.

Proposition 5.9. *A set of points is a Hübler planar set iff it is an a-rank 3 subset.*

Proof. All ranks mentioned in this proof are a-ranks.

Let S be a rank 3 subset (not necessarily a subspace). All the points of S cannot belong to one line (since then S would be a rank 2 subset). Hence condition 1 of the planar set definition is satisfied.

Now take four different points $p_i \in S$, $i \in \{ 1, 2, 3, 4 \}$. Let $\ell_1 = \langle p_1, p_2 \rangle_{\mathrm{a}}$, and let ℓ_2 be the unique line parallel to ℓ_1 with $p_3 \in \ell_2$, and $\ell_3 \,|||\, \ell_1$ with $p_4 \in \ell_3$. Assume that these three lines are all different. If p_1, p_3, and p_4 are all collinear then one of the lines lies between the two other lines and we are done (condition 2), so assume that these points are not collinear. It follows that

$\{ p_1, p_3, p_4 \}$ is a basis for $\langle S \rangle_a$. For simplicity, let us translate the entire problem to the d-submodule $\langle S \rangle_a - p_1$ with basis $\{ b_1, b_2 \} := \{ p_3 - p_1, p_4 - p_1 \}$. Since we are only interested in relations between points and not the actual points themselves this does not make any difference.

Let g be the common generator of the three lines (this generator is unaffected by the translation), and assume that

$$cg = a_1 b_1 + a_2 b_2, \tag{5.2}$$

$c, a_1, a_2 \in \mathbb{Z} \setminus 0$. (If $a_1 = 0$ or $a_2 = 0$, then two of the lines are equal, so this cannot be the case.) One of the lines is between the two others iff there exists three points, one from each line, that are collinear. We can fix one of the points without losing generality, so let us fix 0 (corresponding to $p_1 \in \ell_1$). Hence there are three collinear points of the kind specified iff

$$b_2 + ng \in \langle 0, b_1 + mg \rangle_a = \langle b_1 + mg \rangle_d \tag{5.3}$$

for some $m, n \in \mathbb{Z}$, i.e. iff

$$d(b_2 + ng) = e(b_1 + mg) \tag{5.4}$$

for some $d, e \in \mathbb{Z} \setminus 0$, $m, n \in \mathbb{Z}$.

Using (5.2) the last equation can be rewritten as

$$(a_1 nd - (c + a_1 m)e)b_1 = (a_2 me - (c + a_2 n)d)b_2. \tag{5.5}$$

This equation is equivalent to those above since $c \neq 0$. Since b_1 and b_2 are d-basis elements this equation has a solution iff both coefficients are equal to 0. This yields two equations which are equivalent to (using $c, d, e \neq 0$)

$$\begin{cases} a_1 d + a_2 e = 0, \\ a_1 m + a_2 n = -c. \end{cases} \tag{5.6}$$

Here a_1, a_2, and c are fixed, and we need to find values for d, e, m, n satisfying these equations.

We can always choose $d = a_2$, $e = -a_1$ to satisfy the first equation (note that a_1 and a_2 are nonzero). For the other equation first note that $\gcd(x, y)$ $(x, y \in \mathbb{Z} \setminus 0)$ is the smallest positive integer which can be written in the form $ux + vy$, $u, v \in \mathbb{Z}$ [MB67]. Hence the equation has a solution (m, n) if $\gcd(a_1, a_2) \,|\, -c$. Now note that

$$cg = a_1 b_1 + a_2 b_2 = \gcd(a_1, a_2) \left(\frac{a_1}{\gcd(a_1, a_2)} b_1 + \frac{a_2}{\gcd(a_1, a_2)} b_2 \right). \tag{5.7}$$

Since g is simple this implies, by Lemma 5.3, that $\gcd(a_1, a_2) \,|c$. In other words, one of the three lines lies between the other two lines, and S is a planar set.

Now let R be a planar set. Notice first that a planar set has rank at least 3 since it contains non-collinear points. Assume now that R is a subset of rank 4 or higher. Then there is a basis $B \subseteq R$ of $\langle R \rangle_a$ which has at least four elements. Take any four points from this basis. These points have to satisfy the second condition of the planar set definition. Let ℓ_i, $i \in \{ 1, 2, 3 \}$ be the associated lines. Since two parallel lines span a subspace of rank at most 3, and the four basis points span a subspace of rank 4, this implies that no two of the three lines can be equal. Hence, by the second planar set condition, one of the lines lies between the other two lines. This means that the subspace spanned by the lines is of rank 3, and hence R cannot have rank 4 or higher. □

Corollary 5.10. *A set of points is a Hübler plane iff it is an a-rank 3 subspace (an a-geometry plane).*

Proof. Take an a-rank 3 subspace S. By the previous proposition this is a planar set. We will show that it is Hübler-maximal, and hence a Hübler plane. Let M be the geometry in question. If $S = M$, then S is Hübler-maximal, so assume $S \neq M$ and take any point $p \in M \setminus S$. We have to show that $S \cup p$ is not planar. Since S is a subspace of a-rank 3 we can find two different parallel lines ℓ_1, ℓ_2 entirely contained in S. Choose two different points from ℓ_1 and one point from ℓ_2. These points together with p clearly do not satisfy the second planar set condition; all lines passing through points in ℓ_1 and ℓ_2 are entirely contained in S, so no line (ℓ_1, ℓ_2, or the line through p parallel to those lines) lies between the others. Hence S is Hübler-maximal.

Now take a Hübler plane P. We know that P has a-rank 3, and that all planar sets are contained in a unique Hübler plane. Since P itself is a planar set it cannot be contained in a strictly larger plane; in particular it cannot be contained in an a-rank 3 subspace. Hence P itself has to be a subspace. □

Finally we can turn, via two propositions, to a complete characterisation of Hübler geometries. We begin by giving sufficient conditions for satisfying the first four axioms.

Proposition 5.11. *Every a-geometry over an R-module M of a-rank at least 3 satisfies Hübler's first four axioms, if the lines are taken as the subspaces of a-rank 2 and $|||$ is used for the parallelity relation.*

Proof. Remember that we also have to show that the set of all lines is nonempty.

$\mathcal{L} \neq \emptyset$ This follows since the a-rank of the geometry is greater than 1.

Axiom 1 Two different points p, q are contained in the unique line $\langle p, q\rangle_{\mathrm{a}}$.

Axiom 2 Every a-rank 2 subspace (line) contains at least two points, and since the a-rank of the geometry is at least 3 there are points outside of each line.

Axiom 3 The parallelity equivalence relation is given by $|||$; see Proposition 3.22.

Axiom 4 We have to characterise the translations before showing that this axiom is satisfied. Any function $T : M \to M$ of the form $T(p) = p + q$ for any $q \in M$ is obviously a bijection (its inverse uses $-q$). Assume $T \neq \mathsf{id}$, i.e. $q \neq 0$. We proceed to show that T satisfies the three translation properties:

1. Per definition we have $\ell \,|||\, T(\ell) = \ell + q$ for any line ℓ.
2. Since $q \neq 0$ we have that $T(p) \neq p$ for any point $p \in M$.

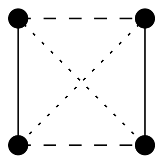

Figure 5.1: The figure shows the four points of $(\mathbb{Z}_2)^2$ equipped with a line set that does not correspond to the a-rank 2 subspaces of the corresponding $\mathbb{Z}$-module (which is of a-rank 1). Lines set in the same style are parallel.

3. Let $D = \{ \langle p, p+q \rangle_{\mathrm{a}} \mid p \in M \}$. Since $q \neq 0$ all these lines (more than one) are well-defined. Furthermore all the lines are parallel since $\langle p, p+q \rangle_{\mathrm{a}} + (p' - p) = \langle p', p'+q \rangle_{\mathrm{a}}$ for arbitrary points $p, p' \in M$. This also shows (by Proposition 3.22) that there is no line $\ell \notin D$ parallel to the lines in D. Hence D is a direction.

We can now show that Axiom 4 is satisfied. For any pair of points $p, q \in M$ there is a translation $T(r) = r + (q - p)$ that maps p to q. (By a Hübler theorem this translation is unique, which means that the translations are exactly the bijections of the form $T(p) = p + q$.) □

Note that all models of Hübler's first four axioms are not necessarily of the kind specified in the preceding proposition. The first four axioms guarantee the existence of an abelian group ($\mathbb{Z}$-module) and the point-translation correspondence that we have used. However, the lines may not correspond to the a-rank 2 subspaces.

Take the $\mathbb{Z}$-module over $(\mathbb{Z}_2)^2$, for instance. Let the line set and the partition of the lines into directions be given by Figure 5.1. Then it is easy to verify that Axioms 1–4 are satisfied. (The three non-identity translations, one for each direction, map every point to the point in the other end of the line that belongs to the direction in question.) The problem is that $(\mathbb{Z}_2)^2$ is of a-rank 1 since $2(m, n) = (0, 0)$ for every point $(m, n) \in (\mathbb{Z}_2)^2$. This module obviously is not torsion free. By going to the $\mathbb{Z}_2$-module over $(\mathbb{Z}_2)^2$ instead, a module which is torsion free, we get a correspondence between the lines and the a-rank 2 subspaces as before. Despite this no full characterisation of Hübler's first four axioms in terms of modules has been discovered.

Let us now give necessary and sufficient conditions for the next three axioms, in the case where the first four axioms are already satisfied and the integral domain is $\mathbb{Z}$.

Proposition 5.12. *Let M be a geometry over a $\mathbb{Z}$-module satisfying the conditions in Proposition 5.11 (and hence Hübler's first four axioms), and let total orders be defined on the points of every line as in Theorem 4.1. Then Hübler's Axioms 5–7 are equivalent to the generator property.*

Proof. First note that one of Hübler's results is that every geometry satisfying the first seven axioms exhibits the generator property (with $R = \mathbb{Z}$).

Now to the other direction. Let p be a point of a line ℓ with generator g. Then $p + g$ and $p - g$ are different from p, and since $g \neq -g$ (by Corollary 5.8) we also have $p + g \neq p - g$. Furthermore we have, by Lemma 5.6, that p lies between $p - g$ and $p + g$ (since $-1 < 0 < 1$). Hence Axiom 5 is satisfied. Now take another point $q \in \ell$. By the generator property we know that $q = p + ng$ for some $n \in \mathbb{Z}$, and by Lemma 5.6 we thus know that there are at most $|n| - 1$ points between p and q, and Axiom 6 is satisfied.

Axiom 7 demands more work. Set $T = \{ 1, 2, 3 \}$. Let ℓ_1, ℓ_2, and ℓ_3 be different, parallel lines, and ℓ and ℓ' lines that have points p_i and p'_i, respectively, in common with all the lines ℓ_i, $i \in T$. Then we have to show that $B(p_1, p_2, p_3)$ holds iff $B(p'_1, p'_2, p'_3)$. Assume that this is not true. Then we can have $B(p'_1, p'_2, p'_3)$ while at the same time (for instance) $B(p_1, p_3, p_2)$. It is enough to show that this particular case leads to a contradiction, the other cases are similar.

Assume that ℓ is generated by t and ℓ' by t'. Furthermore, assume that the three parallel lines have the generator s. See Figure 5.2. By choosing the signs of the generators suitably we have, due to the betweenness constraints given and Lemma 5.6, that

$$\begin{aligned} p_3 &= p_1 + m_1 t, \\ p_2 &= p_3 + m_2 t, \\ p'_2 &= p'_1 + m'_1 t', \text{ and} \\ p'_3 &= p'_2 + m'_2 t', \end{aligned} \tag{5.8}$$

for some $m_1, m_2, m'_1, m'_2 \in \mathbb{Z}^+$. We also have, for some $k_i \in \mathbb{Z}$, $i \in T$, that $p'_i = p_i + k_i s$. Since s and t generate lines that are not parallel the set $B = \{ s, t \}$ is d-independent, with $t' \in \langle B \rangle_{\mathrm{d}}$. Now observe that

$$(m'_1 + m'_2)t' = p'_3 - p'_1 = p_3 - p_1 + (k_3 - k_1)s = m_1 t + (k_3 - k_1)s \tag{5.9}$$

and

$$m'_2 t' = p'_3 - p'_2 = p_3 - p_2 + (k_3 - k_2)s = -m_2 t + (k_3 - k_2)s. \tag{5.10}$$

Here $m'_1 + m'_2 > 0$, $m'_2 > 0$, and $m_1 > 0$, but $-m_2 < 0$, so by the Representation Theorem 3.15 we have a contradiction. Thus Axiom 7 is satisfied. □

Wrapping it all up we get the following theorem, with an easy corollary for the last axiom.

Theorem 5.13. *The models of Hübler's first seven axioms are exactly the a-geometries over $\mathbb{Z}$-modules with the generator property that have a-rank at least 3. (Here lines, parallelity, and order is assumed to be defined as above.)*

Proof. First note that as shown above all Hübler geometries are a-geometries over $\mathbb{Z}$-modules satisfying the generator property. Furthermore note that we have never made any use of Axiom 8 in proving these things. Finally we remark that the a-rank of the geometry has to be at least 3 due to Axiom 2, namely to have at least one point outside of each line.

The other direction of the theorem follows directly from Propositions 5.11 and 5.12. □

Corollary 5.14. *Given a model of the first seven axioms the eighth axiom is equivalent to the plane generator property. (In addition to the other constructs, planes are also assumed to be defined as above.)*

Remember that the plane generator property implies the line one.

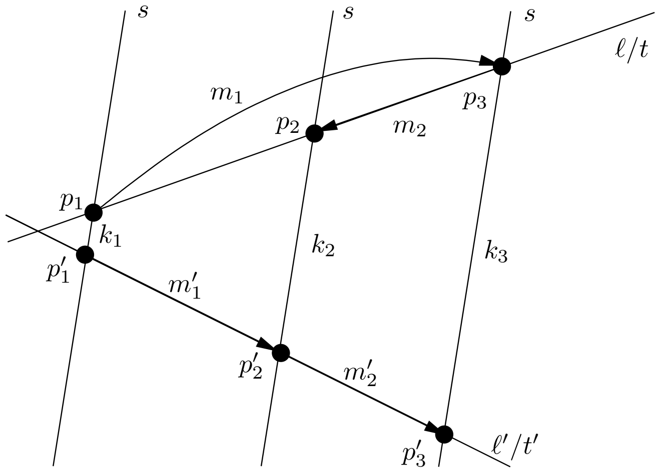

Figure 5.2: The notation used when proving that Axiom 7 is satisfied.

Proof. The eighth axiom implies that each plane is a planar grid, whereby we have the plane generator property.

Now assume that the plane generator property holds. Note that the geometry by the preceding theorem is an a-geometry over a $\mathbb{Z}$-module. Given two different, parallel lines, say ℓ_1 and ℓ_2, we have to show that $\{ \ell \in \mathcal{L} \mid B(\ell_1, \ell, \ell_2) \}$ is finite. Note that all lines between ℓ_1 and ℓ_2 have to lie in the plane $P = \ell_1 \vee \ell_2$ generated by the two parallel lines. (We know from Proposition 3.23 that $\ell_1 \vee \ell_2$ is an a-subspace of rank 3.) Since P by the plane generator property is a planar grid we can apply a Hübler result which states that Axiom 8 is satisfied for all planar grids. □

Of course the plane generator property is just another way of saying that all planes are planar grids. However, this terminology points to the similarity between Axioms 5–7 (in certain cases equivalent to the line generator property) and Axiom 8 (in other cases equivalent to the plane generator property). Axiom 8 was added to the theory by Hübler to ensure a certain level of "discreteness" of the models. It is not known whether even more discreteness can be attained by imposing further restrictions, such as "every geometry of d-rank $\geq n$ has to exhibit the rank n generator property."

If one is willing to accept that a discrete geometry should be finitely generated, which is not at all obvious, then both the line and the plane generator properties can be dropped. This follows since all finitely generated geometries satisfy all the rank n generator properties, up to the rank of the geometry (see Theorem 3.25).

In practice geometries with finite d-rank are much more likely to be used than those with infinite d-rank. However, even if we restrict ourselves to geometries with finite d-rank we cannot say for sure whether Axiom 8/the plane generator property makes any difference or not. This is because it is not established that all a-geometries with finite d-rank are finitely generated. The line generator property is not superfluous since when it is dropped we immediately get a-geometries which have finite d-rank but are not finitely generated, the $\mathbb{Z}$-module over $\mathbb{Q}^n$, $n \in \mathbb{Z}^+$, for instance. The situation when the line generator property holds is currently uncertain. It may be that the rank 1 property implies the rank n one, up to the rank of the module, but this is not at all clear.

5.2 Isomorphism

In this section we do not utilise the identification between points and translations used above, mainly because we want the results of Theorem 5.15 below to be easily comparable with the corresponding t-graph results (see Subsection 2.6.2). For this reason we also go back to the function notation for translations.

Hübler does not define isomorphisms for general axiomatic discrete geometries. Since all Hübler geometries are uniquely characterised by their translation group this definition comes for free though:

Definition 5.1. *Two Hübler geometries are* isomorphic *if their respective translation groups are isomorphic.*

When dealing with two discrete image geometries, the point sets, line sets etc. are indexed: $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{L}_1$, $\mathcal{L}_2$, etc.

The following theorem shows, among other things, that if two geometries are isomorphic, then they are also isomorphic according to the t-graph definition (existence of a betweenness invariant bijection). Some parts of the proof come from the corresponding proofs for t-graphs from Hübler's report.

Theorem 5.15. *Given two discrete image geometries the following statements are equivalent:*

1. *The geometries are isomorphic.*
2. *There are a bijection $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ and a bijection $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ for which $T(p) = q$ iff $\Phi(T)(\varphi(p)) = \varphi(q)$ for all points p and q and all translations T.*
3. *There is a bijection $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$, mapping lines to lines, which is parallelity invariant.*

Furthermore the properties above imply the following properties:

4. *There is a bijection $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ which is betweenness invariant.*
5. *The d-submodule ranks of $\mathcal{T}_1$ and $\mathcal{T}_2$, when viewed as $\mathbb{Z}$-modules, are equal (the geometries have equal "dimension").*

Proof. $1 \Rightarrow 2$ Let $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ be an isomorphism. Fix a point (an origin) $p_o \in \mathcal{P}_1$ and another point $p'_o \in \mathcal{P}_2$. Define $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ by

$$\varphi(p) = \Phi(T)(p'_o), \qquad T(p_o) := p. \tag{5.11}$$

By defining $\psi : \mathcal{P}_2 \to \mathcal{P}_1$ by

$$\psi(p') = \Phi^{-1}(T')(p_o), \qquad T'(p'_o) := p', \tag{5.12}$$

we see that $\psi(\varphi(p)) = p$ and $\varphi(\psi(p')) = p'$, so $\psi = \varphi^{-1}$, and thus φ is a bijection.

Now assume that $T(p) = q$ for some translation $T \in \mathcal{T}_1$ and some points $p, q \in \mathcal{P}_1$. Let T_r be defined by $T_r(p_o) := r$ for any point $r \in \mathcal{P}_1$. Then, using that Φ is a group isomorphism, we have

$$\begin{aligned} \varphi^{-1}\left(\Phi(T)(\varphi(p))\right) &= \varphi^{-1}\left(\Phi(T)\left(\Phi(T_p)(p'_o)\right)\right) \\ &= \varphi^{-1}\left(\Phi(T \circ T_p)(p'_o)\right). \end{aligned} \tag{5.13}$$

By (5.12) we get that

$$\varphi^{-1}\left(\Phi(T \circ T_p)(p'_o)\right) = \Phi^{-1}(T')(p_o) \tag{5.14}$$

where $T'(p'_o) := \Phi(T \circ T_p)(p'_o)$, i.e. $T' = \Phi(T \circ T_p)$. Thus

$$\begin{aligned} \Phi^{-1}(T')(p_o) = \Phi^{-1}\left(\Phi(T \circ T_p)\right)(p_o) &= T(T_p(p_o)) \\ = T(p) &= q \end{aligned} \tag{5.15}$$

or $\Phi(T)(\varphi(p)) = \varphi(q)$. By repeating the argument in the other direction we see that $T(p) = q$ iff $\Phi(T)(\varphi(p)) = \varphi(q)$.

$2 \Rightarrow 1$ It is enough to show that $\Phi(T_1) \circ \Phi(T_2) = \Phi(T_1 \circ T_2)$ for arbitrary translations $T_1, T_2 \in \mathcal{T}_1$, thereby showing that Φ is a group isomorphism. Pick any point $p \in \mathcal{P}_2$ and let $T_2(p) = q$, $T_1(q) = r$. We have

$$\begin{aligned} \left(\Phi(T_1) \circ \Phi(T_2)\right)(\varphi(p)) &= \Phi(T_1)(\varphi(q)) = \varphi(r) \\ &= \varphi\left(\left(T_1 \circ T_2\right)(p)\right) = \Phi(T_1 \circ T_2)(\varphi(p)), \end{aligned} \tag{5.16}$$

and since φ is a bijection we are done.

$2 \Rightarrow 3$ Let φ and Φ be the bijections guaranteed by property 2. We begin by showing that φ maps lines to lines. Take any line $\ell \in \mathcal{L}_1$ with generator $G \in \mathcal{T}_1$ and $p \in \ell$. Since Φ is a group isomorphism (as proved above) we have $\Phi(G)^n = \Phi(G^n)$, whereby $\Phi(G)^n(\varphi(p)) = \varphi(G^n(p))$. Hence $\varphi(\ell) = \{\, \Phi(G)^n(\varphi(p)) \mid n \in \mathbb{Z} \,\}$. This is a line iff $\Phi(G)$ is a simple translation. Assume $\Phi(G) = S^k$ for some simple translation $S \in \mathcal{T}_2$ and integer $k \in \mathbb{Z}$. We have $G = \Phi^{-1}(S)^k$, so since G is simple we have $k = \pm 1$, and $\Phi(G)$ is simple.

It remains to show that φ is parallelity invariant. This follows immediately since the generator G of a line is (implicitly) mapped by φ to $\Phi(G)$, and Φ is a bijection.

$3 \Rightarrow 2$ Let $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ be a parallelity invariant bijection mapping lines to lines. Define $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ by $\Phi(T) = \varphi \circ T \circ \varphi^{-1}$.

Note that given $\ell(p, q) \in \mathcal{L}_1$ we have $\varphi(\ell(p, q)) = \ell(\varphi(p), \varphi(q))$. This follows since $\varphi(\ell(p, q))$ is a line including the points $\varphi(p)$ and $\varphi(q)$ (and these points are different).

First we have to show that Φ is well-defined. It is easy to see that it is type correct, but it is not entirely obvious that it maps translations to translations. Let $T \in \mathcal{T}_1$ be an arbitrary translation. If $T = \mathsf{id}$, then $\Phi(T) = \mathsf{id}$, so assume $T \neq \mathsf{id}$. Since φ and T are bijections it follows that $\Phi(T)$ is a bijection, and since $T \neq \mathsf{id}$ we have $\Phi(T) \neq \mathsf{id}$. Furthermore $\Phi(T)$ satisfies the translation properties:

1. Take any line $\ell' \in \mathcal{L}_2$. We have $\varphi^{-1}(\ell') \in \mathcal{L}_1$ and $T(\varphi^{-1}(\ell')) \,||\, \varphi^{-1}(\ell')$. Since φ is parallelity invariant this implies that $\Phi(T)(\ell') = \varphi(T(\varphi^{-1}(\ell'))) \,||\, \ell'$.
2. Take any point $q' \in \mathcal{P}_2$. Since $\varphi^{-1}(q') \neq T(\varphi^{-1}(q')) = (\varphi^{-1} \circ \Phi(T))(q')$ we get that $q' \neq \Phi(T)(q')$.
3. Take any two different points $p', q' \in \mathcal{P}_2$. We need to show that $\ell(p', \Phi(T)(p')) \,||\, \ell(q', \Phi(T)(q'))$. We have

$$\begin{aligned}
&\varphi^{-1}(\ell(p', \Phi(T)(p'))) = \\
&\ell(\varphi^{-1}(p'), T(\varphi^{-1}(p'))) \;\; || \\
&\ell(\varphi^{-1}(q'), T(\varphi^{-1}(q'))) = \\
&\varphi^{-1}(\ell(q', \Phi(T)(q'))),
\end{aligned} \tag{5.17}$$

and since φ is parallelity invariant this implies that the lines are parallel. We still have to prove that $D := \{\, \ell(p', \Phi(T)(p')) \mid p' \in \mathcal{P}_2 \,\}$ is a direction, so assume that there is a line $\ell' \in \mathcal{L}_2 \setminus D$ which is parallel to the lines in D. Assume further that $r' \in \ell'$. Since $\ell' \,||\, \ell(r', \Phi(T)(r'))$ we get that $\ell' = \ell(r', \Phi(T)(r'))$, and we are done.

Now we know that Φ is well-defined. It is also a bijection, since it has an inverse defined by $\Phi^{-1}(T') = \varphi^{-1} \circ T' \circ \varphi$. Finally it is easy to verify that Φ and φ satisfy the necessary requirements; $T(p) = q$ iff $\Phi(T)(\varphi(p)) = \varphi(q)$, $p, q \in \mathcal{P}_1$.

$2 \Rightarrow 4$ Let φ and Φ be the bijections guaranteed by property 2. Take any three points $p, q, r \in \mathcal{P}_1$ with $B(p, q, r)$. Then there are a translation $T \in \mathcal{T}_1$ and positive integers $i, j \in \mathbb{Z}^+$ such that $q = T^i(p)$ and $r = T^j(q)$. Since Φ is a group isomorphism (as shown above) we know that $\Phi(T)^n = \Phi(T^n)$ for any $n \in \mathbb{Z}$. We get that $\Phi(T)^i(\varphi(p)) = \varphi(q)$ and $\Phi(T)^j(\varphi(q)) = \varphi(r)$, i.e. $B(\varphi(p), \varphi(q), \varphi(r))$.

Because φ and Φ are bijections we also have that $B(p', q', r')$ implies $B(\varphi^{-1}(p'), \varphi^{-1}(q'), \varphi^{-1}(r'))$, $p', q', r' \in \mathcal{P}_2$. Thus φ is a betweenness invariant bijection and the two geometries are isomorphic.

$1 \Rightarrow 5$ This follows immediately since the two translation groups are isomorphic. □

It is currently unknown whether the last two properties in the theorem above are (independently) equivalent to the first three; it is perhaps reasonable to suspect that they are, but it seems to be hard to prove. We do have a partial result, though: If a geometry is finitely generated (like all t-graphs) and has d-rank n, then it is isomorphic to all other geometries satisfying the same two requirements (see Theorem 3.25).

5.3 Square Grid Geometries

The geometries on square grids of any (finite or infinite) dimension/d-rank greater than or equal to two satisfy all eight of Hübler's axioms. We show this to give an example of a discrete geometry. A concrete model may also be of use in counterexamples. Furthermore, these models seem to capture much of what Hübler had in mind when he came up with his axioms. In fact, by Theorem 3.25, all finitely generated Hübler geometries are isomorphic to the square grid geometry with the same d-rank. Finally, models with infinite d-rank show that there are discrete image geometries without finite bases. These are obviously not finitely generated.

The square grid geometries are essentially the standard $\mathbb{Z}$-modules over $\mathbb{Z}^n$. However, to allow infinite dimensions we replace $\mathbb{Z}^n$ by $\mathcal{P} := \mathrm{Dim} \to \mathbb{Z}$. Here Dim is a (possibly infinite) *dimension set* of cardinality at least 2. We use standard lambda notation to describe points. As an example, the zero of the module we are defining is $\lambda n.0$. Let $p, q \in \mathrm{Dim} \to \mathbb{Z}$ and $k \in \mathbb{Z}$. By defining $p + q := \lambda n.p(n) + q(n)$ and $kp := \lambda n.kp(n)$ it is easy to verify that $\mathcal{P} = \mathrm{Dim} \to \mathbb{Z}$ is a torsion free $\mathbb{Z}$-module, and hence an a-geometry. The d-rank of the geometry is obviously equal to the cardinality of Dim. Since the dimension set has cardinality at least 2 this geometry has a-rank at least 3 (the geometry does not consist of a single line).

When Dim is finite we know, by Theorem 3.25, that the geometry satisfies the plane generator property. For completeness we want to show examples of geometries of infinite d-rank, and then we cannot use the theory for finitely generated torsion free modules over PIDs. Even the reader completely uninterested in geometries with infinite d-rank may find something of value below, since the theory is valid also for finite d-rank, and we treat the structures in more depth than above.

Let us first show that the geometry $\mathcal{P}$ defined above has the irreducible element property, thereby characterising its lines. Define a set of *slopes*,

$$\mathcal{S} := \{ p \in \mathcal{P} \setminus 0 \mid \gcd(p) = 1 \} . \tag{5.18}$$

Here $\gcd : \mathcal{P} \setminus 0 \to \mathbb{Z}^+$ is the greatest common divisor of all the components of a point; it is defined by

$$\gcd(p) := \gcd(\{ p(i) \mid i \in \mathrm{Dim} \}) . \tag{5.19}$$

Now assume that $bs = ap$ for some $s \in \mathcal{S}$, $p \in \mathcal{P}$, and $a, b \in \mathbb{Z}$, $a \neq 0$. Since $\gcd(s) = 1$ it immediately follows that $a|b$, so s is irreducible. Furthermore, take any $p \in \mathcal{P} \setminus \mathcal{S}$. If $p = 0$, then it is obviously not irreducible ($2 \times 0 = 3 \times 0$). On the other hand, if $p \neq 0$, then we can construct a point $p' = \lambda n.\frac{p(n)}{\gcd(p)}$ such that $p = \gcd(p)\, p'$, showing yet again that p is not irreducible. We get that $\mathcal{S}$ is the set of all irreducible points, and furthermore the geometry satisfies the irreducible element property. Hence (by Proposition 3.27) $\mathcal{S}$ is the set of line generators.

It requires more work to show that the plane generator property is satisfied. In fact, we will show directly that Axiom 8 holds. (The proof below can probably be simplified considerably.) Denote the set of all lines by $\mathcal{L}$, and let us use the notation $\ell_p(s)$ for the line passing through p having slope s. Let two different, parallel lines be given, say $\ell_1 = \ell_{p_1}(s)$ and $\ell_2 = \ell_{p_2}(s)$. We have to show that the set $B = \{ \ell \in \mathcal{L} \mid B(\ell_1, \ell, \ell_2) \}$ is finite.

For the notation used, see Figure 5.3. We have that $p_2 - p_1 = kt$ for some $k \in \mathbb{Z} \setminus 0$ and $t \in \mathcal{S}$. The intuition is that every line ℓ between ℓ_1 and ℓ_2 has to pass through exactly one point x in the shaded area of the figure. Using the

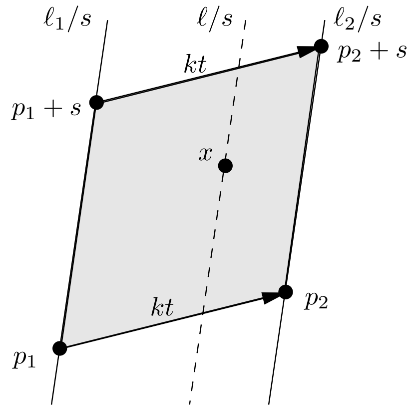

Figure 5.3: The notation used when proving that the plane generator property holds.

associated vector space $F(\mathcal{P}) = \text{Dim} \to \mathbb{Q}$, a vector space over $\mathbb{Q}$, we can state this more precisely as

$$B \subseteq \{\, \ell_x(s) \mid x \in A \,\}, \tag{5.20}$$

where

$$A = \{\, x \in \mathcal{P} \mid x = p_1 + \alpha s + \beta kt,\ \alpha \in [0,1),\ \beta \in (0,1) \,\}. \tag{5.21}$$

(The intervals are intervals in $\mathbb{Q}$.) This remains to be proved, though.

Lemma 5.16. $B \subseteq \{\, \ell_x(s) \mid x \in A \,\}$.

Proof. For every line $\ell \in B$ we have $p_1 + k't' \in \ell$ and $p_1 + (k'+k'')t' \in \ell_2$ for some $t' \in \mathcal{S}$ and $k', k'' \in \mathbb{Z}^+$. Thus

$$(k'+k'')t' = kt + ms \tag{5.22}$$

for some $m \in \mathbb{Z}$, and we have

$$k't' = \frac{k'}{k'+k''}kt + \frac{k'm}{k'+k''}s. \tag{5.23}$$

Here $\frac{k'}{k'+k''} \in (0,1)$. Furthermore the inequality

$$\frac{k'm}{k'+k''} + m' \in [0,1) \tag{5.24}$$

has a unique solution $m' \in \mathbb{Z}$. Since s generates ℓ we still have

$$p_1 + k't' + m's = p_1 + \frac{k'}{k'+k''}kt + \left(\frac{k'm}{k'+k''} + m'\right) s \in \ell. \tag{5.25}$$

Hence

$$p_1 + \alpha s + \beta kt \in \ell \tag{5.26}$$

holds for at least one pair (α, β), $\alpha \in [0,1)$, $\beta \in (0,1)$. In fact, since $\{\, s, t \,\}$ is independent this solution is unique. Thus $B \subseteq \{\, \ell_x(s) \mid x \in A \,\}$. □

Now it only remains to show that A is finite. We begin with another result.

Lemma 5.17. *Given $s, t \in \mathcal{S}$ with $s \neq \pm t$, there are two members $n_1, n_2 \in \mathrm{Dim}$ such that $s(n_1)t(n_2) \neq s(n_2)t(n_1)$.*

Proof. Assume that this is not the case. Then $s(n_1)t(n_2) = s(n_2)t(n_1)$ for all $n_1, n_2 \in \mathrm{Dim}$. This implies that $s(n_1)t = t(n_1)s$. Furthermore, because $s \in \mathcal{S}$ we can choose n_1 such that $s(n_1) \neq 0$. This implies that $t(n_1) \neq 0$. Now, because $\gcd(s) = 1$ we get that $s(n_1) | t(n_1)$, so $t = \frac{t(n_1)}{s(n_1)} s$. This can only be the case if $t = \pm s$, which is a contradiction. □

Lemma 5.18. *A is finite.*

Proof. First note that

$$|A| = |\{\, x \in \mathcal{P} \mid x = \alpha s + \beta k t,\ \alpha \in [0,1),\ \beta \in (0,1) \,\}|\,. \tag{5.27}$$

By Lemma 5.17 we have that there are $n_1, n_2 \in \mathrm{Dim}$ such that $s(n_1)t(n_2) \neq s(n_2)t(n_1)$. Let $\mathcal{P}' := \{\, n_1, n_2 \,\} \to \mathbb{Z}$ define a new square grid geometry, and let $s', t' \in \mathcal{P}'$ be defined by

$$s' := \lambda n.s(n), \qquad t' := \lambda n.t(n). \tag{5.28}$$

Because $s'(n_1)t'(n_2) \neq s'(n_2)t'(n_1)$ we have that $\{\, s', t' \,\}$ is independent. Let

$$A' := \{\, x \in \mathcal{P}' \mid x = \alpha s' + \beta k t',\ \alpha \in [0,1),\ \beta \in (0,1) \,\}\,. \tag{5.29}$$

Observe that the independence of s' and t' implies that

$$|A| \leq |A'|\,, \tag{5.30}$$

because every pair (α, β) corresponding to a (unique) member of A also corresponds to a unique member of A'. Finally, using pair notation for functions,

$$\begin{aligned} A' \subseteq \{\{\, \langle n_1, k_1 \rangle,\ \langle n_2, k_2 \rangle \,\} \mid k_1, k_2 \in \mathbb{Z},\ |k_1| < |s'(n_1)| + |kt'(n_1)|\,, \\ |k_2| < |s'(n_2)| + |kt'(n_2)|\}\,, \end{aligned} \tag{5.31}$$

which shows that A' is finite. Hence A is finite. □

This concludes the proof showing that Axiom 8 is satisfied. We also have an immediate corollary.

Corollary 5.19. *There are Hübler geometries of any d-rank larger than or equal to 2.*

5.4 Conclusions and Future Work

As noted in the introduction, a good axiomatisation of discrete geometry should allow many possible models. In this light it is discouraging to find that Hübler's geometries have a relatively limited set of models, only $\mathbb{Z}$-modules satisfying some criteria. We noted already in the conclusions of Chapters 3 and 4 that modules over ordered domains are not suited for modelling (discrete) geometry in general. Hübler shows himself that his axioms disallow cyclic translations (necessary for e.g. cylindrical models) and finite models. Furthermore we have not ruled out the possibility that all Hübler geometries of the same d-rank are isomorphic; that is an interesting question yet to be answered, and a positive answer would show once and for all that Hübler's axioms are too restrictive.

When reading Hübler's report it is easy to get the impression that he has worked hard to make sure that almost nothing besides ordinary square grid geometries satisfy his axioms. This may not be very surprising, considering that his main focus is plane image analysis. Maybe it is possible to get a more general framework by weakening, changing, or removing some of the axioms, something which Hübler himself indicates. At least Hübler's work gives some ideas on how to characterise discreteness. Whether rank n generator properties are the right way to go remains to be discovered.

Now on to some comments that may not be very important if this framework is not going to be used, but nevertheless may be interesting. Note first that even though the framework is not as general as we would want it to be, it might still be useful in some cases. After all, cylindrical geometries are not in very common use, and finite geometries can presumably be treated as finite subsets of infinite geometries. When algorithms are developed for this variant of finite geometry it may lead to problems, though, since (intermediate) values computed may not be restricted to the finite subset.

We could easily have extended the results on convexity from Chapter 4 to Hübler geometries, and related those results to Hübler's own. However, this is so straightforward that we leave it as an exercise for the interested reader. Note that our work on convexity based on half-spaces is more general than Hübler's work on strict convexity, which is restricted to planes.

No direct reference to oriented matroids is made in this chapter, since this is not necessary. Of course the results of Chapter 4 imply that all Hübler geometries have an oriented matroid structure.

We have not characterised all axioms independently. Axiom 8 is characterised by itself, and Axioms 5–7 are characterised as one entity. For completeness it would be nice to characterise Axioms 1–4 as well, if possible. They would have to be characterised all together, since we do not get a module until after Axiom 4. Another interesting question is to which degree Hübler's axioms are independent, and whether the rank 1 generator property implies all higher ones. The last statement would imply that all geometries of a fixed, finite d-rank are isomorphic, and is hence related to the question about how restrictive Hübler's axioms are.

Chapter 6

Axiomatic Oriented Projective Geometry

This brief chapter indicates some possible ways in which to axiomatise oriented projective geometry. The first approach is to use involution-OMs whose underlying matroids are projective. It is shown that all Stolfi OPGs (see Section 2.7) satisfy these requirements; the Stolfi OPGs form part of a set of models based on torsion free modules over ordered domains. Another approach uses projective, and possibly also simple, oriented matroids (see the conclusion of this chapter).

6.1 Projective Involution-OMs

Let us begin by giving an explicit characterisation of Stolfi OPG convex closures.

Proposition 6.1. *Let M be a Stolfi OPG. Denote the convex closure of this geometry by $[\cdot]$. Then $[\cdot] : \wp(M) \to \wp(M)$ has the explicit representation*

$$[S] = \left\{ \left[\!\!\left[\sum_{i=1}^{n} a_i x_i \right]\!\!\right] \,\middle|\, [\![x_i]\!] \in S,\ a_i \in \mathbb{R}^+,\ n \in \mathbb{Z}^+ \right\} \setminus [\![0]\!] .$$

Proof. Denote the right hand side of the equation by $Q(S)$.

As noted in Section 2.7 a subset $S \subseteq M$ is convex iff $[\![x+y]\!] \in S$ for all independent points $[\![x]\!], [\![y]\!] \in S$. Two points are independent if they are not equal or antipodal.

Given two independent points $p, q \in Q(S)$ with $p = [\![\sum_{i=1}^{m} a_i x_i]\!]$ and $q = [\![\sum_{i=1}^{n} b_i y_i]\!]$ we have to show that $r := [\![\sum_{i=1}^{m} a_i x_i + \sum_{i=1}^{n} b_i y_i]\!] \in Q(S)$. (All sums are assumed to be of the form given in the definition of $Q(S)$.) Since p and q are independent we know that $r \neq [\![0]\!]$, which is all we need. Hence $Q(S)$ is convex.

We also have to show that $Q(S)$ is the minimal convex set containing S. We do this by showing, by induction on n, that every point $[\![\sum_{i=1}^{n} a_i x_i]\!] \in Q(S)$ has to be in $[S]$. When $n = 1$ this is obvious. Now assume that the property holds for all sums with n elements, $n \in \mathbb{Z}^+$. Take a sum $p := \left[\!\left[\sum_{i=1}^{n+1} a_i x_i\right]\!\right] \in Q(S)$ and let $q := [\![\sum_{i=1}^{n} a_i x_i]\!]$. If $q = [\![0]\!]$ then $p = [\![x_{n+1}]\!] \in [S]$, so assume $q \neq [\![0]\!]$, i.e. $q \in Q(S)$. By the induction hypothesis we get that $q \in [S]$. If q and $r := [\![a_{n+1}x_{n+1}]\!]$ are independent then this implies that $p \in [S]$. If on the other hand the two points are dependent, then we have two different cases:

$r = q$ We immediately get $p = q \in [S]$.

$r = \neg q$ This implies that $\neg q = [\![x_{n+1}]\!] \in S$. We either get $p = q$ or $p = \neg q$ (since $p \in Q(S)$ implies that $p \neq [\![0]\!]$). In either case we have $p \in [S]$.

The proposition follows by the induction principle. □

Now we can show one way in which Stolfi OPGs are related to oriented matroids. There is no point in restricting attention only to Stolfi's models, though.

Theorem 6.2. *Let M be a torsion free R-module, where R is an ordered domain. Let M' be the quotient set $(M \setminus 0)/{\sim}$, where the equivalence relation $\sim$ is defined by $m \sim m'$ iff there is some $r, r' \in R^+$ with $rm = r'm'$. Denote the equivalence class containing $m \in M \setminus 0$ by $[\![m]\!]$, and define $[\![0]\!] := \{ 0 \}$. Define $\neg : M' \to M'$ by $\neg [\![m]\!] := [\![-m]\!]$ and $[\cdot] : \wp(M') \to \wp(M')$ by*

$$[S] := \left\{ \left[\!\!\left[\sum_{i=1}^{n} a_i s_i \right]\!\!\right] \middle| [\![s_i]\!] \in S,\ a_i \in R^+,\ n \in \mathbb{Z}^+ \right\} \setminus [\![0]\!] .$$

Then $(M', [\cdot], \neg)$ is an involution-OM whose underlying matroid has a lattice of subspaces which is isomorphic to that of $M/\langle\emptyset\rangle_{\mathrm{d}}$, and hence is projective.

Proof. It is easy to verify that $\sim$ is an equivalence relation, and that $m \neq \neg m$ holds for all $m \in M'$ iff M is torsion free.[1] Furthermore we immediately get $\neg\neg m = m$, $[\emptyset] = \emptyset$, and $[\neg S] = \neg[S]$ for all $m \in M'$, $S \subseteq M'$. By construction $[\cdot]$ is increasing, monotone, and finitary, and it is easy to verify that it is idempotent.

It remains to prove Properties 4 and 5. Assume that $p = [\![x]\!] \in [S \cup \neg p]$ for some $p \in M'$, $S \subseteq M'$. We get that $p = [\![\sum_{i=1}^n a_i s_i - ax]\!]$ for some $[\![s_i]\!] \in S$, $a_i \in R^+$, $a \in R^+ \cup 0$, and $n \in \mathbb{Z}^+$. (Since $p \notin [\neg p] = \{ \neg p \}$ we know that $n \neq 0$.) This implies that $[\![(1 + a)x]\!] = [\![\sum_{i=1}^n a_i s_i]\!]$, i.e. $p \in [S]$.

Now assume that $q \in [S \cup \neg p] \setminus [S]$ for some $S \subseteq M'$, $p, q \in M'$, $p = [\![x]\!]$, $q = [\![y]\!]$, $q \neq \neg p$. We get that $[\![y]\!] = [\![\sum_{i=1}^n a_i s_i - ax]\!]$ for some $[\![s_i]\!] \in S$, $a, a_i \in R^+$, and $n \in \mathbb{N}$. Assume now that $[\![s_j]\!] = p$ for some j. (We can assume without limitation that this is true for at most one element $[\![s_j]\!]$.) This implies that $a_j < a$, otherwise we get $q \in [S]$. We now have either $[\![(a - a_j)x]\!] = [\![\sum_{i \neq j} a_i s_i - y]\!]$ or $[\![ax]\!] = [\![\sum_{i=1}^n a_i s_i - y]\!]$, in both cases with no occurrence of $[\![x]\!]$ on the right hand side. In other words $p \in [(S \cup \neg q) \setminus p]$. Hence we know that M' is an involution-OM.

We will now verify that M' has an underlying matroid whose lattice of subspaces is isomorphic to that of the d-matroid over M, which is shown to

[1] Note that $m \neq \neg m$ is not part of the original system, it was added to simplify things. We could probably have simplified things in a similar manner in earlier chapters by disallowing modules that are not torsion free.

be modular in Theorem 3.13. We do this indirectly; the (projective) canonical geometry $M'' := M/\langle\emptyset\rangle_{\mathrm{d}}$ associated with the d-matroid M has a lattice of subspaces which is isomorphic to that of M.

Note that, since M is torsion free, $\langle\emptyset\rangle_{\mathrm{d}} = \{\, 0 \,\}$. For all $m \in M \setminus 0$ define

$$[\![m]\!] := \{\, m' \in M \setminus 0 \,|\, r, r' \in R \setminus 0,\ rm = r'm' \,\}\,. \tag{6.1}$$

We get that $M'' = \{\, [\![m]\!] \,|\, m \in M \setminus 0 \,\}$. Now note that every point $[\![m]\!] \in M''$ corresponds uniquely to two points in M', $[\![m]\!]$ and $[\![-m]\!]$. Let μ be the canonical projection from M' to M'', i.e. $\mu([\![m]\!]) = [\![m]\!]$, and let μ^{-1} be the preimage of μ (modified for set arguments as in Section 3.7). Verifying that $\mathrm{cl}(S) := [S \cup \neg S] = \mu^{-1}\big(\langle\mu(S)\rangle_{\mathrm{d}}^{M''}\big)$ is now straightforward using the explicit characterisations of the closure operators. (The canonical geometry closure operator $\langle\cdot\rangle_{\mathrm{d}}^{M''}$ is defined as in (2.1), but using $\langle\cdot\rangle_{\mathrm{d}}$ instead of cl.) This shows that the two closure operators yield isomorphic lattices of subspaces. Since (M', cl) is the underlying matroid of M' we are done. □

Corollary 6.3. *Every Stolfi OPG M yields an involution-OM $(M, [\cdot], \neg)$ whose underlying matroid is projective.*

Proof. Follows from the definition of a Stolfi OPG and Proposition 6.1. □

To see that not all the models introduced in the theorem above are isomorphic to a Stolfi OPG just note that the $\mathbb{Z}$-module over $\mathbb{Z}^2$ yields an involution-OM with a countably infinite ground set; no Stolfi OPG has this property.

6.2 Conclusions and Future Work

The preceding results show that it may be possible to define an oriented projective geometry as an involution-OM whose underlying matroid is projective. Of course more work needs to be done to determine if or under which extra requirements all the important practical properties of Stolfi OPGs (see Section 1.3) are retained. At least all new models introduced in the theorem above are very similar in structure to the Stolfi OPGs, and hence they probably have most of the Stolfi OPGs' properties. Another point to be made is that although we have already extended the set of OPGs considerably, it is not clear that any of the new models can be called discrete. Take the model based on the $\mathbb{Z}$-module over $\mathbb{Z}^2$, for instance. Although $\mathbb{Z}^2$ is discrete the quotient construction makes the points in some sense densely packed.

Another way to go would be to use the other definition of oriented matroids. A natural proposal is to define OPGs as oriented matroids that are projective, and perhaps also simple. If we do not insist on the matroids being simple we get that all oriented d-matroids (see Theorem 4.11) are models; Theorem 3.13 shows that all d-matroids are projective.

Adding in the simpleness requirement makes it harder to find a model, since this rules out all two-sided models. A suggestion made by Mike Smyth is that half a sphere would do the trick. We give a sketch of the idea: The points on the unit sphere of $\mathbb{R}^3$ is a standard model of the real projective plane, but in this model each point is identified with its antipodal point. Instead take exactly half a sphere, i.e. the upper open hemisphere and half the "equator," open in one endpoint and closed in the other, and use the restrictions of all the great circles of the sphere as lines. Since we have retained all points, lines, and in fact all the structure of the original projective plane we get that this half-sphere is also a projective plane. Each line naturally partitions the half-sphere into two possibly empty sets, and it is easy to see that the cocircuits so obtained satisfy the oriented matroid axiom.

Given the two different proposals above for axiomatising oriented projective geometry the next step is naturally to compare and relate the two systems. Such an examination would probably need to work out the relationship between the underlying notions of oriented matroid first, something which does not seem to have been done yet. It may also be that another definition of oriented matroids is better suited than the two we have considered here.

An interesting question is what kind of duality the OPGs defined above support. As noted in Section 2.7 many projective geometries satisfy some kind of duality. It is currently not clear to what degree this applies to the general notions of projective geometry used here, though. More research is needed. Furthermore an OPG duality should be stronger than the ordinary projective geometry dualities, since it ought to include orientation. We know that Stolfi OPGs have that kind of duality, and it can probably be extended to all the models introduced in Theorem 6.2 as well.

Chapter 7

Conclusions and Future Work: Summary

Note that there is a section "Conclusions and Future Work" towards the end of each chapter with original material. This chapter contains just a brief summary of what has already been stated in those conclusions, together with some general remarks.

Modules over integral and ordered domains have been shown to be useful for characterising Hübler's geometry. However, even with some discreteness assumption added (perhaps some kind of generator property) they do not provide a framework for discrete geometry general enough for our purposes. Examples of geometries that are hard or impossible to treat within this framework include many finite geometries and the geometry of a cylinder.

Having said this there may be more specialised situations where modules can be used, and some of our results may be useful. As an example we have proved several new results about Hübler's system (which has been used by Hübler himself to prove a practical result regarding lines in the digital plane [Hüb89]). Furthermore we have shown quite clearly that torsion free modules over integral (ordered) domains work just as well as vector spaces over (ordered) fields as models of infinite (oriented) matroids.

Despite the drawbacks with Hübler's system matroids and oriented matroids may still be of use in a general framework for discrete geometry, since these systems have many models that are not based on modules. Since oriented matroids include the concepts order and convexity they are probably more useful than ordinary matroids.

Oriented projective geometry (OPG) is the area with least coverage in this report. Drawing on experiences from previous chapters it was easy to prove that all torsion free modules over ordered domains yield models of one of the proposed axiomatisations of OPG. These models include Stolfi's OPGs which have been useful in practice. There is still some way to go until we know if any of the axiomatisations have the properties that one can expect from an OPG, though. Then there is even more work to do to get a characterisation of *discrete* OPGs, a characterisation that preferably should be useful in practice, if this is at all possible.

The common theme of this discussion is that more work needs to be done.

The previous chapters have put forward several possible directions for future work, based on the results in this text, and hopefully this report can serve as a basis for future investigations.

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Appendix A

Errors in Hübler's Thesis

During my work I have found some errors in Hübler's report [Hüb89]. For reference some of these are mentioned below. I have not checked all parts of the report carefully, there may still be other errors.

Page 49 "Der so fixierte Nachbarschaftsgraphen erfüllt zwar alle Eigenschaften von Folgerung 3.2, . . ." No, not property c. There are several displacements mapping a_i to a_{i+1}, $i \in \mathbb{Z}$, e.g. both the one mapping b_i to b_{i+1} and c_i to c_{i+1} and the one mapping b_i to c_{i+1} and c_i to b_{i+1}.

Lemma 3.2, pages 52–53 This lemma states that all members of a t-graph basis are simple. This is however wrong. Take the t-graph on the two-dimensional square grid, for instance. The translations mapping points one step to "the right," two steps "up" and three steps "down" together constitute a basis, but they are not all simple. The error in the proof lies in the assumption that $1 - n \cdot n_i$ divides $n_j \cdot n$ for all $j \in \{ 1, \dots, \ell \}$, $j \neq i$.

Folgerung 3.10, pages 67–71 In the proof of a $\Rightarrow$ b an argument valid only in two dimensions is used, and in the proof of b $\Rightarrow$ c the case where $g_1 \,||\, g_3$ is not covered. My proof of Theorem 5.15, which is the equivalent for discrete image geometries, seems to take care of the last flaw, though.

Satz 4.3, pages 84–88 One simple mistake on the first page; $A \circ B = ID$ is not generally true, but it does not matter because $A \circ B = A$ or $A \circ B = B$.

There is another mistake at the beginning of the second page, ". . . die Gerade $g' := A \circ B(g(p, q))$ eine von $g(p, q)$ verschiedene, aber zu $g(p, q)$ parallele Gerade." I can give a counterexample on the two-dimensional square grid t-graph: Let A be the translation mapping points one step "up and to the left," and let B map one step to "the right." Then any "vertical" line is mapped onto itself.

However, the property that $A \circ B(p) \neq p$ for any point p can still be proved. Assume that $A \circ B(p) = q \neq p$, while at the same time $A \circ B(r) = r$. Then we know that $B(r) = s \neq r$ for some point s, and $A(s) = r$. Since $g(r, p) \,||\, g(r, q)$ we get that the two lines are equal. Now assume that $B(p) = t$, $A(t) = q$. We get that $g(p, t) \,||\, g(r, s)$, and $g(s, r) \,||\, g(t, q)$, so p, q, t, r, s all have to belong to the same line. Now take any point outside this line, say u. Assume that $A \circ B(u) \neq u$. By reasoning as above we get that $u \in g(r,s)$, a contradiction, so $A \circ B(u) = u$ for all points outside $g(r,s)$. We get that $g(p,u) \,||\, g(q,u)$, implying that $p = q$, a contradiction.

Lemma 4.3, pages 95–96 The lemma states that there are no cyclic translations. This is not exactly true, because id is cyclic according to the definition. However, both earlier and in this lemma Hübler acts as if id was not covered by the definition.

Folgerung 4.12, page 101 Add $M \subseteq N$ to the set conditions.

Page 103 Perhaps the intention is for one of the $\bigvee$ really to be a $\bigwedge$.

4.7. Ein wichtiges Beispiel, pages 115–120 The motivating example for Axiom 8 is completely flawed. First there is a simple typo in the definition of the point set. I am certain that the definition should be $\mathcal{P} := \bigcup_{i \in \mathbb{N}} P_i$, not $\mathcal{P} := \bigcap_{i \in \mathbb{N}} P_i$, since the second form implies that $\mathcal{P} = \mathbb{Z}^2$.

Secondly, the presented "model" is not a model. Axiom 6 fails since all points e_1^i belong to the same line, with all points e_1^i, $i \geq 1$ lying between e_1^0 and $-e_1^0 + e_2$. Hübler's mistake lies in "Behauptung 2," which does not hold. To see this, note that $e_1^i \in P_j$ iff $j \geq i$.

Finally, the "model" is not densely embedded in $\mathbb{R}^2$. It is easy to check that there are no points within a distance $\frac{1}{9}$ of $(\frac{1}{6}, \frac{1}{6})$. When showing that the "model" is dense Hübler for some reason switches to another point set; the original set is defined using three parameters, i, k, and j, while the second one uses four, i, k, u, and v. By the way, also this new "model" fails to be a model, in the same way as above.